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DISCONTINUOUS SOLUTIONS IN L^∞ FOR HAMILTON-JACOBI EQUATIONS***

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Abstract

An approach is introduced to construct global discontinuous solutions in L^∞ for Hamilton-Jacobi equations. This approach allows the initial data only in L^∞ and applies to the equations with nonconvex Hamiltonians. The profit functions are introduced to formulate the notion of discontinuous solutions in L^∞. The existence of global discontinuous solutions in L^∞ is established. These solutions in L^∞ coincide with the viscosity solutions and the minimax solutions, provided that the initial data are continuous. A prototypical equation is analyzed to examine the L^∞ stability of our L^∞ solutions. The analysis also shows that global discontinuous solutions are determined by the topology in which the initial data are approximated.

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§1. Introduction

We are concerned with global discontinuous solutions in L^∞ of the Cauchy problem for Hamilton-Jacobi equations

$$u_t + H(t, x, u, Du) = 0, \qquad x \in \mathbb{R}^n,\ 0 \le t \le T, \tag{1.1}$$

$$u(0, x) = \varphi(x). \tag{1.2}$$

This problem for continuous solutions has been extensively studied in many relevant articles such as Hopf[20], Lax[24], Douglis[12], Fleming[15], Kruzkhov[23], Friedman[17], Krassovski-Subbotin[22], Crandall-Lions[9], Crandall-Evans-Lions[10], Lions-Souganidis[26], Capuzzo Dolcetta-Lions[6], Subbotin[30], and Crandall-Ishii-Lions[11]. For more complete references, we refer to some recent monographs of Benton[5], Lions[25], Fleming-Soner[16], Barles[2], Bardi-Capuzzo Dolcetta[1], and Subbotin[29]. The theory of continuous viscosity solutions

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has been established for Hamilton-Jacobi equations, since Crandall and Lions introduced the notion in [9].

Our main interest is to explore possible approaches to construct discontinuous solutions in L^∞ for Hamilton-Jacobi equations. Constructing such solutions is important from both theoretical and practical points of view. The experimental and theoretical results in game theory, control theory, optics, and conservation laws indicate the significance and necessity to understand the discontinuous solutions.

Ishii[21], Barles-Perthame[3], Barron-Jensen[4], and Subbotin[29] made efforts in studying the discontinuous solutions. Their notions are in the context of semicontinuous solutions. In Ishii[21], the Perron method was introduced to show the existence of possible semicontinuous solutions provided that either semicontinuous supsolutions or subsolutions exists, while the data were assumed continuous. Barles-Perthame[3] studied Bellman equations associated with optimal control problems and showed the uniqueness of upper-semicontinuous viscosity solutions under some compatibility conditions. Barron-Jensen[4] showed the existence and uniqueness of upper-semicontinuous solutions with upper-semicontinuous initial data for Lipschitz convex Hamiltonians. Subbotin[29] defined the semicontinuous solutions as the limits of continuous solution sequences and showed the existence and uniqueness of such solutions, provided there exists a supsolution. For general non-convex Hamiltonians, one can not expect semicontinuous solutions to exist in general, which motivates us to explore new possible approaches for constructing discontinuous solutions of (1.1)–(1.2) beyond the class of semicontinuous functions.

There are basically two ways to study the continuous solutions of Hamilton-Jacobi equations. One way is the viscosity method as mentioned above. The other is based on the characteristics. In the latter, Subbotin introduced the notion of minimax solutions via differential inclusions to study Hamilton-Jacobi equations (see [29,30]). Motivated by the earlier works, we propose an approach to solve the existence of discontinuous solutions in L^∞ for Hamilton-Jacobi equations with general L^∞ initial data. Our approach and ideas are quite general. This approach allows the initial data only in L^∞ and applies to non-convex Hamiltonians. One of the main ingredients in our approach is to introduce the profit functions for constructing discontinuous solutions in L^∞ .

The philosophy behind our approach for discontinuous solutions is very natural from game-theoretical point of view. It is known that a solution of Hamilton-Jacobi equations is a value function of a two-player, zero-sum differential game. Our profit functions can be roughly considered as the deposits subtracted from the payoff functions. Our solutions can be interpreted as follows: The supsolutions satisfy one player's will, while the subsolutions satisfy the other's; the exact solutions must fulfill both players' wills. The profit functions are introduced to justify whether the solutions fulfill the players' wills. The theme of two-player, zero-sum games is that each player maximizes what he obtains and minimizes what he losses. Our profit functions exactly follow this theme even for discontinuous solutions.

From Section 2 to Section 4, we present our theory in L^∞. In Section 2, appropriate terminologies are introduced to define and study the profit functions. In Section 3, we present the notion of solutions in L^∞ and show the existence of global solutions by taking the infimum of the supsolution set or the supremum of the subsolution set. In Section 4, we show that the L^∞ solutions coincide with the viscosity solutions introduced in Crandall-Lions[9] and the minimax solutions in Subbotin[29,30], provided the initial data are continuous.

In Section 5, a prototypical equation is studied to analyze the stability of discontinuous solutions in L^∞. Two lemmas on the measurability of a given set are shown in Appendix. The analysis in Section 5 indicates that our solutions are stable with respect to the initial data in L^∞. Moreover, from the game-theoretical point of view, the initial data are generally obtained in the essential sense, which is another point for us to consider our solutions in the L^∞ topology. Our analysis also shows that global discontinuous solutions are sensitive to the topology in which the initial data are approximated in some cases. Even for these cases, our solutions can produce all other possible solutions as observed by sacrificing the L^∞ stability and approximating the initial data via other appropriate topologies if one wishes. It would be interesting to study further the behavior of the discontinuous solutions in L^∞ established here.

§2. Profit Functions and Their Regularity

To display our ideas and methods in a clear setting, we make the following assumptions on the Hamiltonian $H(t,x,z,p)$ of the Cauchy problem (1.1)–(1.2):

(A1) $H(t,x,z,p)$ is continuous in (t,x,z,p) and increasing in z;

(A2) $|H(t,x,z,p_1)-H(t,x,z,p_2)|\le C_0(1+|x|)|p_1-p_2|$, and
$|H(t,x,z,0)|\le C_0(1+|x|+|z|)$, for all $t\in(0,T]$;

(A3) $|H(t,x_1,z,p)-H(t,x_2,z,p)|\le \lambda(L)(1+|p|)|x_1-x_2|$, where $|x_1|,|x_2|\le L$;

(A4) $|H(t,x,z_1,p)-H(t,x,z_2,p)|\le C_0(1+|x|+|p|)|z_1-z_2|$.

Remark 2.1. The proof given below shows that hypotheses (A2)-(A4) can be weakened. Typical examples of such Hamiltonians include:

$$H(t,x,z,p)=|p|,\quad H(t,x,z,p)=\sqrt{1+|p|^2},\quad\text{and}\quad H(t,x,z,p)=x|p|.$$

See Glimm et al[18], Sethian[28], and the references cited therein. In particular, the Hamiltonian $H=|p|$ was first introduced in combustion by Landau as a flame propagation model (see [28]).

We first introduce some notations and definitions. Denote

$$B^d(x,r)=\Big\{y\in\mathbb{R}^d\ \Big|\ \Big(\sum_{i=1}^d (y_i-x_i)^2\Big)^{\frac12}<r\Big\}.$$

Define the essential infimum and supremum of an $L^\infty_{\rm loc}(\mathbb{R}^d)$ function $v(x)$ at every point $x\in\mathbb{R}^d$:

$$I(v)(x)\equiv\sup_{A\in S_x}\operatorname*{ess\,inf}_{y\in A} v(y),\quad S(v)(x)\equiv\inf_{A\in S_x}\operatorname*{ess\,sup}_{y\in A} v(y),$$

where

$$S_x=\Big\{A\subset\mathbb{R}^d\text{ measurable}\ \Big|\ \lim_{r\to0}\frac{m(A\cap B^d(x,r))}{m(B^d(x,r))}=1\Big\}.$$

It is clear that $I(v)(x)$ and $S(v)(x)$ are well defined at every point $x\in\mathbb{R}^d$, and $I(v)(x)=S(v)(x)$ almost everywhere.

Now we introduce the winning and losing profit functions.

Definition 2.1. *Fix $\tau \in [0,T]$ and $p(t,x) \in C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)$. Given a measurable function v and a position (or value) function f, we define the winning and losing profit functions:*

$$\Lambda_-^v(t,x,(\tau,f,p)) = \inf\{S(v)(x(\tau)) - z(\tau)\,|\,(x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}, \tag{2.1}$$

$$\Lambda_+^v(t,x,(\tau,f,p)) = \sup\{I(v)(x(\tau)) - z(\tau)\,|\,(x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}, \tag{2.2}$$

where $\mathrm{Sol}(t,f(t,x),p)$ *denotes the set of solutions:*

$$(x(\cdot),z(\cdot)) : [\tau,t] \to \mathbb{R}^n\times\mathbb{R} \qquad \text{for} \quad t \geq \tau$$

of the characteristic inclusions, $(\dot{x}(\cdot),\dot{z}(\cdot)) \in E(t,x,z,p)$ satisfying the conditions: $x(t)=x$, $z(t)=f(t,x)$, where

$$E(t,x,z,p) \equiv \{(h,g)\in\mathbb{R}^n\times\mathbb{R}\mid\ |h|\leq C_0(1+|x|), g=\langle h,p\rangle - H(t,x,z,p)\}.$$

Remark 2.2. In (2.1), the quantity

$$z(\tau) = f(t,x) + \int_t^\tau (\langle\dot{x},p\rangle - H(s,x,z,p))\,ds$$

is the payoff functional. For a given strategy p of the second player, the winning profit function Λ_-^v for the second player, which is the initial deposit subtracted by the payoff, is minimized by the first player. Λ_+^v is the losing profit function of the first player maximized by the second player. The definition of $\Lambda_\pm^v$ exactly follows the theme of game theory.

Remark 2.3. To establish the equivalence of the L^∞ notion defined later in this paper and the L^∞ viscosity notion which will be introduced in [8], we may define the following less restrictive notion of profit functions:

$$\Lambda_-^v(t,x,(\tau,f,p)) = \inf\{I(v)(x(\tau)) - z(\tau)\,|\,(x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}, \tag{2.3}$$

$$\Lambda_+^v(t,x,(\tau,f,p)) = \sup\{S(v)(x(\tau)) - z(\tau)\,|\,(x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}. \tag{2.4}$$

In this paper, our analysis is always based on the definitions in (2.1) and (2.2), which are more restrictive. It is straightforward to check that all the results hold in this paper if (2.1) and (2.2) are replaced by (2.3) and (2.4).

Lemma 2.1. *Fix $\tau\in[0,T]$ and $p(t,x)\in C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)$. Then, for any locally measurable function $h(t,x)\geq 0$ and any point $x\in B(0,r)$,*

$$h(t,x) \leq \Lambda_-^v(t,x,(\tau,f,p)) - \Lambda_-^v(t,x,(\tau,f+h,p)) \leq e^{C(t-\tau)}h(t,x), \tag{2.5}$$

where C depends only on C_0, T, and $\|p\|_C$.

Proof. For any $\epsilon>0$, there exists a solution $(x_\epsilon(\cdot),z_\epsilon(\cdot))\in\mathrm{Sol}(t,f(t,x),p)$ with $x(t)=x$ such that

$$\Lambda_-^v(t,x,(\tau,f,p)) \geq S(v)(x_\epsilon(\tau)) - z_\epsilon(\tau) - \epsilon,$$

where z_ϵ satisfies the differential equation

$$\dot{z}_\epsilon(s) = \langle\dot{x}_\epsilon(s), p(s,x_\epsilon(s))\rangle - H(s,x_\epsilon(s),z_\epsilon(s),p(s,x_\epsilon(s)))$$

with $z_\epsilon|_{s=t}=f(t,x)$.

By Picard's Theorem, there exists a solution on $[\tau,t]$ of the following differential equation:

$$\dot{z}_h(s) = \langle\dot{x}_\epsilon(s), p(s,x_\epsilon(s))\rangle - H(s,x_\epsilon(s),z_h(s),p(s,x_\epsilon(s)))$$

with data:

$$z_h(t) = f(t,x) + h(t,x) \geq f(t,x) = z_\epsilon(t).$$

Since $z \to H(s,x,z,p)$ is increasing,

$$z_h(\tau) - z_\epsilon(\tau) = h(t,x) + \int_t^\tau (H(s,x_\epsilon,z_\epsilon,p) - H(s,x_\epsilon,z_h,p))\,ds \geq h(t,x).$$

Thus, we have

$$\begin{aligned}\Lambda_-^v(t,x,(\tau,f+h,p)) &\leq S(v)(x_\epsilon(\tau)) - z_h(\tau)\\ &\leq S(v)(x_\epsilon(\tau)) - z_\epsilon(\tau) - h(t,x)\\ &\leq \Lambda_-^v(t,x,(\tau,f,p)) - h(t,x) + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, $\Lambda_-^v(t,x,(\tau,f+h,p)) \leq \Lambda_-^v(t,x,(\tau,f,p)) - h(t,x)$. Thus we proved the first inequality.

Now we show the second inequality. For any $\epsilon > 0$, there is a solution $(x_h(\cdot), z_h(\cdot)) \in \mathrm{Sol}(t,(f+h)(t,x),p)$ with $x(t) = x$ such that

$$\Lambda_-^v(t,x,(\tau,f+h,p)) \geq S(v)(x_h(\tau)) - z_h(\tau) - \epsilon,$$

where $|\dot{x}_h| \leq C(1+|x_h|)$, and

$$\dot{z}_h(s) = \langle \dot{x}_h(s), p(s,x_h(s))\rangle - H(s,x_h(s),z_h(s),p(s,x_h(s))), \quad \tau < s < t,$$

with data

$$z_h|_{s=t} = f(t,x) + h(t,x), \qquad x_h|_{s=t} = x.$$

On the other hand, there exists a unique solution of the following Cauchy problem

$$\dot{z}(s) = \langle \dot{x}_h(s), p(s,x_h(s))\rangle - H(s,x_h(s),z(s),p(s,x_h(s))), \qquad z|_{s=t} = f(t,x).$$

By (A4) and the Gronwall inequality, we know

$$|z_h(\tau) - z(\tau)| \leq e^{C(t-\tau)}h(t,x),$$

where C is independent of v, f, τ, and t. Then we have

$$\begin{aligned}\Lambda_-^v(t,x,(\tau,f,p)) &\leq S(v)(x_h(\tau)) - z(\tau)\\ &\leq S(v)(x_h(\tau)) - z_h(\tau) + e^{C(t-\tau)}h(t,x)\\ &\leq \Lambda_-(t,x,(\tau,f+h,p)) + e^{C(t-\tau)}h(t,x) + \epsilon.\end{aligned}$$

Therefore

$$\Lambda_-^v(t,x,(\tau,f,p)) \leq \Lambda_-^v(t,x,(\tau,f+h,p)) + e^{C(t-\tau)}h(t,x).$$

This completes the proof of Lemma 2.1.

Remark 2.4. Similarly, for Λ_+^v, we have

$$h(t,x) \leq \Lambda_+^v(t,x,(\tau,f,p)) - \Lambda_+^v(t,x,(\tau,f+h,p)) \leq e^{C(t-\tau)}h(t,x), \tag{2.6}$$

where C depends only on C_0, T, and $\|p\|_C$.

Before we study the properties of winning and losing profit functions, we first state the following simple fact which can be proved by the Gronwall inequality.

Suppose that $(x_j(\cdot), z_j(\cdot)), j = 1,2$, are the two solutions of the characteristic inclusions:

$$|\dot{x}_j| \leq C(1+|x_j|), \qquad \dot{z}_j = \langle \dot{x}_j, p\rangle - H(t,x_j,z_j,p),$$

with $x_1(\cdot) = x_2(\cdot)$, $|z_1(t_0) - z_2(t_0)| \leq \epsilon$, $|x_1(t_0)| \leq M$, where $p(t,x) \in C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)$ and

$0 \le \tau \le t_0 \le T$. Then

$$|z_1(\tau) - z_2(\tau)| \le C\epsilon, \tag{2.7}$$

$$|z_1(\tau) - z_1(t_0)| \le C|\tau - t_0|, \tag{2.8}$$

where C depends only on M, T, and p.

Now we check whether our definition of winning and losing profit functions is well-defined; that is, given a measurable position function, whether the associated profit functions are measurable. For this purpose, we introduce a useful lemma from the measure theory whose proof will be given in Appendix.

Lemma 2.2. *Suppose that $A \subset B^d(0, M) \subset \mathbb{R}^d$ enjoys the pointwise nondegenerate density property: For each $x \in A$, there exists a measurable subset $A_x \subset A, x \in A_x$, such that*

$$\limsup_{r\to 0} \frac{m(A_x \cap B^d(x,r))}{m(B^d(x,r))} > 0. \tag{2.9}$$

Then A is measurable.

Based on Lemma 2.2, we now show that any profit function associated with a continuous position function is measurable. First we show that, at a given time, the profit function is measurable if the position function is continuous. To do so, we introduce some simple facts about essential supremum and essential infimum, and the preservation of nondegenerate density by bi-Lipschitz homeomorphism.

Lemma 2.3. *Suppose $v \in L^\infty_{\rm loc}(\mathbb{R}^d)$. Then, for a fixed point x and any $\epsilon > 0$,*

$$\limsup_{r\to 0} \frac{m(\{y \in \mathbb{R}^d \,|\, v(y) \ge S(v)(x) - \epsilon\} \cap B^d(x,r))}{m(B^d(x,r))} > 0, \tag{2.10}$$

$$\limsup_{r\to 0} \frac{m(\{y \in \mathbb{R}^d \,|\, v(y) \le I(v)(x) + \epsilon\} \cap B^d(x,r))}{m(B^d(x,r))} > 0. \tag{2.11}$$

Proof. On the contrary,

$$\limsup_{r\to 0} \frac{m(\{y \in \mathbb{R}^d \,|\, v(y) \ge S(v)(x) - \epsilon\} \cap B^d(x,r))}{m(B^d(x,r))} = 0.$$

Set $B \equiv \{y \in \mathbb{R}^d \,|\, v(y) < S(v)(x) - \epsilon\}$. Then

$$B \in S_x = \Big\{A \subset \mathbb{R}^n \text{ measurable} \,\Big|\, \lim_{r\to 0} \frac{m(A \cap B^d(x,r))}{m(B^d(x,r))} = 1\Big\}.$$

By the definition of $S(v)$, we have $S(v)(x) \le \operatorname*{esssup}_B v \le S(v)(x) - \epsilon$, which is a contradiction. Fact (2.9) can be similarly proved.

Lemma 2.4. *Suppose U and V are open sets in $\mathbb{R}^d$. Let $f : U \to V$ be a bi-Lipschitz homeomorphism. If $x \in A \subset \bar{A} \subset U$ with*

$$\limsup_{r\to 0} \frac{m(A \cap B^d(x,r))}{m(B^d(x,r))} > 0,$$

then $f(x)$ is a point in $f(A) \subset f(\bar{A}) \subset V$ with

$$\limsup_{r\to 0} \frac{m(f(A) \cap B^d(f(x),r))}{m(B^d(f(x),r))} > 0.$$

Proof. Since f is a bi-Lipschitz homeomorphism, there exists $K > 1$ such that

$$\|Df\|_{L^\infty(\bar{A})} + \|Df^{-1}\|_{L^\infty(f(\bar{A}))} \le K.$$

It is obvious that, for any $r > 0$ and any measurable set $B \subset U$,

$$B^d(f(x), r/K) \subset f(B^d(x,r)) \subset B^d(f(x), Kr),$$
$$(dK^d)^{-1} m(f(B)) \le m(B) \le dK^d m(f(B)).$$

Thus, for any $r > 0$ and measurable set $B \subset U$, one has

$$\frac{m(f(B) \cap B^d(f(x), r))}{m(B^d(f(x), r))} \ge \frac{m(f(B \cap B^d(x,r)))}{m(B^d(f(x), r))} \ge \frac{m(B \cap B^d(x,r))}{d^2 K^{2d} m(B^d(x,r))}.$$

The above inequalities imply the lemma.

Lemma 2.5. *Suppose that v is a locally bounded measurable function and $p(t,x)$ is continuous. Then, for any position function $f(t,x) \in C([0,T] \times \mathbb{R}^n)$, the corresponding profit function, as a function of x, $\Lambda_-^v(t, x, (\tau, f, p)) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for any $\tau \le t \le T$.*

Proof. It suffices to show that, for any fixed $t \in [\tau, T]$, $\alpha \in \mathbb{R}^1$, and $M > 0$, the set $D = B^n(0, M) \cap \{x \mid \Lambda_-^v < \alpha\}$ is measurable. Let us look at a point $x_0 \in D$. By the definition of $\Lambda_-^v(t, x, (\tau, f, p))$, there exists a solution $(x_\epsilon(\cdot), z_\epsilon(\cdot)) \in \mathrm{Sol}(t, f(t, x_0), p)$ with $x_\epsilon(t) = x_0$, $z_\epsilon(t) = f(t, x_0)$ such that

$$\Lambda_-^v(t, x_0, (\tau, f, p)) \ge S(v)(x_\epsilon(\tau)) - z_\epsilon(\tau) - \frac{\epsilon}{2}, \qquad t \ge \tau,$$

where $\epsilon = \alpha - \Lambda_-^v(t, x_0, (\tau, f, p))$.

Consider the following differential equations

$$\dot{x}(s) = \frac{1 + |x(s)|}{1 + |x_\epsilon(s)|} \dot{x}_\epsilon(s),$$
$$\dot{z}(s) = \langle \dot{x}(s), p(s, x(s)) \rangle - H(s, x(s), z(s), p(s, x(s))),$$

where $x(t) = x$ and $z(t) = f(t, x)$.

It is obvious that the above differential equations induce a bi-Lipschitz homeomorphism shown by the Gronwall inequality. By Lemmas 2.3-2.4, we know that there is a measurable set A_{x_0} with nondegenerate density at x_0 such that $x \in A_{x_0} \subset D$. Lemma 2.2 ensures that D is measurable which means that $\Lambda_-^v(t, x, (\tau, f, p))$ is measurable and locally integrable.

Now we prove that $\Lambda_-^v(t, x, (\tau, f, p))$ is measurable in both time and space variables if the position function f is continuous.

Lemma 2.6. *Suppose that v is a locally bounded measurable function and $p(t,x)$ is continuous. Then, for any position function $f(t,x) \in C([0,T] \times \mathbb{R}^n)$, the corresponding profit function $\Lambda_-^v(t, x, (\tau, f, p)) \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^n)$.*

Proof. It suffices to show that, for each $\alpha \in \mathbb{R}^1$ and $M > 0$, the set

$$E = B^{n+1}(0, M) \cap \{(t,x) \mid \Lambda_-^v < \alpha\} \quad \text{is measurable.}$$

We are going to show that each point $(t_0, x_0) \in E$ enjoys the pointwise nondegenerate density property. Set

$$E_r = B^{n+1}((t_0, x_0), r) \cap \Big\{(t,x) \,\Big|\, |x - x_0| \le \frac{C}{2}(t_0 - t)\Big\}.$$

If we can show that $E_r \subset E$ for some small r, then E is measurable by Lemma 2.2.

By the definition of $\Lambda_-^v(t, x, (\tau, f, p))$, there exists a solution

$$(x_\epsilon(\cdot), z_\epsilon(\cdot)) \in \mathrm{Sol}(t_0, f(t_0, x_0), p)$$

with $x_\epsilon(t_0)=x_0$, $z_\epsilon(t_0)=f(t_0,x_0)$ such that

$$\Lambda_-^v(t_0,x_0,(\tau,f,p)) \geq S(v)(x_\epsilon(\tau)) - z_\epsilon(\tau) - \frac{\epsilon}{4}, \qquad t_0 \geq \tau,$$

where $\epsilon = \alpha - \Lambda_-^v(t_0,x_0,(\tau,f,p))$.

For every point $(t,x)\in E_r$, consider the following characteristic path $(x(s),z(s))$:

$$\dot{x} = \begin{cases} \frac{x-x_0}{t-t_0}, & \text{if } t\geq s\geq t_0,\\ \dot{x}_\epsilon, & \text{if } t_0 \geq s \geq \tau,\end{cases}$$
$$\dot{z} = \langle \dot{x}, p\rangle - H(t,x,z,p), \qquad t\geq s\geq \tau,$$

with $x(t)=x$, $z(t)=f(t,x)$.

Since f is continuous, by (2.7) and (2.8), there is $r_0>0$ such that, for any $(t,x)\in E_{r_0}$, $S(v)(x(\tau)) - z(\tau) \leq \alpha - \frac{\epsilon}{4}$ with $x(t)=x$, $z(t)=f(t,x)$. Thus $\Lambda_-^v(t,x,(\tau,f,p)) \leq \alpha - \frac{\epsilon}{4}$ for all $(t,x)\in E_{r_0}$. That is, $E_{r_0}\subset E$. Therefore, E is measurable.

Indeed, there is an intrinsic regularity relation between the position function $f(t,x)$ and the profit function $\Lambda_-^v(t,x,(\tau,f,p))$, which is stated in the following lemma.

Lemma 2.7. *Let $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$. Let $v(x)$ be a locally bounded measurable function. Then*

(i) *For any position function $f(t,x)\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n)$ satisfying $\sup\limits_{[\tau,T]}\|f(t,\cdot)\|_{L^\infty(A)}<\infty$, with any bounded measurable set A, $\Lambda_-^v(t,x,(\tau,f,p))\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n)$.*

(ii) *Suppose that $g(t,x)\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n)$ satisfies $\sup\limits_{[\tau,T]}\|g(t,\cdot)\|_{L^\infty(A)}<\infty$ for any bounded measurable set A. Then there exists a unique $f(t,x)\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n)$ with $\sup\limits_{[\tau,T]}\|f(t,\cdot)\|_{L^\infty(A)}<\infty$ for any bounded measurable set A such that*

$$g(t,x) = \Lambda_-^v(t,x,(\tau,f,p)) \qquad \textit{for all} \ \ t\in[\tau,T],$$

and, in particular, if $g(t,x)\equiv 0$, then $f(\tau,x)=v(x)$, a.e.

Proof. (i) There exists a sequence of continuous functions $\{f_k\}\subset C([0,T]\times\mathbb{R}^n)$ such that $f_k\to f$ in $L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n)$. Lemma 2.6 ensures that

$$\Lambda_-^v(t,x,(\tau,f_k,p))\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n).$$

The fact (2.7)–(2.8) shows that $\Lambda_-^v(t,x,(\tau,f_k,p))$ are uniformly bounded. By employing the continuity property of integral and (2.5), we know

$$\Lambda_-^v(t,x,(\tau,f_k,p)) \to \Lambda_-^v(t,x,(\tau,f,p)), \qquad \text{in} \qquad L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n).$$

Therefore, we have

$$\Lambda_-^v(t,x,(\tau,f,p))\in L^1_{\rm loc}([\tau,T]\times\mathbb{R}^n).$$

(ii) By appealing to the Gronwall inequality and Assumption (A4), we know that, given a bounded measurable set A, there exists a constant $M_A>0$ such that

$$\Lambda_-^v(t,x,(\tau,-M_A,p)) \geq \|g\|_{L^\infty(A)} \geq -\|g\|_{L^\infty(A)} \geq \Lambda_-^v(t,x,(\tau,M_A,p))$$

for any $t\in[\tau,T]$, where $A\subset B^n(0,r)$.

Consider the set

$$S_{g,A} = \{f\in L^1([\tau,T]\times A) \mid \Lambda_-^v(t,x,(\tau,f,p)) \leq g(t,x), \quad \text{a.e.}\}.$$

By the above inequalities and Lemma 2.6, $S_{g,A}$ is not empty. For any $f \in S_{g,A}$, define

$$J(f) = \int_\tau^T \int_A \Lambda_-^v(t, x, (\tau, f, p))\, dxdt.$$

It is obvious that $S_{g,A}$ is a closed set in $L^1(A)$, and

$$J(f) \le \int_\tau^T \int_A g(t,x)\, dxdt \equiv I_g \quad \text{for any } f \in S_{g,A}.$$

Claim. For any $f \in S_{g,A}$ with $J(f) < J_g$, there exists $f_1 \in S_{g,A}$ such that $f_1 \ge f$ and $J(f_1) > J(f)$.

This fact can be seen as follows. Since $J(f) < J_g$, there exists $B \subset [\tau, T] \times A$ with $m(B) > 0$ and $\delta > 0$ such that

$$\Lambda_-^v(t, x, (\tau, f, p)) < g(t,x) - \delta, \qquad (t,x) \in B.$$

Define

$$f_1 = \begin{cases} f, & \text{if } (t,x) \in A\backslash B, \\ f - e^{-C(t-\tau)}\delta, & \text{if } (t,x) \in B, \end{cases}$$

where C is the constant in (2.5). By (2.5), $J(f_1) > J(f)$ and $\Lambda_-^v(t, x, (\tau, f_1, p)) \le g(t,x)$. Thus, the claim holds.

By the claim above, there exists a sequence of decreasing measurable functions $\{f_k\}_{k=1}^\infty \subset S_{g,A}$ such that $J(f_k) > J_g - \frac{1}{k}$ and $-M_A \le f_k \le M_A$. By the monotone convergence theorem, there exists $\tilde{f} \in L^1([\tau, T] \times A)$ such that

$$f_k(t,x) \to \tilde{f}(t,x) \in L^1_{\text{loc}}([\tau, T] \times \mathbb{R}^n), \quad \Lambda_-^v(t, x, (\tau, \tilde{f}, p)) = g(t,x), \quad \text{a.e. } (t,x).$$

Similarly, for any fixed $t \in [\tau, T]$, we can show that there exists $f(t,\cdot) \in L^1(A)$ such that $\Lambda_-^v(t, x, (\tau, f, p)) = g(t,x)$, a.e. x, by replacing $J(f)$ above via

$$\tilde{J}(f) = \int_A \Lambda_-^v(t, x, (\tau, f, p))\, dx \quad \text{for any fixed } t \in [\tau, T].$$

Then

$$f(t,x) = \tilde{f}(t,x), \qquad \text{a.e. } (t,x),$$

which is in $L^1([\tau, T] \times A)$. By (2.5), f is unique. It follows that $\sup\limits_{[\tau,T]} \|f(t,\cdot)\|_{L^\infty(A)} < \infty$.

Similarly, for the losing profit function, we have

Lemma 2.8. *Let $p(t,x) \in C(\mathbb{R}^1 \times \mathbb{R}^n; \mathbb{R}^n)$. Let $v(x)$ be a locally bounded measurable function. Then*

(i) *For any position function $f(t,x) \in L^1_{\text{loc}}([\tau, T] \times \mathbb{R}^n)$ satisfying*

$$\sup_{[\tau,T]} \|f(t,\cdot)\|_{L^\infty(A)} < \infty,$$

with any bounded measurable set A,

$$\Lambda_+^v(t, x, (\tau, f, p)) \in L^1_{\text{loc}}([\tau, T] \times \mathbb{R}^n).$$

(ii) *Suppose that $g(t,x) \in L^1_{\text{loc}}([\tau, T] \times \mathbb{R}^n)$ satisfies $\sup\limits_{[\tau,T]} \|g(t,\cdot)\|_{L^\infty(A)} < \infty$, for any bounded measurable set A. Then there exists a unique $f(t,x) \in L^1_{\text{loc}}([\tau, T] \times \mathbb{R}^n)$ with $\sup\limits_{[\tau,T]} \|f(t,\cdot)\|_{L^\infty(A)} < \infty$ for any bounded measurable set A such that*

$$g(t,x) = \Lambda_+^v(t, x, (\tau, f, p)) \qquad \textit{for any } t \in [\tau, T], \text{ a.e. } x \in \mathbb{R}^n,$$

and, in particular, if $g(t,x)\equiv 0$*, then* $f(\tau,x)=v(x)$, a.e.

It follows from Lemma 2.7 (Lemma 2.8, respectively) that there is a unique locally bounded measurable function $u_-^\varphi((t,x),p)$ ($u_+^\varphi((t,x),p)$, respectively) satisfying

$$\Lambda_-^\varphi(t,x,(0,u_-^\varphi((t,x),p),p))=0, \tag{2.12}$$

$$\Lambda_+^\varphi(t,x,(0,u_+^\varphi((t,x),p),p))=0, \tag{2.13}$$

respectively, for $(t,x)\in[0,T]\times\mathbb{R}^n$, where $\varphi(x)$ is a locally bounded measurable function. It is easy to see that

$$u_-^\varphi((0,x),p)=\varphi(x)=u_+^\varphi((0,x),p). \tag{2.14}$$

§3. Existence of Discontinuous Solutions in L^∞

First we define the supsolution set and the subsolution set for the Cauchy problem (1.1)–(1.2) in terms of profit functions. Then we present the existence proof.

Let

$$W=\{u(t,x)\in L^\infty_{\rm loc}([0,T]\times\mathbb{R}^n)\,|\,u(t,\cdot)\in L^\infty_{\rm loc}(\mathbb{R}^n)\text{ for every }t\in[0,T]\}.$$

Denote by S^u the set of supsolutions $w(t,x)\in W$ which satisfy

(i) For any $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$,

$$\Lambda_-^\varphi(t,x,(0,w,p))\le 0 \tag{3.1}$$

for almost everywhere $(t,x)\in[0,T]\times\mathbb{R}^n$. Additionally, for every $t\in[0,T]$, (3.1) holds for almost everywhere $x\in\mathbb{R}^n$.

(ii) The semigroup property: For every $\tau\in[0,T]$,

$$\Lambda_-^{w(\tau,x)}(t,x,(\tau,w,p))\le 0 \tag{3.2}$$

for almost everywhere $(t,x)\in[\tau,T]\times\mathbb{R}^n$. Additionally, for every $t\in[\tau,T]$, (3.2) holds for almost everywhere $x\in\mathbb{R}^n$.

Similarly, S^l denotes the set of subsolutions $w\in W$ which satisfy

(i) For any $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$,

$$\Lambda_+^\varphi(t,x,(0,w,p))\ge 0 \tag{3.3}$$

for almost everywhere $(t,x)\in[0,T]\times\mathbb{R}^n$. Additionally, for every $t\in[0,T]$, (3.3) holds for almost everywhere $x\in\mathbb{R}^n$.

(ii) Furthermore, for every $\tau\in[0,T]$,

$$\Lambda_+^{w(\tau,x)}(t,x,(\tau,w,p))\ge 0 \tag{3.4}$$

for almost everywhere $(t,x)\in[\tau,T]\times\mathbb{R}^n$. Additionally, for every $t\in[\tau,T]$, (3.4) holds for almost everywhere $x\in\mathbb{R}^n$.

It implies from the definition of S^u with the aid of (2.5) that, for any $w\in S^u$ and $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$, $w(t,x)\ge u_-^\varphi((t,x),p)$ a.e. in $[0,T]\times\mathbb{R}^n$. Similarly, for any $w\in S^l$ and $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$, $w(t,x)\le u_+^\varphi((t,x),p)$ a.e. in $[0,T]\times\mathbb{R}^n$.

Definition 3.1. *We say that u is a solution of the Cauchy problem* (1.1)–(1.2) *if u belongs to S^u and S^l simultaneously.*

Condition (i) for S^u and S^l contains the exact information how the solution u is determined by the initial data $\varphi(x)$.

To study the perturbation of characteristic paths, we first recall the definition of weak isotopy.

Definition 3.2. *Suppose that $O \subset \mathbb{R}^d$ is a domain. A Lipschitz continuous map $x : [0,T] \times O \to \mathbb{R}^d$ is called a weak isotopy if*

(i) $x(0) = I$;

(ii) *$x(\tau)$ is a bi-Lipschitz continuous homeomorphism for any $\tau \in [0,T]$ with uniform Lipschitz constant independent of τ.*

Remark 3.1. For the definition of isotopy, see [19] or [27]. For the purpose of this paper, we need to study the bi-Lipschitz homeomorphism which preserves nondegenerate measure so that we introduce the weak version of isotopy above.

A typical example of weak isotopy is given by the following lemma, which can be proven by the Gronwall inequality forward and backward.

Lemma 3.1. *Suppose that $f_j : [0,T] \to \mathbb{R}^n, j = 1,2,$ are bounded measurable functions and $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous. Then the following differential equation generates weak isotopy over $[0,T]$, $\dot{x} = g(x)f_1(t) + f_2(t)$.*

The following lemma establishes a nice property of weak isotopy, which is the preservation of nondegenerate measure.

Lemma 3.2. *Suppose $x_0 \in B \subset x(T)O$ and $\limsup\limits_{r\to 0} \frac{m(B\cap B^d(x_0,r))}{m(B^d(x_0,r))} > 0$, where $O \subset \mathbb{R}^d$ is a domain. Then*

$$\limsup_{r\to 0} \frac{m(x^{-1}((0,T),B) \cap B^{d+1}(x^{-1}(T)x_0,r))}{m(B^{d+1}(x^{-1}(T)x_0,r))} > 0, \tag{3.5}$$

where $x^{-1}((0,T),B) = \{(t,y) \in (0,T) \times \mathbb{R}^d \,|\, x(T)x^{-1}(t)y \in B\}$.

Proof. By property (ii) of weak isotopy, there exists $K > 1$ such that

$$K^{-1}|x(\tau)x_1 - x(\tau)x_2| \le |x_1 - x_2| \le K|x(\tau)x_1 - x(\tau)x_2|,$$

where x_1, $x_2 \in O$, $\tau \in [0,T]$, and K is independent of τ. It is obvious that, for any $r > 0$, $B^d(x_0,Kr) \subset x(T)O$ and any measurable set $U \subset x(T)O$,

$$x^{-1}(T)(B^d(x_0,r/K)) \subset B(x^{-1}(T)x_0,r) \subset x^{-1}(T)(B^d(x_0,Kr)),$$
$$K^{-1}m(U) \le m(x^{-1}(T)U) \le Km(U).$$

By assumption, there exist $\delta > 0$ and a sequence $\{r_k\}_{k=1}^{\infty}$, with $r_k \to 0$ as $k \to \infty$, such that

$$\frac{m(B \cap B^d(x_0,r_k))}{m(B^d(x_0,r_k))} \ge \delta.$$

Note that

$$x^{-1}((0,T),B \cap B^d(x_0,r_k)) \subset x^{-1}((0,T),B^d(x_0,r_k)) \subset (0,T) \times (x(t)x^{-1}(T)x_0,Kr_k).$$

Therefore

$$\begin{aligned}
&m(x^{-1}((0,T), B\cap B^d(x,r_k)\cap B^{d+1}(x^{-1}(T)x_0, 2Kr_k)))\\
\geq &\int_0^{\sqrt{3}Kr_k} m(x(\tau)x^{-1}(T)(B\cap B^d(x_0,r_k)))\,d\tau\\
\geq &\sqrt{3}Kr_kK^{-d}m(B\cap B^d(x,r_k))\\
\geq &\sqrt{3}r_kK^{-d+1}\delta m(B^d(x,r_k))\\
\geq &\sqrt{3}K^{-d+1}\delta(2K)^{-d-1}\frac{\omega_d}{\omega_{d+1}}m(B^{d+1}(x^{-1}(T)x_0, 2Kr_k)),
\end{aligned}$$

which implies that (3.5) holds.

We now show that S^u is not empty. In the proof later on, we denote by $L(f)$ and $L(B)$ the Lebesgue set of measurable function f and the subset of points of density 1 of measurable set B, respectively.

Lemma 3.3. *For fixed $p'(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$, $u_+^\varphi((t,x),p')\in S_0^u$. More precisely, u_+^φ satisfies that, for any $p(t,x)\in C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$ and $0\leq\tau\leq T$, and for every point $(t,x)\in L(u_+^\varphi)$ which is the set of Lebesgue points of u_+^φ,*

$$\Lambda_-^{u_+^\varphi}(t,x,(\tau,u_+^\varphi,p))\leq 0. \tag{3.6}$$

And, for every $t\geq\tau$, (3.6) holds for almost everywhere $x\in\mathbb{R}^n$.

Proof. First we show that, for any p and p' in $C(\mathbb{R}^1\times\mathbb{R}^n;\mathbb{R}^n)$ and any measurable function f,

$$\mathrm{Sol}(t,f(t,x),p)\cap\mathrm{Sol}(t,f(t,x),p')\neq\emptyset.$$

Consider the following functions

$$\begin{aligned}
g^c(t,x,z,p,p') &= (p'-p)\frac{H(t,x,z,p')-H(t,x,z,p)}{|p'-p|^2},\\
h^c(t,x,z,p,p') &= \langle g^c(t,x,z,p,p'),p'\rangle - H(t,x,z,p')\\
&= \langle g^c(t,x,z,p,p'),p\rangle - H(t,x,z,p).
\end{aligned}$$

By Assumption (A2), $|g^c(t,x,z,p,p')|\leq C(1+|x|)$. Therefore, we have

$$(g^c(t,x,z,p,p'),h^c(t,x,z,p,p'))\in E(t,x,z,p')\cap E(t,x,z,p),$$

which implies $\mathrm{Sol}(t,f(t,x),p)\cap\mathrm{Sol}(t,f(t,x),p')\neq\emptyset$.

It is obvious that $u_+^\varphi\in W$. Assume that u_+^φ does not satisfy inequality (3.6) at some point $(t_0,x_0)\in L(u_+^\varphi)$. That is, there exist $\tau_0\leq t_0$ and $\delta>0$ such that

$$\Lambda_-^{u_+^\varphi}(t_0,x,(\tau_0,u_+^\varphi,p))>\delta>0.$$

By the definition of Λ_-, for any $(x(\cdot),z(\cdot))\in\mathrm{Sol}(t,u_+^\varphi(x_0),p)$,

$$S(u_+^\varphi)(\tau_0,x(\tau_0))-z(\tau_0)>\delta>0,$$

where $z(\tau_0)=u_+^\varphi(x)+\int_{t_0}^{\tau_0}(\langle\dot{x},p\rangle-H(t,x,z,p))\,dt$ and $x(t_0)=x_0$. Thus

$$S(u_+^\varphi)(\tau_0,x(\tau_0))-z(\tau_0)>\delta>0,$$

where $(x(\cdot),z(\cdot))\in\mathrm{Sol}(t,u_+^\varphi(x_0),p)\cap\mathrm{Sol}(t,u_+^\varphi(x_0),p')$. Let (x^0,z^0) be a characteristic path in $\mathrm{Sol}(t,u_+^\varphi(x_0),p)\cap\mathrm{Sol}(t,u_+^\varphi(x_0),p')$. Consider the characteristic flow issued from the neighborhood of (t_0, x_0) given by the following differential equations

$$\dot{x}(s) = \frac{1 + |x(s)|}{1 + |x^0(s)|}\dot{x}^0(s), \tag{3.7}$$

$$\dot{z}(s) = \langle \dot{x}(s), p(s, x(s))\rangle - H(s, x(s), z(s), p(s, x(s))), \tag{3.8}$$

where $x(t_0) = x$ and $z(t_0) = u_+^\varphi(t_0, x)$. By Lemma 3.1, $\dot{x}$ generates a weak isotopy. With the aid of Lemmas 2.3 and 3.2, and the explicit formula of $\dot{x}$, there exists a measurable set A of non-zero measure, which has positive density at (t_0, x_0), such that, for any $(t, x) \in A$, $u_+^\varphi(\tau_0, x(\tau_0)) - z(\tau_0) > \frac{\delta}{2} > 0$, where $\dot{x}$ and $\dot{z}$ are defined by (3.7) and (3.8). By the definition of u_+^φ and (2.6), we know $\Lambda_+^\varphi(\tau_0, y, (0, g, p')) \geq \delta/2 > 0$, where $g(y) = u_+^\varphi(x^{-1}(\tau_0)(y), p') + \int_{t_0}^{\tau_0}(\langle \dot{x}, p'\rangle - H(s, x, z, p'))\,ds$ for $y \in x(t_0)A$. Thus

$$\Lambda_+^\varphi(t_0, x, (0, u_+^\varphi((t, x), p'), p')) \geq \delta/2 > 0, \quad (t, x) \in A.$$

This is a contradiction to the definition of $u_+^\varphi((t, x), p')$. This proves (3.6) in (t, x). By the same contradiction argument as above with the aid of Lemmas 2.3–2.4 instead of Lemma 3.2, we can show for every $t \geq \tau$, (3.6) holds for almost everywhere x.

Based on a given element $w \in S^u$, we can produce another one in S^u.

Lemma 3.4. *Given $w(t, x) \in S^u$ and $p(t, x) \in C(\mathbb{R}^1 \times \mathbb{R}^n; \mathbb{R}^n)$, we define*

$$\hat{w}(t, x) = \begin{cases} w(t, x), & \text{if } (t, x) \in [0, s] \times \mathbb{R}^n, \\ u_+^{w(s,x)}(t, x), & \text{if } (t, x) \in [s, T] \times \mathbb{R}^n, \end{cases}$$

where $u_+^{w(s,x)}$ satisfies

$$\Lambda_+^{w(s,x)}(t, x, (s, u_+^{w(s,x)}, p)) = 0.$$

Then $\hat{w} \in S^u$.

It is easy to show $\hat{w} \in S^u$, with the aid of the proof of Lemma 3.3 and by the definition of Λ_+^v.

Now we are ready to prove the main result of this paper.

Theorem 3.1. *Given a locally bounded measurable function $\varphi(x)$, there exists a unique minimal element of S^u, that is, the solution of the Cauchy problem* (1.1)–(1.2).

Proof. Step 1. If u_1 and u_2 belong to S^u, then $\min\{u_1, u_2\} \equiv u_1 \wedge u_2 \in S^u$. By (2.5) for Λ_-^v, it is easy to see the following relations hold:

$$\max\{\Lambda_-^v(t, x, (\tau, f_1, p)), \Lambda_-^v(t, x, (\tau, f_2, p))\} = \Lambda_-^v(t, x, (\tau, f_1 \wedge f_2, p)),$$

$$\Lambda_-^{f_1 \wedge f_2}(t, x, (\tau, f_1 \wedge f_2, p)) \leq \max\{\Lambda_-^{f_1}(t, x, (\tau, f_1, p)), \Lambda_-^{f_2}(t, x, (\tau, f_2, p))\}.$$

Then the above inequality ensures that $u_1 \wedge u_2 \in S^u$.

Step 2. Suppose that $\{u_\beta\}_{\beta \in \mathcal{B}}$ is an ordered family in S^u: for any u_1, $u_2 \in \{u_\beta\}$, either $u_1 \leq u_2$ or $u_2 \leq u_1$. Then there exists $w \in S^u$ such that $w \leq u_\beta$ for $\beta \in \mathcal{B}$. First we define a sequence of domains A_k:

$$A_k = \{(t, x) \,|\, |x| \leq (k + 1)e^{2C(T-t)} - 1, 0 \leq t \leq T\}.$$

Let $A_k(t) = A_k \cap (\{t\} \times \mathbb{R}^n)$. For each $\beta \in \mathcal{B}$, we define $I_{k,\beta} = \int_{A_k} u_\beta\, dxdt$ for each $k \in N$, and, for each t, $I_{k,\beta}(t) = \int_{A_k(t)} u_\beta\, dx$ for each $k \in N$.

Choose $p(t, x) \in C(\mathbb{R}^1 \times \mathbb{R}^n; \mathbb{R}^n)$ such that, for any $\beta \in \mathcal{B}$, $u_\beta \geq u_-^\varphi((t, x), p)$. Denote $I_k = \int_{A_k} u_-^\varphi((t, x), p)\, dxdt$ and $I_k(t) = \int_{A_k(t)} u_-^\varphi((t, x), p)\, dx$. Thus $I_{k,\beta} \geq I_k$ and $I_{k,\beta}(t) \geq I_k(t)$.

For each fixed k, the family of $I_{k,\beta}$ ($I_{k,\beta}(t)$ respectively) has a lower bound I_k ($I_k(t)$ respectively). There exists a decreasing sequence $\{u_{k,l}\}_{l=1}^{\infty} \subset \{u_\beta\}$ such that $\lim\limits_{l\to\infty} I_{k,l} = \inf\limits_{\beta\in\mathcal{B}} I_{k,\beta}$,

Note that $\{u_\beta\}_{\beta\in\mathcal{B}}$ is decreasing. Define $u_l = \min\{u_{1,l}, \cdots, u_{l,l}\}$. Then $\{u_l\}_{l=1}^{\infty}$ is decreasing and has a lower bound $u_-^{\varphi}((t,x),p)$. By the monotone convergence theorem, there exists a measurable function $\bar{w}$ such that $\lim\limits_{l\to\infty} u_l = \bar{w}$, a.e. in A_k.

Denote by w the pointwise limit of $\{u_{k,l}\}_{l=1}^{\infty}$. We know that $w = \bar{w}$ a.e. in A_k is measurable. By the monotone convergence theorem, for each t, there is a measurable function in $x \in \mathbb{R}^n$ such that $\lim\limits_{l\to\infty} u_l(t) = \tilde{w}(t)$ a.e. in $A_k(t)$. We know $w = \tilde{w}$ a.e. in $A_k(t)$. Thus w is measurable in $x \in \mathbb{R}^n$ for every t.

Assumption (A2) and inequality (2.5) ensure that (3.1)–(3.2) hold for w in A_k. Therefore, w is what we want.

Step 3. With the aid of Steps 1-2, Zorn's lemma ensures that there is a unique minimal element u in S^u. With the aid of Lemma 3.4 and inequality (2.6), we can show that (3.3) and (3.4) hold for u.

Remark 3.2. With the aid of the differential inclusion theory, the existence results in this paper may be extended to the case that $H(t,x,z,p)$ is discontinuous with respect to (t,x) and is piecewise continuous with respect to p.

§4. Consistency

It has been shown in [29, 30] that the minimax solutions are equivalent to the viscosity solutions, provided that the initial data are continuous. In this section we show that our solutions coincide with the minimax solutions, provided that the initial data are continuous.

Let the following functions

$$(t,x,y) \to p_\pm(t,x,y) : (0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$(t,x,y) \to p(t,x,y) : (0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

be locally Lipschitz continuous.

For the purpose of the proof of consistency, we need to establish the following lemma on the existence of mutual tracking trajectories of the characteristic inclusions.

Lemma 4.1. *Suppose $\varphi(x)$ is continuous. Let $w_\pm$ and u be the L^∞ supsolution (subsolution) and minimax solution of* (1.1)–(1.2)*, respectively. Then, for every point* $(t_0, x_0) \in L(w_\pm)$*, there exist solutions of the systems of differential inclusions*

$$(\dot{x}, \dot{z}_\pm) \in E(t,x,z_\pm,p_\pm(t,x,y)), \quad (\dot{y},\dot{z}) \in E(t,y,z,p(t,x,y)),$$

that satisfy the initial conditions

$$(x(t_0), z_\pm(t_0)) = (x_0, w_\pm(t_0,x_0)), \quad (y(t_0), z(t_0)) = (x_0, u(t_0,x_0))$$

and the inequalities $\pm(z_\pm(0) - \varphi(x(0))) \geq 0, \quad \pm(z(0) - \varphi(y(0))) \leq 0,$ *respectively.*

Proof. We prove only for the "+" case; the proof for the "−" case is the same.

Let $y_0(t)$ satisfy $y_0(t) = x_0$ for $0 \leq t \leq t_0$. By the continuity of φ, there exists a solution (x_1, z_+^1) of characteristic inclusion with $p_+^1(t,x) = p(t,x,y_0(t))$:

$$|\dot{x}_1| \leq C(1+|x_1|), \qquad \dot{z}_+^1 = \langle \dot{x}_1, p_+^1\rangle - H(t,x_1,z_+^1,p_+^1)$$

with initial conditions $x_1(t_0) = x_0$, $z_+^1(t_0) = w_+(t_0, x_0)$ such that $z_+^1(0) \geq \varphi(x_1(0))$. Then, by the definition of minimax solutions, there exists a solution (y_1, z^1) of characteristic inclusion with $p^1(t, y) = p(t, x_1(t), y)$:

$$|\dot{y}_1| \leq C(1 + |y_1|), \qquad \dot{z}^1 = \langle \dot{y}_1, p^1 \rangle - H(t, y_1, z^1, p^1),$$

with initial conditions $y_1(t_0) = x_0$, $z^1(t_0) = u(t_0, x_0)$ such that $z^1(0) \leq \varphi(y_1(0))$.

$$\cdots, \qquad \cdots, \qquad \cdots$$

Continuing recursively, we obtain four sequences p_+^k, p^k, $\{(x_k, z_+^k)\}$, and $\{(y_k, z^k)\}$ satisfying

$$p_+^k(t, x) = p(t, x, y_{k-1}(t)), \qquad p^k(t, y) = p(t, x_k(t), y),$$
$$|\dot{x}_k| \leq C(1 + |x_k|), \qquad \dot{z}_+^k = \langle \dot{x}_k, p_+^k \rangle - H(t, x_k, z_+^k, p_+^k),$$
$$|\dot{y}_k| \leq C(1 + |y_k|), \qquad \dot{z}^k = \langle \dot{y}_k, p^k \rangle - H(t, y_k, z^k, p^k),$$

with initial conditions $x_k(t_0) = y_k(t_0) = x_0$, $z_+^k(t_0) = w_+(t_0, x_0)$, and $z^k(t_0) = u(t_0, x_0)$ such that

$$z_+^k(0) \geq \varphi(x_k(0)), \qquad z^k(0) \leq \varphi(y_k(0)).$$

Since $|x_k| + |y_k| \leq (|x_0| + 1)e^{CT}$ for all k when $0 \leq t \leq t_0$, there exists $M > 0$ such that $|p_+^k| + |p^k| \leq M$ for all k. Thus (x_k, z_+^k) and (y_k, z^k) are uniformly Lipschitz continuous. By the compactness of the sequences and the continuity of φ, there exist subsequences that converge to (x, z_+) and (y, z), respectively, which are our desired trajectories.

Based upon Lemma 4.1, we can prove the following theorem.

Theorem 4.1. *Assume that $\varphi(x)$ is continuous. Let $u(t, x)$ be an L^∞ supsolution of (1.1)–(1.2) and $v(t, x)$ the continuous minimax solution. Then $u(t, x) \geq v(t, x)$ almost everywhere.*

Proof. Let $L(u)$ be the Lebesgue set of u. It suffices to show that, for $(t_0, x_0) \in L(u)$, $u(t_0, x_0) \geq v(t_0, x_0)$.

On the contrary, $u(t_0, x_0) \leq v(t_0, x_0) - \delta$, where $\delta > 0$. Denote by $X(t, x_0)$ the set of absolute continuous functions $x(\cdot) : [t, T] \to R^n$, which satisfy the differential inequality $|\dot{x}(t)| \leq C(1 + |x(t)|)$ and the initial condition $x(t) = x_0$. Define

$$S = \{x(\tau) \mid \tau \in [t, T], x(\cdot) \in X(t, x_0)\}.$$

By the Gronwall inequality, we know $S \subset B^n(0, (1 + |x_0|)e^{2CT})$. Let λ be the Lipschitz constant in Assumption (A3) where $L = (|x_0| + 1)e^{2CT}$.

Set

$$\eta_k(t) \equiv \frac{e^{\lambda t} - \frac{1}{k}}{\frac{1}{k}}, \qquad r_k(x, y) \equiv \sqrt{\frac{1}{k^4} + |x - y|^2},$$
$$\beta_k(t, x, y) \equiv \eta_k(t) r_k(x, y), \qquad L_k(t, x, y, \mu, \nu) = \beta_k(t, x, y) + \mu - \nu,$$

where $(t, x, y, \mu, \nu) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$.

Consider the derivative of the Lyapunov function L_k with respect to the system of differential inclusions:

$$\begin{aligned} &|\dot{x}| \leq C(1 + |x|), \qquad \dot{\mu} = \langle \dot{x}, p \rangle - H(t, x, \mu, p), \\ &|\dot{y}| \leq C(1 + |y|), \qquad \dot{\nu} = \langle \dot{y}, p \rangle - H(t, y, \nu, p), \\ &p = -D_x \beta_k(t, x, y) = D_y \beta_k(t, x, y) = \eta_k(t) \frac{y - x}{r_k(t, x, y)}, \end{aligned} \tag{4.1}$$

with initial conditions:

$$x(t_0) = y(t_0) = x_0, \qquad \mu(t_0) = u(t_0, x_0), \qquad \nu(t_0) = v(t_0, x_0).$$

Note that

$$(t_0, x_0, \mu_0, \nu_0) \in N \equiv \{(x, y, \mu, \nu) \in S \times S \times \mathbb{R} \times \mathbb{R} \mid \mu \le \nu\}.$$

Let the trajectory $(x(t), y(t), \mu(t), \nu(t))$ of the above differential inclusions go within the set N on the interval $[\tau, t_0] \subset [0, t_0]$. The straightforward computation yields

$$\begin{aligned}
\frac{dL_k(t)}{dt} &= \frac{\partial \beta_k}{\partial t} + \langle D_x\beta_k, \dot{x}\rangle + \langle D_y\beta_k, \dot{y}\rangle + \dot{\mu} - \dot{\nu} \\
&= \lambda k e^{\lambda t} r_k(x, y) - H(t, x, \nu, p) + H(t, y, \mu, p) \\
&\ge \lambda k e^{\lambda t} r_k(x, y) - \lambda(1 + |p|)|x - y| \\
&\ge \lambda r_k(x, y)\left(k e^{\lambda t} - 1 - |p|\right) \\
&= \lambda r_k(x, y)\left(k e^{\lambda t} - 1 - \eta_k(t)\frac{|x - y|}{r_k(x, y)}\right) \\
&\ge \lambda r_k(x, y)\left(k e^{\lambda t} - 1 - \eta_k(t)\right) = 0.
\end{aligned}$$

Since $L_k(t_0) = \frac{1}{k^2}\eta_k(t_0) + \mu(t_0) - \nu(t_0) \le \frac{1}{k^2}\eta_k(t_0) - \delta < 0$ and, for almost all $t \in [\tau, t_0]$, $\dot{L}_k[t] \ge 0$, we obtain that the trajectory stays in the domain N for all $t \in [0, t_0]$.

By Lemma 4.1, there is a trajectory of System (4.1), $(x_k(\cdot), y_k(\cdot), \mu_k(\cdot), \nu_k(\cdot))$ such that

$$\mu_k(0) \ge u(0, x_k(0)), \qquad \nu_k(0) \le v(0, x_k(0)).$$

Since inequality $\dot{L}_k[t] \ge 0$ holds for any solution of System (4.1), we have

$$\begin{aligned}
L_k[t_0] &= \frac{1}{k^2}\eta_k(t_0) + u(t_0, x_0) - v(t_0, x_0) \ge L_k[0] \\
&\ge \eta_k(0) r_k(x_k(0), y_k(0)) + u(0, x_k(0)) - v(0, y_k(0)) \\
&\ge \varphi(x_k(0)) - \varphi(y_k(0)).
\end{aligned}$$

Note that

$$\lim_{k\to\infty} \frac{1}{k^2}\eta_k(t_0) = 0, \quad \lim_{k\to\infty} \eta_k(0) \to \infty.$$

It is obvious that

$$|\varphi(x_k(0)) - \varphi(y_k(0))| \le \max_{(x,y)\in S\times S} |u(x) - v(y)| \equiv M < \infty.$$

Therefore, from the inequality

$$\eta_k(0) r(x_k(0), y_k(0)) \le \frac{1}{k^2}\eta_k(t_0) + u(x_0, y_0) - v(x_0, y_0) + M,$$

we obtain

$$|x_k(0) - y_k(0)| \to 0 \quad \text{as} \quad k \to \infty.$$

Furthermore, passing to the limit as $k \to \infty$ in the following inequality by invoking the continuity of φ:

$$\frac{1}{k^2}\eta_k(t_0) + u(t_0, x_0) - v(t_0, x_0) \ge \varphi(x_k(0)) - \varphi(y_k(0)),$$

we have $u(t_0, x_0) \ge v(t_0, x_0)$. This leads to a contradiction to the assumption. The proof is completed.

Similarly, with the help of Lemma 4.2, we have

Theorem 4.2. *Assume that $\varphi(x)$ is continuous. Let $u(t,x)$ be the L^∞ subsolution of* (1.1)–(1.2)*, and $v(t,x)$ the continuous minimax solution. Then $u(t,x) \le v(t,x)$ almost everywhere.*

Therefore, the L^∞ solutions coincide with the continuous minimax solutions when the initial data are continuous. Consequently, the L^∞ solutions consist with the continuous viscosity solutions.

§5. An Example

In this section we examine some interesting phenomena and study the L^∞ stability of discontinuous solutions of the Cauchy problem:

$$u_t - \alpha|Du| = 0, \qquad x \in \mathbb{R}^1, \quad \alpha > 0, \tag{5.1}$$

$$u(0,x) = \varphi(x). \tag{5.2}$$

Before we examine the Cauchy problem (5.1)-(5.2), we first recall the definition of countably piecewise continuous functions. We say that a function f is essentially continuous on a set A if, after replacing function values over a zero-measure subset of A, the function f is continuous on A in the classical sense. A function $f(x)$ is called countably piecewise continuous function over $\mathbb{R}^d$ if there exists a countable open decomposition $\{\Omega_k\}_{k=1}^\infty$ of $\mathbb{R}^d$ satisfying

$$\mathbb{R}^d = \bigcup_{k=1}^\infty \bar{\Omega}_k, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \ne j,$$

such that $f(x)$ is essentially continuous on every set Ω_k for any $k \in N$.

Denote by $L_p^\infty(\mathbb{R}^d)$ the subset of $L_{\rm loc}^\infty(\mathbb{R}^d)$ consisting of countably piecewise continuous functions. It is easy to verify that $L_p^\infty(\mathbb{R}^d)$ is a subspace of $L_{\rm loc}^\infty(\mathbb{R}^d)$ under the L^∞ norm. Denote by $L_{cp}^\infty(\mathbb{R}^d)$ the closure of $L_p^\infty(\mathbb{R}^d)$ under the L^∞ norm.

It is obvious that $\sin((\sin(x^{-1}))^{-1}) \in L_{cp}^\infty(\mathbb{R}^1)$. The following $L^\infty(\mathbb{R}^1)$ functions

$$\beta_k(x) = \begin{cases} 1, & \text{if } x \in (q_i - \frac{1}{2^{k+i}}, q_i + \frac{1}{2^{k+i}}) \text{ for any } i \in N, \\ 0, & \text{otherwise} \end{cases}$$

do not belong to $L_{cp}^\infty(\mathbb{R}^1)$, where $q_i \in Q$.

Theorem 5.1. *Let $\varphi(x) \in L_{cp}^\infty(R^1)$. Then the solution $u(t,x)$ to* (5.1)–(5.2) *is given by the following formula*

$$u(t,x) = \operatorname*{esssup}_{C(t,x)} \varphi(y), \quad \text{a.e. } in\ [0,T) \times \mathbb{R}^1, \tag{5.3}$$

where $C(t,x) = \{y \in \mathbb{R}^1 \,|\, |\frac{y-x}{t}| \le \alpha\}$.

Proof. It suffices to show that (5.3) holds at each point $(t_0,x_0) \in L(u(t,x)) \cap ([0,T) \times B^1(0,m))$. Denote by $X(t,x)$ the set of absolutely continuous functions $x(\cdot) : [0,t] \to R^1$ which satisfy the differential inequality $|\dot{x}(t)| \le C(1+|x(t)|)$ and the initial condition $x(t) = x$. Define

$$S = \bigcup_{(t,x)\in[0,T]\times B^1(0,m)} \{x(0) \mid x(\cdot) \in X(t,x)\}.$$

By the Gronwall inequality, we know $S \subset B^2(0,(m+1)e^{2CT})$. Thus there exists $M>0$ such that $-M < \operatorname*{esssup}_{y\in S}\varphi(y) < M$.

Step 1. For every point $(t_0,x_0)\in L(u(t,x))\cap([0,T)\times B^1(0,m))$, $u(t_0,x_0)\le \operatorname*{esssup}_{C(t_0,x_0)}\varphi(y)$, where $C(t_0,x_0)=\{y\in\mathbb{R}^1 \,|\, |\frac{y-x_0}{t_0}|\le\alpha\}$.

Consider a sequence of continuous functions defined as follows:

$$p_k(t,x)=\begin{cases} l(x-x_0), & \text{if } T\ge t>t_0-\frac1k,\\ l(x-x_0-\alpha(t-t_0+\frac1k)), & \text{if } x-x_0-\alpha(t-t_0+\frac1k)<0,\ t\le t_0-\frac1k,\\ l(x-x_0+\alpha(t-t_0+\frac1k)), & \text{if } x-x_0+\alpha(t-t_0+\frac1k)>0,\ t\le t_0-\frac1k,\\ 0, & \text{elsewhere,}\end{cases}$$

where $l=2k^4M$ for each k. For any $x(\tau)\in X(\tau_1,x_1)$ with $x_0+\alpha(\tau-t_0+\frac1k)\ge x(\tau)$ for $\tau_1\le\tau\le\tau_2$, one has

$$\begin{aligned}\int_{\tau_2}^{\tau_1}(\langle\dot x,p_k\rangle+\alpha|p_k|)\,d\tau &= \int_{\tau_2}^{\tau_1}\langle\dot x-\alpha,p_k\rangle\,d\tau\\ &= k^4M\Big(x(\tau)-x_0-\alpha\Big(\tau-t_0+\frac1k\Big)\Big)^2\Big|_{\tau=\tau_2}^{\tau=\tau_1}.\end{aligned}$$

For any $x(\tau)\in X(\tau_1,x_1)$ with $x_0-\alpha(\tau-t_0+\frac1k)\le x(\tau)$ for $\tau_1\le\tau\le\tau_2$, one has

$$\begin{aligned}\int_{\tau_2}^{\tau_1}(\langle\dot x,p_k\rangle+\alpha|p_k|)\,d\tau &= \int_{\tau_2}^{\tau_1}\langle\dot x+\alpha,p_k\rangle\,d\tau\\ &= k^4M\Big(x(\tau)-x_0+\alpha\Big(\tau-t_0+\frac1k\Big)\Big)^2\Big|_{\tau=\tau_2}^{\tau=\tau_1}.\end{aligned}$$

Thus, for any $x(\tau)\in X(t_1,x_1)$ with $x_0+\alpha(t_1-t_0+\frac1k)\le x_1\le x_0-\alpha(t_1-t_0+\frac1k)$ and $t_1<t_0$, direct computation shows that

(i) If $x(0)\le x_0-\alpha(t_0-\frac1k)$, then

$$\int_{t_1}^{0}(\langle\dot x,p_k\rangle+\alpha|p_k|)\,d\tau = k^4M\Big(x(0)-x_0+\alpha\Big(t_0-\frac1k\Big)\Big)^2.$$

(ii) If $x(0)\ge x_0+\alpha(t_0-\frac1k)$, then

$$\int_{t_1}^{0}(\langle\dot x,p_k\rangle+\alpha|p_k|)\,d\tau = k^4M\Big(x(0)-x_0-\alpha\Big(t_0-\frac1k\Big)\Big)^2.$$

(iii) If $x_0-\alpha(t_0-\frac1k)\le x(0)\le x_0+\alpha(t_0-\frac1k)$, then $\int_{t_1}^0(\langle\dot x,p_k\rangle+\alpha|p_k|)\,d\tau=0$.

Since $(t_0,x_0)\in L(u)$, for any $\epsilon>0$, there exists K (a positive integer) such that, for any $k\ge K$, there is $(t_k,x_k)\in L(u)$ with

$$t_0-\frac1k>t_k>t_0-\frac2k,\quad x_0+\alpha\Big(t_k-t_0+\frac1k\Big)\le x_k\le x_0-\alpha\Big(t_k-t_0+\frac1k\Big),$$

satisfying $|u(t_k,x_k)-u(t_0,x_0)|\le\epsilon$. By the definition of subsolution, (2.2), and the above equalities, one has $u(t_0,x_0)\le \operatorname*{esssup}_{y\in C(t_0,x_0)}\varphi(y)+\epsilon$ by letting $k\to\infty$. Since ϵ is arbitrary, Step 1 is completed.

Step 2. For every point $(t,x)\in L(u(t,x))\cap([0,T)\times B^1(0,m))$,

$$u(t_0,x_0)\ge \operatorname*{esssup}_{C(t_0,x_0)}\varphi(y),$$

where $C(t_0,x_0)=\{y\in\mathbb{R}^1 \,|\, |\frac{y-x_0}{t_0}|\le\alpha\}$.

Claim. For any $\epsilon>0$, there exists an interval $B^1(x_\epsilon,r_\epsilon)\subset C(t_0,x_0)$ such that

$$\|\varphi-\operatorname*{esssup}_{C(t_0,x_0)}\varphi(y)\|_{L^\infty(B^1(x_\epsilon,r_\epsilon))}\le\epsilon.$$

This can be seen as follows. There exists a measurable set A with $m(A)>0$ such that

$$\|\varphi-\operatorname*{esssup}_{C(t_0,x_0)}\varphi(y)\|_{L^\infty(A)}\le\frac{1}{4}\epsilon,$$

while there exists an L^∞_p function $g(x)$ such that

$$\|g-\varphi\|_{L^\infty(C(t_0,x_0))}\le\frac{1}{4}\epsilon.$$

Thus there exists a continuity point x_ϵ of $g(x)$ with $x_\epsilon\in A$ satisfying

$$|g(x)-\operatorname*{esssup}_{C(t_0,x_0)}\varphi(y)|\le\frac{3}{4}\epsilon,$$

for any $x\in B^1(x_\epsilon,r_\epsilon)$. Thus, on $B^1(x_\epsilon,r_\epsilon)$, the claim holds.

Consider the following continuous function:

$$p(t,x)=-2Mk\Big(x-x_\epsilon-\frac{x_0-x_\epsilon}{t_0}t\Big).$$

Note that, for any $0\le\tau_0\le t_0$,

$$\begin{aligned}\int_{\tau_0}^0(\langle\dot{x},p\rangle+\alpha|p|)\,d\tau&=\int_{\tau_0}^0\langle\dot{x}-\gamma,p\rangle\,d\tau+\int_{\tau_0}^0(\langle\gamma,p\rangle+\alpha|p|)\,d\tau\\&\le-Mk(x(\tau)-x_\epsilon-\gamma\tau)^2|_{\tau=\tau_0}^{\tau=0},\end{aligned}$$

where $\gamma=\frac{x_0-x_\epsilon}{t_0}$. Using the fact that (2.1) holds for (t_0,x_0) with the inequalities above, one has

$$u(t_0,x_0)\ge\operatorname*{esssup}_{y\in C(t_0,x_0)}\varphi(y)-C\epsilon,$$

by letting $k\to\infty$, where $C>1$. Since ϵ is arbitrary, Step 2 is established. This completes the proof of Theorem 5.1.

Remark 5.1. In the above proof, we know that Step 1 holds for any L^∞ initial data. The fact that initial data belong to L^∞_{cp} was utilized in Step 2. The proof of Step 2 implies that even (5.3) holds when $\varphi(x)=\beta_k(x)$, which is not in L^∞_{cp}. The unique solution to (5.1)–(5.2) with $\varphi(x)=\beta_k(x)$ is 1 almost everywhere. Thus this solution does not have weak L^1 continuity in time at the initial time. In fact, neither the continuous viscosity solutions nor minimax solutions are stable with respect to initial data in the L^1 sense. For L^∞_{cp} initial data, the L^∞ solutions may possess L^1 continuity in time. Moreover, the L^∞ solutions may preserve L^∞_{cp} space.

Corollary 5.1. *For any two solutions $u_1(t,x)$ and $u_1(t,x)$ with L^∞_{cp} initial data $\varphi_1(x)$ and $\varphi_2(x)$, respectively,*

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^\infty}\le\|\varphi_1-\varphi_2\|_{L^\infty},\quad\text{a.e. } t>0.$$

Remark 5.2. The corollary just indicates that the L^∞ solutions are unique and L^∞ stable with respect to L^∞_{cp} initial data. Our solutions can be also employed to construct other possible solutions as observed by sacrificing the L^∞ stability. For example, consider

(5.1) with initial data

$$\varphi_k(x) = \begin{cases} 1, & \text{if } -\frac{1}{k} \le x \le \frac{1}{k}, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 5.1, we know that

$$u_k(t,x) = \begin{cases} 1, & \text{if } -\frac{1}{k} - \alpha t \le x \le \frac{1}{k} + \alpha t, \text{ for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

is the solution for each positive integer k. So $\{u_k\}_{k=1}^{\infty}$ converges in L^1 to the following function

$$u(t,x) = \begin{cases} 1, & \text{if } -\alpha t \le x \le \alpha t, \text{ for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

which is Barron-Jensen's upper-semicontinuous solution (see [4]) to (5.1) with initial data

$$\varphi(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

This example also shows that discontinuous solutions are very sensitive to the topology in which the initial data are approximated. Our solutions are L^∞ stable as indicated above and can also produce other solutions as observed by approximating the initial data via appropriate topologies.

Remark 5.3. In this paper we present a notion of discontinuous solutions in L^∞ and prove the existence of L^∞ solutions. It is important to know the relation between the notion introduced here and the notion of viscosity solutions and to understand the uniqueness of solutions in L^∞. In [8], we will clarify the relation between the two notions and compare the L^∞ notion with the other existing notions for discontinuous solutions. We will also present further studies of L^∞ solutions for a class of Hamiltonians, which are important in differential game theory and control theory.

Appendix

Proof of Lemma 2.2. In the proof, we denote by $L(B)$ the subset of points of density 1 for a given measurable set B. Let $A = A^0 \neq \emptyset$. Consider

$$S_0 = \{D \subset A^0 \mid D \text{ is measurable and } m(D) > 0\}.$$

By assumption, S_0 is not empty. We choose a set $D_0 \in S_0$ such that

$$m(D_0) > \frac{3}{4} \sup_{D \in S_0} m(D) > 0,$$

and $B^0 \backslash D_0$ is measurable, where $B_0 = B^d(0, M)$. Then $B^1 \equiv L(B^0 \backslash D_0)$ is a set of density 1 for $B^0 \backslash D_0$ and $m(B^1) = m(B^0 \backslash D_0)$.

Let $A^1 = (A^0 \backslash D_0) \cap B^1$. If $A^1 = \emptyset$, stop. If $A^1 \neq \emptyset$, consider

$$S_1 = \{D \subset A^1 \mid D \text{ is measurable and } m(D) > 0\}.$$

Then S_1 is not empty. This can be seen as follows. Take a point $x \in A^1$. By the definition of nondegenerate density property, there exists a measurable set $A_x \subset A$, a sequence $\{r_k\}_{k=1}^{\infty} \equiv G$ with $\lim_{k\to\infty} r_k = 0$, and $\delta > 0$ such that

$$\frac{m(A_x \cap B^d(x, r_k))}{m(B^d(x, r_k))} > \delta, \qquad \text{for } r_j \in \{r_k\}_{k=1}^{\infty}.$$

By the definition of points of density 1, there is $j \in N$ such that

$$\frac{m(B^1 \cap B^d(x, r_k))}{m(B^d(x, r_k))} > 1 - \frac{\delta}{4}, \qquad \text{for } k \geq j \text{ and } r_k \in G,$$

which yields $m(A_x \cap B^1) > 0$. While $A_x \cap B^1 \subset A^1$, then S_1 is nonempty. Then we can choose a set $D_1 \subset S_1$ such that $m(D_1) \geq \frac{3}{4} \sup_{D \in S_1} m(D)$.

Recursively repeating the above procedure, we obtain the following four sequences $\{D_k\}_{k=0}^{\infty}$, $\{B^k\}_{k=0}^{\infty}$, $\{A^k\}_{k=0}^{\infty}$, and $\{S_k\}_{k=0}^{\infty}$ satisfying

(i) $B^k \subset B^{k-1}, D_k \subset A^k \subset B^k$, and D_k, B^k are measurable;

(ii) $A^k \subset A^{k-1} \backslash D_{k-1}$ and $m(A^{k-1} \backslash (A^k \cup D_{k-1})) = 0$, and $D_k \cap \bigcup_{j=1}^{k-1} D_j = \emptyset$;

(iii) $S_k = \{D \subset A^k \mid D \text{ is measurable and } m(D) > 0\}$;

(iv) $\sup_{D \in S_{k-1}} m(D) \geq \sup_{D \in S_k} m(D)$, $m(D_k) > \frac{3}{4} \sup_{D \in S_k} m(D)$.

Since $m(B^0)$ is finite and $\bigcup_k D_k \subset B^0$, $m(D_k) \to 0$ as $k \to \infty$. By Property (iv), $\sup_{D \in S_k} m(D) \to 0$ as $k \to \infty$. Also $A \backslash \bigcup_{k=0}^{\infty} D_k \subset B^0 \backslash \bigcup_{k=0}^{\infty} D_k$. Consider

$$S_\infty = \Big\{ D \subset A \backslash \bigcup_{k=0}^{\infty} D_k \,\Big|\, \text{D is measurable and } m(D) > 0 \Big\}.$$

It is obvious that

$$\sup_{D \in S_k} m(D) \geq \sup_{D \in S_\infty} m(D), \quad \text{for each } k.$$

Then $\sup_{D \in S_\infty} m(D) = 0$, which means $S_\infty = \emptyset$. Using the nondegenerate density property of A and the definition of points of density 1, one can argue by a contradiction to show

$$\Big(A \backslash \bigcup_{k=0}^{\infty} D_k \Big) \bigcap L \Big(B^0 \backslash \bigcup_{k=0}^{\infty} D_k \Big) = \emptyset.$$

Then $m\Big(A \backslash \bigcup_{k=0}^{\infty} D_k \Big) = 0$. Therefore, A is measurable. This completes the proof.

We here mention a necessary and sufficient condition of measurability of a given set in $\mathbb{R}^d$ implied by Lemma 2.2.

Lemma. *A set $A \subset \mathbb{R}^d$ is measurable if and only if there is a zero-measure set $B \subset A$ such that every point $x \in A \backslash B$ satisfies nondegenerate density property.*

Proof. Sufficiency. As proved by Lemma 2.2, $A \backslash B$ is measurable. Thus $A = B \cup (A \backslash B)$ is measurable.

Necessity. The characteristic function 1_A is measurable and locally integrable. By Lebesgue theorem, for almost all $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} 1_A(y)\, dy = 1_A(x).$$

It implies that, modulo a zero-measure set, every point of A has density 1.

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