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DEGENERATED HOMOCLINIC BIFURCATIONS WITH HIGHER DIMENSIONS***

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Abstract

The degenerated homoclinic bifurcation for high dimensional system is considered. The existence, uniqueness, and incoexistence of the 1-homclinic orbit and 1-periodic orbit near Γ are studied under the nonresonant condition. Complicated bifurcation pattern is described under the resonant condition.

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§1. Introduction and Hypotheses

In recent years, with the development of nonlinear science and the deep study of chaotic phenomena, an increasingly large number of papers are devoted to the bifurcation problems of homoclinic and heteroclinic orbits in high dimensional space (see [1–14]). Due to the difficulty encountered, unfortunately, only a few (e.g. [1, 13, 14]) are concerned with the periodic orbits bifurcated from singular loops. Papers [1, 13] discussed the problem of the homoclinic loop bifurcation in high dimension with codimension 2, that is, the system has resonant eigenvalues and the homoclinic loop $\Gamma = \{z = r(t) : t \in \mathbf{R},\ r(\pm\infty) = 0\}$ satisfies the nondegenerated condition $\operatorname{codim}(T_{r(t)}W^u + T_{r(t)}W^s) = 1$.

In this paper, the periodic and homoclinic orbits produced from the degenerated homoclinic bifurcation are considered, which means we assume $\operatorname{codim}(T_{r(t)}W^u + T_{r(t)}W^s) = 2$. Results corresponding to nonresonant and resonant conditions are obtained. The method to establish a system of local coordinates near the homoclinic loop suggested and used in [13, 14] is simplified here.

Consider the following C^r system

$$\dot{z} = f(z) + \varepsilon g(z, \mu, \varepsilon), \tag{1.1}$$

where $r \geq 4$, $z \in \mathbf{R}^{m+n}$, $\mu \in \mathbf{R}^k$, $0 \leq |\varepsilon| \ll 1$, $f(0) = 0$, $g(0, \mu, \varepsilon) = 0$.

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We need the following assumptions.

(H1) For $\varepsilon = 0$, System (1.1) has a homoclinic loop $\Gamma = \{z = r(t) : t \in \mathbf{R}\}$ with $r(\pm\infty) = 0$. The stable manifold W^s and the unstable manifold W^u of $z = 0$ are m-dimensional and n-dimensional, respectively. Moreover, The linearization $Df(0)$ at the equilibrium O has simple real eigenvalues λ_1, $-\lambda_2$, λ_3 and $-\lambda_4$ such that any remaining eigenvalue λ of $Df(0)$ satisfies either $\mathrm{Re}\lambda > \lambda_5 > \lambda_3 > \lambda_1 > 0$, or $\mathrm{Re}\lambda < -\lambda_6 < -\lambda_4 < -\lambda_2 < 0$ for some positive numbers λ_5 and λ_6. For any $p \in \Gamma$, $\mathrm{codim}(T_pW^u + T_pW^s) = 2$.

(H2) Define $e^{\pm} = \lim\limits_{t\to\pm\infty} \dot{r}(-t)/|\dot{r}(-t)|$. Then, $e^+ \in T_0W^u$ and $e^- \in T_0W^s$ are unit eigenvectors corresponding to λ_1 and $-\lambda_2$, respectively.

Let W^{ss} and W^{uu} be the strong stable manifold and the strong unstable manifold of $z = 0$, respectively, $\overline{e}^+$ and $\overline{e}^-$ be unit eigenvectors corresponding to λ_3 and $-\lambda_4$, $W^{uu+} \subset W^{uu}$ and $W^{ss-} \subset W^{ss}$ be the one-dimensional solution manifolds tangent to $\overline{e}^+$ and $\overline{e}^-$ at $z = 0$, respectively, $W^{uuu} \subset W^{uu}$ be the $(n-2)$-dimensional solution manifold tangent to the generalized eigenspace corresponding to those eigenvalues with larger real part than λ_5, and $W^{sss} \subset W^{ss}$ be the $(m-2)$-dimensional solution manifold tangent to the generalized eigenspace corresponding to those eigenvalues with smaller real part than $-\lambda_6$. Then, we have $T_0W^{uu} = T_0W^{uuu} \oplus T_0W^{uu+}$, $T_0W^{ss} = T_0W^{sss} \oplus T_0W^{ss-}$.

(H3)
$$\lim_{t\to+\infty} (T_{r(t)}W^s \cap T_{r(t)}W^u) = e^- \oplus \overline{e}^-,$$
$$\lim_{t\to-\infty} (T_{r(t)}W^s \cap T_{r(t)}W^u) = e^+ \oplus \overline{e}^+.$$

(H4)
$$\mathrm{span}\,(T_{r(t)}W^u,\ T_{r(t)}W^s,\ e^+,\ \bar{e}^+) = \mathbf{R}^{m+n},\quad t \gg 1,$$
$$\mathrm{span}\,(T_{r(t)}W^u,\ T_{r(t)}W^s,\ e^-,\ \bar{e}^-) = \mathbf{R}^{m+n},\quad t \ll -1.$$

We say Γ is degenerate if $\dim(T_{r(t)}W^u \cap T_{r(t)}W^s) > 1$. In degenerate cases, the patten of bifurcation will be much more complicated. It is easy to see that under the hypothesis (H1), the hypothesis (H2) is generic, and so are the hypotheses (H3) and (H4). Hypothesis (H4) is equivalent to

$$T_{r(t)}W^u \to T_oW^{uuu} \oplus e^- \oplus \bar{e}^- \quad \text{as} \quad t \to +\infty,$$
$$T_{r(t)}W^s \to T_oW^{sss} \oplus e^+ \oplus \bar{e}^+ \quad \text{as} \quad t \to -\infty.$$

This is called the strong inclination property.

§2. Local Coordinates

Our study will be based on the analysis of the Poincaré map defined on some local transversal section of Γ. For the establishment of the Poincaré map, we should choose a suitable coordinate system. Consider System (1.1) under the hypotheses (H1)–(H4). Suppose that the neighborhood U is small enough. Then we can introduce a C^r change such that System (1.1) has the following form in U:

$$\begin{aligned} \dot{x} &= [\lambda_1(\varepsilon) + \cdots]\, x, & \dot{y} &= [-\lambda_2(\varepsilon) + \cdots]\, y, \\ \dot{\bar{u}} &= [\lambda_3(\varepsilon) + \cdots]\, \bar{u}, & \dot{\bar{v}} &= [-\lambda_4(\varepsilon) + \cdots]\, \bar{v}, \\ \dot{u} &= [B_1(\varepsilon) + \cdots]\, u, & \dot{v} &= [-B_2(\varepsilon) + \cdots]\, v, \end{aligned} \tag{2.1}$$

where $\lambda_i(0) = \lambda_i$ for $i = 1, 2, 3, 4$, $\mathrm{Re}\sigma(B_1(\varepsilon)) > \lambda_3$ and $\mathrm{Re}\sigma(-B_2(\varepsilon)) < -\lambda_4$ for $|\varepsilon|$ small enough. In other words, we have straightened the following manifolds in U,

$$\Gamma \cap W^u_{\mathrm{loc}} = \{y = 0, \bar{u} = 0, \bar{v} = 0, u = 0, v = 0\},\ W^{uu+}_{\mathrm{loc}} = \{x = 0, y = 0, \bar{v} = 0, u = 0, v = 0\},$$
$$\Gamma \cap W^s_{\mathrm{loc}} = \{x = 0, \bar{u} = 0, \bar{v} = 0, u = 0, v = 0\},\ W^{ss+}_{\mathrm{loc}} = \{x = 0, y = 0, \bar{u} = 0, u = 0, v = 0\},$$
$$W^{uuu}_{\mathrm{loc}} = \{x = 0, y = 0, \bar{u} = 0, \bar{v} = 0, v = 0\},\ W^{sss}_{\mathrm{loc}} = \{x = 0, y = 0, \bar{u} = 0, \bar{v} = 0, u = 0\}.$$

Here, $u \in \mathbf{R}^{n-2}$, $v \in \mathbf{R}^{m-2}$, and (2.1) is C^{r-1}.

Taking a time translation if necessary, we may assume $r(-T) = (\delta, 0, 0, 0, 0^*, 0^*)^*$, $r(T) = (0, \delta, 0, 0, 0^*, 0^*)^*$, where δ is small enough such that $\{(x, y, \bar{u}, \bar{v}, u, v) : |x|, |y|, |\bar{u}|, |\bar{v}|, |u|, |v| < 3\delta/2\} \subset U$. Let $A(t) = Df(r(t))$. Consider the linear system

$$\dot{z} = A(t)z \tag{2.2}$$

and its adjoint system

$$\dot{z} = -A^*(t)z. \tag{2.3}$$

Now we choose solutions of (2.2) as following:

$$z_1(t), z_2(t) \in (T_{r(t)}W^s)^c \cap (T_{r(t)}W^u)^c,$$
$$z_3(t) = -\dot{r}(t)/|\dot{r}(T)|, \quad z_4(t) \in T_{r(t)}W^s \cap T_{r(t)}W^u,$$
$$z_5(t) = (z_5^1(t), \cdots, z_5^{n-2}(t)) \in T_{r(t)}W^{uuu} \subset (T_{r(t)}W^s)^c \cap T_{r(t)}W^u,$$
$$z_6(t) = (z_6^1(t), \cdots, z_6^{m-2}(t)) \in T_{r(t)}W^{sss} \subset T_{r(t)}W^s \cap (T_{r(t)}W^u)^c,$$

$$z_1(T) = (1, 0, 0, 0, 0, w_{16}^*)^*, \quad z_2(T) = (\tilde{w}_{21}, 0, 1, 0, 0, w_{26}^*)^*, \quad z_3(T) = (0, 1, 0, 0, 0, 0)^*,$$
$$z_4(-T) = (0, 0, 1, 0, 0, 0)^*, \quad z_5(-T) = (0, 0, 0, 0, I, 0)^*, \quad z_6(T) = (0, 0, 0, 0, 0, I)^*$$

such that $Z(t) = (z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t))$ is a fundamental solution matrix.

Proposition 2.1. *If* (H1)–(H4) *are valid, then there exist constant vectors w_{16}, w_{26} and $\tilde{w}_{21}$ such that the followings are true:*

$$z_1(-T) = (w_{11}, w_{12}, w_{13}, w_{14}, w_{15}^*, 0)^*, \quad z_2(-T) = (w_{21}, w_{22}, w_{23}, w_{24}, w_{25}^*, 0)^*,$$
$$z_3(-T) = (w_{31}, 0, 0, 0, 0, 0)^*, \quad z_4(T) = (0, w_{42}, 0, w_{44}, 0, 0)^*,$$
$$z_5(T) = (w_{51}^*, w_{52}^*, w_{53}^*, w_{54}^*, w_{55}^*, w_{56}^*)^*, \quad z_6(-T) = (w_{61}^*, w_{62}^*, w_{63}^*, w_{64}^*, w_{65}^*, w_{66}^*)^*,$$

where $w_{31} < 0$, $w_{44} \neq 0$, $\det w_{55} \neq 0$, $\det w_{66} \neq 0$ and either $w_{12}w_{24} \neq 0$, $w_{22} = 0$ or $w_{12} = 0$, $\tilde{w}_{21} = 0$, $w_{14}w_{22} \neq 0$. Moreover, for δ small enough, $|w_{1i}w_{12}^{-1}| \ll 1$ for $i \neq 2$, $|w_{2i}w_{24}^{-1}| \ll 1$ for $i \neq 4$, $|\tilde{w}_{21}w_{24}^{-1}| \ll 1$, $|w_{42}w_{44}^{-1}| \ll 1$, $|w_{5i}w_{55}^{-1}| \ll 1$ for $i \neq 5$, $|w_{6i}w_{66}^{-1}| \ll 1$ for $i \neq 6$.

Proof. The existence of $z_5(t)$ and $z_6(t)$ with given values at T and $-T$ is clear. By the definition of $z_3(t)$, we have $w_{31} < 0$ immediately. Now let $\bar{z}_1(t)$ be a solution of (2.2) with $\bar{z}_1(T) = (1, 0, 0, 0, 0^*, 0^*)^*$. Then, $z_1(t) = \bar{z}_1(t) + z_6(t)w_{16}$ is also a solution of (2.2) with $z_1(T) = (1, 0, 0, 0, 0^*, w_{16}^*)^*$. Denote $\bar{z}_1(-T) = (\bar{w}_{11}, \bar{w}_{12}, \bar{w}_{13}, \bar{w}_{14}, \bar{w}_{15}^*, \bar{w}_{16}^*)^*$ and take $w_{16} = -w_{66}^{-1}\bar{w}_{16}$. Then we get $z_1(-T)$ as desired in case $\det w_{66} \neq 0$.

Now by the definition, $z_1(t) \in (T_{r(t)}W^u)^c \cap (T_{r(t)}W^s)^c$, we get $(w_{12})^2 + (w_{14})^2 \neq 0$. First assume $w_{12} \neq 0$. Then, similar to the procedure for getting the desired $z_1(t)$, we see there is a vector $\bar{w}_{26}$ such that there exists a solution $\bar{z}_2(t)$ satisfing $\bar{z}_2(T) = (0, 0, 1, 0, 0^*, \bar{w}_{26}^*)^*$ and $\bar{z}_2(-T) = (\bar{w}_{21}, \bar{w}_{22}, \bar{w}_{23}, \bar{w}_{24}, \bar{w}_{25}^*, 0^*)^*$. Since $w_{12} \neq 0$, we can define $z_2(t) = \bar{z}_2(t) + z_1(t)\tilde{w}_{21}$ with $\tilde{w}_{21} = -\bar{w}_{22}w_{12}^{-1}$ and $w_{26} = \bar{w}_{26} + \tilde{w}_{21}w_{16}$ such that $z_2(-T) = (w_{21}, 0, w_{23}, w_{24}, w_{25}^*, 0^*)^*$.

That $z_4(T)$ has the expression $(0, w_{42}, 0, w_{44}, 0^*, 0)^*$ is simply because $T_{r(t)}W^s \cap T_{r(t)}W^u$ is an invariant subspace of (2.2) and becomes the $y - \bar{v}$ plane as $t \geq T$.

A simple computation shows that $\det Z(T) = -w_{44} \det w_{55}$, which turns out that $w_{44} \neq 0$ and $\det w_{55} \neq 0$.

Now we show $\det w_{66} \neq 0$. In fact, if $\det w_{66} = 0$, then, due to $\dim W^{sss} = \mathrm{rank} z_6(T) = \mathrm{rank} z_6(-T)$, we have $T_{r(-T)}W^{sss} \cap \mathrm{span}\{T_{r(-T)}W^u, e^-, \bar{e}^-\} \neq \emptyset$. Notice that hypothesis (H3) means $\dim(T_{r(-T)}W^s \cap \mathrm{span}\{T_{r(-T)}W^u, e^-, \bar{e}^-\}) \geq 3$, which turns out that $\dim(\mathrm{span}\{T_{r(-T)}W^u, T_{r(-T)}W^s, e^-, \bar{e}^-\}) < n + m$. It contradicts hypothesis (H4) .

$w_{24} \neq 0$ is a direct consequence of $\det Z(-T) = -w_{12}w_{24}w_{31} \det w_{66} \neq 0$.

If $w_{12} = 0$, then $w_{14} \neq 0$, and we simply take $z_2(t) = \bar{z}_2(t)$. This means $\tilde{w}_{21} = 0$. Now it follows from $\det Z(-T) = w_{14}w_{22}w_{31} \det w_{66} \neq 0$ that $w_{22} \neq 0$.

The remainder is easy to check by using the expressions of $A(t)$ at $t = +\infty$ and $-\infty$, we omit the detail. The proof is finished.

Generically, we have $w_{12} \neq 0$. Since the discussion is similar for the case $w_{12} = 0$ (then $w_{14}w_{22} \neq 0$ by Proposition 2.1), we restrict ourselves to the case $w_{12} \neq 0$ in this paper.

Denote $r(t) = (r_1(t), r_2(t), r_3(t), r_4(t), r_5^*(t), r_6^*(t))^*$, $w_{12} = \Delta|w_{12}|$. We say that Γ is nontwisted as $\Delta = 1$, and twisted as $\Delta = -1$.

In the following, we regard $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$, $z_5(t)$, $z_6(t)$ as a local coordinate system along Γ. Denote $\Psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t), \psi_6(t)) = (Z^{-1}(t))^*$. Due to [15], we have

Proposition 2.2. $\Psi(t)$ *is a fundamental solution matrix of* (2.3)*. Moreover,* $\psi_1(t)$*,* $\psi_2(t)$ $\in (T_{r(t)}W^s)^c \cap (T_{r(t)}W^u)^c$ *are bounded and tend to zero exponentially as* $t \to \pm\infty$*.*

§3. Poincaré Map and Its Associated Successor Function

Now we set up the Poincaré map. Set

$$n_5 = (n_5^1, \cdots, n_5^{n-2})^*, \quad n_6 = (n_6^1, \cdots, n_6^{m-2})^*,$$
$$s(t) = r(t) + z_1(t)n_1 + z_2(t)n_2 + z_4(t)n_4 + z_5(t)n_5 + z_6(t)n_6.$$

Let

$$S_0 = \{z = s(T) : |x|, |y|, |\overline{u}|, |\overline{v}|, |u|, |v| < 3\delta/2\},$$
$$S_1 = \{z = s(-T) : |x|, |y|, |\overline{u}|, |\overline{v}|, |u|, |v| < 3\delta/2\}$$

be cross sections of Γ at $t = T$ and $t = -T$, respectively, where δ is small enough such that S_0, $S_1 \subset U$. At first, we set up a map F_1 from S_1 to S_0 which is defined by the orbits of (1.1). Secondly, we consider the map F_2 from S_0 to S_1 induced by the orbits of (2.1) in U. Thirdly, by combining the two maps we get the Poincaré map $F = F_1 \circ F_2$: $S_0 \mapsto S_0$. Finally, the associated successor function is given.

Let $z = s(t)$ be a solution of (1.1). Substituting it into (1.1), we get

$$\dot{r}(t) + \dot{Z}(t)(n_1, n_2, 0, n_4, n_5^*, n_6^*)^* + Z(t)(\dot{n}_1, \dot{n}_2, 0, \dot{n}_4, \dot{n}_5^*, \dot{n}_6^*)^*$$
$$= f(r(t)) + A(t)Z(t)(n_1, n_2, 0, n_4, n_5^*, n_6^*)^* + \varepsilon g(r(t), \mu, 0) + \text{h.o.t.}$$

By $\dot{r}(t) = f(r(t))$ and $\dot{Z}(t) = A(t)Z(t)$, it reads as

$$Z(t)(\dot{n}_1, \dot{n}_2, 0, \dot{n}_4, \dot{n}_5^*, \dot{n}_6^*)^* = \varepsilon g(r(t), \mu, 0) + \text{h.o.t.}$$

Multiplying two sides of the equation by $\Psi^*(t)$ and using $\Psi^*(t)Z(t) = I$, we obtain

$$\dot{n}_i = \varepsilon \psi_i^*(t) g(r(t), \mu, 0) + \text{h.o.t.}, \quad i = 1, 2, 4, 5, 6. \tag{3.1}$$

System (3.1) yields the map $F_1 : S_1 \mapsto S_0$ defined by $(n_1(-T), n_2(-T), n_4(-T), n_5^*(-T), n_6^*(-T))^* \longmapsto (n_1(T), n_2(T), n_4(T), n_5^*(T), n_6^*(T))^*$,

$$n_i(T) = n_i(-T) + \varepsilon M_i(\mu) + \text{h.o.t.}, \tag{3.2}$$

where $M_i(\mu) = \int_{-T}^{T} \psi_i^*(t) g(r(t), \mu, 0) dt$, $i = 1, 2, 4, 5, 6$.

Proposition 3.1. *For* $i = 1, 2, 4, 5, 6$, $M_i(\mu) = \int_{-\infty}^{+\infty} \psi_i^*(t) g(r(t), \mu, 0) dt$.

Proof. It suffices to show $\psi_i^*(t) g(r(t), \mu, 0) = 0$ for $|t| > T$. Clearly, $r(t) = (0,\ r_2(t),\ 0,\ 0,\ 0^*,\ 0^*)^*$ and $|r_2(t)| < \delta$ for $t > T$. Owing to $\psi_i^*(T) z_3(T) = 0$ for $i \neq 3$ and solving (2.3) we get the y-component of $\psi_i(t)$ is equal to zero for $t > T$. Meanwhile, (2.1) implies that $g(r(t), \mu, 0) = (0, g_2(r(t), \mu, 0), 0, 0, 0^*, 0^*)^*$ for $t > T$. Thus we have $\psi_i^*(t) g(r(t), \mu, 0) = 0$ for $t > T$. Similarly, we have $\psi_i^*(t) g(r(t), \mu, 0) = 0$ for $t < -T$. Thus the proof is complete.

Functions $M_1(\mu)$, $M_2(\mu)$, $M_4(\mu)$, $M_5(\mu)$ and $M_6(\mu)$ are called Melnikov functions.

Now we consider the map $F_2 : S_0 \to S_1$, $q_0(x_0, y_0, \bar{u}_0, \bar{v}_0, u_0^*, v_0^*)^* \to q_1(x_1, y_1, \bar{u}_1, \bar{v}_1, u_1^*, v_1^*)^*$ which is induced by the orbit of (2.1) in U, where $u_i = (u_i^1, \cdots, u_i^{n-2})^*$, $v_i = (v_i^1, \cdots, v_i^{m-2})^*$ for $i = 0, 1$.

First assume $\lambda_1 \leq \lambda_2$. In order to guarantee the differentiability of the map at the origin, we set $s = e^{-\lambda_1(\varepsilon)\tau}$, where τ is the flying time from $q_0(x_0, y_0, \bar{u}_0, \bar{v}_0, u_0^*, v_0^*)^*$ to $q_1(x_1, y_1, \bar{u}_1, \bar{v}_1, u_1^*, v_1^*)^*$.

If we apply the method of [2] or successive substitutions with initial solution value

$$\begin{aligned} x &= e^{\lambda_1(\varepsilon)(t-T-\tau)} x_1, & y &= e^{-\lambda_2(\varepsilon)(t-T)} y_0, \\ \bar{u} &= e^{\lambda_3(\varepsilon)(t-T-\tau)} \bar{u}_1, & \bar{v} &= e^{-\lambda_4(\varepsilon)(t-T)} \bar{v}_0, \\ u &= e^{B_1(\varepsilon)(t-T-\tau)} u_1, & v &= e^{-B_2(\varepsilon)(t-T)} v_0 \end{aligned} \tag{3.3}$$

to (2.1), and neglect the higher order terms for δ sufficiently small, then we get

$$\begin{aligned} x_0 &= e^{-\lambda_1(\varepsilon)\tau} x_1 = s x_1, & y_1 &= e^{-\lambda_2(\varepsilon)\tau} y_0 = s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} y_0, \\ \bar{u}_0 &= e^{-\lambda_3(\varepsilon)\tau} \bar{u}_1 = s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)} \bar{u}_1, & \bar{v}_1 &= e^{-\lambda_4(\varepsilon)\tau} \bar{v}_0 = s^{\lambda_4(\varepsilon)/\lambda_1(\varepsilon)} \bar{v}_0, \\ u_0 &= e^{-B_1(\varepsilon)\tau} u_1 = s^{B_1(\varepsilon)/\lambda_1(\varepsilon)} u_1, & v_1 &= e^{-B_2(\varepsilon)\tau} v_0 = s^{B_2(\varepsilon)/\lambda_1(\varepsilon)} v_0. \end{aligned} \tag{3.4}$$

Here we have used the fact that

$$\begin{aligned} &x \sim O(\delta) e^{\lambda_1(t-T-\tau)}, \quad y \sim O(\delta) e^{-\lambda_2(t-T)}, \quad \bar{u} \sim O(\delta) e^{\lambda_3(t-T-\tau)}, \\ &\bar{v} \sim O(\delta) e^{-\lambda_4(t-T)}, \quad u \sim O(\delta) e^{B_1(t-T-\tau)}, \quad v \sim O(\delta) e^{-B_2(t-T)}. \end{aligned}$$

Now we seek the new coordinates of q_0 and q_1. Let

$$\begin{aligned} q_0 &= (x_0, y_0, \bar{u}_0, \bar{v}_0, u_0^*, v_0^*)^* = r(T) + Z(T)(n_{01}, n_{02}, 0, n_{04}, n_{05}^*, n_{06}^*)^*, \\ q_1 &= (x_1, y_1, \bar{u}_1, \bar{v}_1, u_1^*, v_1^*)^* = r(-T) + Z(-T)(n_{11}, n_{12}, 0, n_{14}, n_{15}^*, n_{16}^*)^*. \end{aligned}$$

Then, using $r(T) = (0, \delta, 0, 0, 0^*, 0^*)^*$ and $r(-T) = (\delta, 0, 0, 0, 0^*, 0^*)^*$, we get

$$\begin{aligned} (n_{01}, n_{02}, 0, n_{04}, n_{05}^*, n_{06}^*)^* &= Z^{-1}(T)(x_0, y_0 - \delta, \bar{u}_0, \bar{v}_0, u_0^*, v_0^*)^*, \\ (n_{11}, n_{12}, 0, n_{14}, n_{15}^*, n_{16}^*)^* &= Z^{-1}(-T)(x_1 - \delta, y_1, \bar{u}_1, \bar{v}_1, u_1^*, v_1^*)^*. \end{aligned} \tag{3.5}$$

Let

$$a_1 = w_{11} - w_{21}w_{24}^{-1}w_{14}, \quad a_3 = w_{13} - w_{23}w_{24}^{-1}w_{14}, \quad a_5 = w_{15} - w_{25}w_{24}^{-1}w_{14},$$
$$b_1 = w_{51} - \tilde{w}_{21}w_{53}, \quad b_2 = w_{52} - w_{42}w_{44}^{-1}w_{54},$$
$$b_6 = w_{56} - w_{16}w_{51} - (w_{26} - \tilde{w}_{21}w_{16})w_{53}, \quad c_1 = w_{61} - w_{21}w_{24}^{-1}w_{64},$$
$$c_3 = w_{63} - w_{23}w_{24}^{-1}w_{64}, \quad c_4 = w_{64} - w_{14}w_{12}^{-1}w_{62}, \quad c_5 = w_{65} - w_{25}w_{24}^{-1}w_{64},$$

where $\|a_i w_{12}^{-1}\| \ll 1$ for $i = 1, 3, 5$, $\|b_i w_{55}^{-1}\| \ll 1$ for $i = 1, 2, 6$, $\|c_i w_{66}^{-1}\| \ll 1$ for $i = 1, 3, 4, 5$ as δ small enough. Due to the hypothesis $w_{12} \neq 0$ and Proposition 2.1, we have

$$\begin{aligned} &n_{01} = x_0 - \tilde{w}_{21}\bar{u}_0 - b_1 w_{55}^{-1}u_0, \quad n_{02} = \bar{u}_0 - w_{53}w_{55}^{-1}u_0, \\ &y_0 \approx \delta, \quad n_{04} = w_{44}^{-1}(\bar{v}_0 - w_{54}w_{55}^{-1}u_0), \\ &n_{05} = w_{55}^{-1}u_0, \quad n_{06} = -w_{16}x_0 - (w_{26} - w_{16}\tilde{w}_{21})\bar{u}_0 - b_6 w_{55}^{-1}u_0 + v_0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &n_{11} = w_{12}^{-1}(y_1 - w_{62}w_{66}^{-1}v_1), \quad n_{12} = w_{24}^{-1}(-w_{14}w_{12}^{-1}y_1 + \bar{v}_1 - c_4 w_{66}^{-1}v_1), \\ &x_1 \approx \delta, \ n_{14} = -a_3 w_{12}^{-1}y_1 + \bar{u}_1 - w_{23}w_{24}^{-1}\bar{v}_1 - (c_3 - a_3 w_{12}^{-1}w_{62})w_{66}^{-1}v_1, \\ &n_{15} = -a_5 w_{12}^{-1}y_1 + u_1 - w_{25}w_{24}^{-1}\bar{v}_1 - (c_5 - a_5 w_{12}^{-1}w_{62})w_{66}^{-1}v_1, \ n_{16} = w_{66}^{-1}v_1. \end{aligned} \tag{3.7}$$

Now we have defined the map $F_1 : S_1 \to S_0$ by (3.2) and the map $F_2 : q_0 \in S_0 \to q_1 \in S_1$ by (3.4), (3.6) and (3.7). Let $F_1(q_1) = q_2 = r(T) + Z(T)(n_{21}, n_{22}, 0, n_{24}, n_{25}, n_{26})$. Then (3.2) reads as

$$n_{2i} = n_{1i} + \varepsilon M_i(\mu) + \text{h.o.t.}, \quad i = 1, 2, 4, 5, 6. \tag{3.8}$$

Thus we get the Poincaré map $F = F_1 \circ F_2 : q_0 \in S_0 \to q_2 \in S_0$,

$$F((n_{01}, n_{02}, n_{04}, n_{05}^*, n_{06}^*)^*) = (n_{21}, n_{22}, n_{24}, n_{25}^*, n_{26}^*)^*,$$

where $(n_{i1}, n_{i2}, n_{i4}, n_{i5}^*, n_{i6}^*)^*$ for $i = 0, 1, 2$ are given by (3.6), (3.7) and (3.8). Explicitly, if we substitute (3.4) (3.6) and (3.7) into (3.8) and neglect the higher order terms (compared with $O(s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)})$ or $O(s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)})$), then the Poincaré map $F : S_0 \to S_0$ is given by

$$\begin{aligned} &n_{21} = w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} + \varepsilon M_1(\mu) + \text{h.o.t.}, \\ &n_{22} = -w_{24}^{-1}w_{14}w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} + w_{24}^{-1}s^{\lambda_4(\varepsilon)/\lambda_1(\varepsilon)}\bar{v}_0 + \varepsilon M_2(\mu) + \text{h.o.t.}, \\ &n_{24} = -a_3 w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} - w_{23}w_{24}^{-1}s^{\lambda_4(\varepsilon)/\lambda_1(\varepsilon)}\bar{v}_0 + \bar{u}_1 + \varepsilon M_4(\mu) + \text{h.o.t.}, \\ &n_{25} = -a_5 w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} - w_{25}w_{24}^{-1}s^{\lambda_4(\varepsilon)/\lambda_1(\varepsilon)}\bar{v}_0 + u_1 + \varepsilon M_5(\mu) + \text{h.o.t.}, \\ &n_{26} = w_{66}^{-1}s^{B_2(\varepsilon)/\lambda_1(\varepsilon)}v_0 + \varepsilon M_6(\mu) + \text{h.o.t.}, \end{aligned} \tag{3.9}$$

and its associated successor function

$$G(s, \bar{v}_0, v_0, \bar{u}_1, u_1) = F(q_0) - q_0 = (n_{21}, n_{22}, n_{24}, n_{25}^*, n_{26}^*)^* - (n_{01}, n_{02}, n_{04}, n_{05}^*, n_{06}^*)^*$$

is given by

$$\begin{aligned} &G_1 = w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} - \delta s + \tilde{w}_{21}s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)}\bar{u}_1 + \varepsilon M_1(\mu) + \text{h.o.t.}, \\ &G_2 = -w_{24}^{-1}w_{14}w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} - s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)}\bar{u}_1 + \varepsilon M_2(\mu) + \text{h.o.t.}, \\ &G_4 = -a_3 w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} + \bar{u}_1 - w_{44}^{-1}\bar{v}_0 + \varepsilon M_4(\mu) + \text{h.o.t.}, \\ &G_5 = -a_5 w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} + u_1 - w_{25}w_{24}^{-1}s^{\lambda_4(\varepsilon)/\lambda_1(\varepsilon)}\bar{v}_0 + \varepsilon M_5(\mu) + \text{h.o.t.}, \\ &G_6 = -v_0 + w_{16}\delta s + (w_{26} - w_{16}\tilde{w}_{21})s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)}\bar{u}_1 + \varepsilon M_6(\mu) + \text{h.o.t.} \end{aligned} \tag{3.10}$$

Then assume $\lambda_1 > \lambda_2$. In this case, we take $s = e^{-\lambda_2(\varepsilon)\tau}$. We should now use $s^{\lambda/\lambda_2(\varepsilon)}$ to replace $s^{\lambda/\lambda_1(\varepsilon)}$ in (3.4), (3.9) and (3.10) for $\lambda = \lambda_1, \cdots, \lambda_4$, B_1, B_2, respectively.

Remark 3.1. In the following, we always assume that functions F and G have been differentiablly extended to some neighborhood of $s = 0$.

§4. Nonresonant Homoclinic Bifurcations

In this section, we consider the nonresonant case $\lambda_1 \neq \lambda_2$. We only study the case $\lambda_1 < \lambda_2$, and the results also apply to the case $\lambda_1 > \lambda_2$ if we change t to $-t$.

Now we use (3.10) to study the existence and the uniqueness of the 1-homoclinic orbit and the 1-periodic orbit. Consider the solutions of the equation

$$G(s, \bar{v}_0, v_0, \bar{u}_1, u_1) = 0. \tag{4.1}$$

The degeneracy of the homoclinic loop Γ implies that $\tilde{G} = \partial G(s, \bar{v}_0, v_0, \bar{u}_1, u_1)/\partial(s, \bar{v}_0, v_0, \bar{u}_1, u_1)$ is degenerate at $(s, \bar{v}_0, v_0, \bar{u}_1, u_1) = 0$. Thus the implicit function theorem is not valid in this case. But, the last three equations of (4.1): $G_4 = 0$, $G_5 = 0$, $G_6 = 0$ always have a unique solution $\bar{u}_1 = \bar{u}_1(\varepsilon, \mu, s, \bar{v}_0) = O(\varepsilon) + O(\bar{v}_0) + o(s)$, $u_1 = u_1(\varepsilon, \mu, s, \bar{v}_0) = O(\varepsilon) + o(s)$, $v_0 = v_0(\varepsilon, \mu, s, \bar{v}_0) = O(\varepsilon) + O(s)$ for $|\varepsilon|$, s, $|\bar{v}_0|$ sufficiently small. Substituting it into $G_1 = 0$ and $G_2 = 0$, we see $G = 0$ is equivalent to $G_1 = 0$, $G_2 = 0$, that is,

$$\begin{aligned} & w_{12}^{-1}\delta s^{\lambda_2(\varepsilon)/\lambda_1(\varepsilon)} - \delta s + \tilde{w}_{21} s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)}\bar{u}_1 + \varepsilon M_1(\mu) + \text{h.o.t} = 0, \\ & - w_{24}^{-1} w_{14} w_{12}^{-1} \delta s^{\lambda_2(\varepsilon)/\lambda_1(s)} - s^{\lambda_3(\varepsilon)/\lambda_1(\varepsilon)}\bar{u}_1 + \varepsilon M_2(\mu) + \text{h.o.t} = 0. \end{aligned} \tag{4.2}$$

If there is a $\mu = \bar{\mu}$ such that

$$(M_1(\bar{\mu}), M_2(\bar{\mu})) = (0, 0), \quad \text{rank}(\partial(M_1(\mu), M_2(\mu))/\partial\mu|_{\mu=\bar{\mu}}) = 2, \tag{4.3}$$

then, by a scale transformation $s \to \varepsilon s$, $\bar{v}_0 \to \varepsilon \bar{v}_0$, we can apply the implicit function theorem to claim that there is a $(k-2)$-dimensional surface $\Sigma_1 = \Sigma_1(s, \bar{v}_0, \varepsilon) \subset \mathbf{R}^k$ in the neighborhood of $\bar{\mu}$ such that (4.2) has a solution $(s, \bar{v}_0)$ satisfying $0 \le s \ll |\varepsilon|$ and $|\bar{v}_0| \ll |\varepsilon|$ for $|\varepsilon|$ small enough and $\mu \in \Sigma_1$. That is, (1.1) has a homoclinic orbit near Γ for $|\varepsilon|$ small enough and $\mu \in \Sigma_1(0, \bar{v}_0, \varepsilon)$ and a periodic orbit near Γ as $\mu \in \Sigma_1(s, \bar{v}_0, \varepsilon)$ for $s > 0$. Moreover, (4.3) means $\partial M_1(\bar{\mu})/\partial\mu \neq 0$. Then it follows from the first equation of (4.2) that $(\partial M_1(\bar{\mu})/\partial\mu)(\partial\mu/\partial s) = \delta + \text{h.o.t.}$ for $|\varepsilon|$ and s small enough, which means that at least one component of μ is monotonic with respect to s for fixed ε and $\bar{v}_0$. It turns out that System (1.1) has a unique 1-homoclinic orbit or a unique 1-perioddic orbit $\Gamma_{\varepsilon\ \mu}$ as $\mu \in \Sigma_1$ such that the fourth component of $\Gamma_{\varepsilon\ \mu} \cap S_0$ is $\bar{v}_0$.

Now we assume that there is a $\mu = \bar{\mu}$ such that $M_1(\bar{\mu}) \neq 0$, $M_2(\bar{\mu}) = 0$ and $\partial M_2(\bar{\mu})/\partial\mu \neq 0$. Then we have $s = \varepsilon\delta^{-1}M_1(\mu) + \text{h.o.t.}$, and the implicit function theorem says that there is a $(k-1)$-demensional surface $\Sigma_2 = \Sigma_2(\bar{v}_0, \varepsilon) \subset R^k$ in the neighborhood of $\bar{\mu}$ such that the second equation of (4.2) has a solution $\bar{v}_0$ for $\mu \in \Sigma_2$ and $0 < |\varepsilon| \ll 1$ satisfying $\varepsilon M_1(\bar{\mu}) > 0$. In this case, System (1.1) has a periodic orbit near Γ. Furthermore, if $\partial M_1(\bar{\mu})/\partial\mu \neq 0$, then the periodic orbit near Γ is unique for fixed $\bar{v}_0$.

If $M_2(\bar{\mu}) \neq 0$, then the first equation of (4.2) has solution $s = O(\varepsilon) + o(|v_0|)$, and the second equation of (4.2) has the form $M_2(\mu) = o(\varepsilon) + o(|v_0|)$. Obviously, it has no solution for $0 \neq |\varepsilon|$, $|v_0|$, $|\mu - \bar{\mu}| \ll 1$.

Thus, we have shown the following theorem.

Theorem 4.1. *Suppose that hypotheses* (H1)–(H4) *are valid and* $w_{12} \neq 0$, $\lambda_1 < \lambda_2$. *Then the following are true.*

(1) *If there is a* $\mu = \bar{\mu}$ *such that* $(M_1(\bar{\mu}), M_2(\bar{\mu})) = (0,0)$, $\operatorname{rank}(\partial(M_1, M_2)/\partial\mu|_{\mu=\bar{\mu}}) = 2$, *then there exists a* $(k-2)$*-dimensional surface* $\Sigma_1 = \Sigma_1(s, \bar{v}_0, \varepsilon)$ *in the neighborhood of* $\bar{\mu}$ *such that* (1.1) *has a unique* 1*-homoclinic orbit near* Γ *as* $\mu \in \Sigma_1(0, \bar{v}_0, \varepsilon)$ *for* $|\varepsilon|$ *small enough and fixed* $|\bar{v}_0| \ll 1$, *and a unique* 1*-periodic orbit near* Γ *as* $\mu \in \Sigma_1(s, \bar{v}_0, \varepsilon)$ *for* $0 < s \ll 1$, $0 < |\varepsilon| \ll 1$ *and fixed* $|\bar{v}_0| \ll 1$. *Moreover, the* 1*-homoclinic orbit and the* 1*-periodic orbit near* Γ *cannot coexist.*

(2) *If there is a* $\mu = \bar{\mu}$ *such that* $M_1(\bar{\mu}) \neq 0$, $M_2(\bar{\mu}) = 0$ *and* $\partial M_2(\bar{\mu})/\partial\mu \neq 0$, *then there exists a* $(k-1)$*-dimensional surface* $\Sigma_2 = \Sigma_2(\bar{v}_0, \varepsilon)$ *near* $\bar{\mu}$ *such that System* (1.1) *has a* 1*-periodic orbit near* Γ *as* $\mu \in \Sigma_2(\bar{v}_0, \varepsilon)$ *for* $|\bar{v}_0|$ *and* $|\varepsilon|$ *sufficiently small and* $\varepsilon M_2(\bar{\mu}) > 0$. *Moreover, if* $\partial M_1(\bar{\mu})/\partial\mu \neq 0$, *then the above* 1*-periodic orbit is unique for fixed* $\bar{v}_0$.

(3) *If* $M_2(\bar{\mu}) \neq 0$, *then System* (1.1) *has no* 1*-homoclinic orbit or* 1*-periodic orbit near* Γ *as* $|\varepsilon| > 0$ *and* $|\mu - \bar{\mu}|$ *sufficiently small.*

§5. Resonant Homoclinic Bifurcations

At last, we consider the homoclinic bifurcation with resonant eigenvalues $\lambda_1 = \lambda_2 :=\lambda$. This is one kind of bifurcation with codimension 3. For conciseness, we may assume

$$\lambda_1(\varepsilon) \equiv \lambda, \qquad \lambda_2(\varepsilon) = \lambda + \varepsilon\lambda, \quad 0 < \varepsilon \ll 1. \tag{5.1}$$

In this case, the successor function has the following form

$$\begin{aligned}
G_1 &= \delta(w_{12}^{-1}s^{1+\varepsilon} - s) + \tilde{w}_{21}s^{\lambda_3(\varepsilon)/\lambda}\bar{u}_1 + \varepsilon M_1(\mu) + \text{h.o.t.},\\
G_2 &= -w_{24}^{-1}w_{14}w_{12}^{-1}\delta s^{1+\varepsilon} - s^{\lambda_3(\varepsilon)/\lambda}\bar{u}_1 + \varepsilon M_2(\mu) + \text{h.o.t.},\\
G_4 &= -a_3w_{12}^{-1}\delta s^{1+\varepsilon} + \bar{u}_1 - w_{44}^{-1}\bar{v}_0 + \varepsilon M_4(\mu) + \text{h.o.t.},\\
G_5 &= (-a_5w_{12}^{-1}\delta s^{1+\varepsilon} + u_1 - w_{25}w_{24}^{-1}s^{\lambda_4(\varepsilon)/\lambda}\bar{v}_0 + \varepsilon M_5(\mu) + \text{h.o.t.},\\
G_6 &= -v_0 + w_{16}\delta s + (w_{26} - w_{16}\tilde{w}_{21})s^{\lambda_3(\varepsilon)/\lambda}\bar{u}_1 + \varepsilon M_6(\mu) + \text{h.o.t.}
\end{aligned} \tag{5.2}$$

As before, the equations $G_4 = 0$, $G_5 = 0$, $G_6 = 0$ always have solution $\bar{u}_1 = \bar{u}_1(\varepsilon, \mu, s, \bar{v}_0)$, $u_1 = u_1(\varepsilon, \mu, s, \bar{v}_0)$, $v_0 = v_0(\varepsilon, \mu, s, \bar{v}_0)$ for s, ε and $|\bar{v}_o|$ sufficiently small. Substituting it into $G_1 = 0$, $G_2 = 0$, we have

$$s^{1+\varepsilon} = w_{12}[s - \delta^{-1}\varepsilon M_1(\mu)] + \text{h.o.t.}, \tag{5.3}$$

$$w_{14}s^{1+\varepsilon} = w_{24}w_{12}\delta^{-1}\varepsilon M_2(\mu) + \text{h.o.t.} \tag{5.4}$$

Set

$$N(s) = s^{1+\varepsilon}, \qquad L(s) = w_{12}[s - \delta^{-1}\varepsilon M_1(\mu))] + \text{h.o.t.} \tag{5.5}$$

Proposition 5.1. *Suppose that* (H1)–(H4) *and* (5.1) *hold,* $w_{12} \neq 0$. *Then* $L(s)$ *is tangent to* $N(s)$ *if and only if* $M_1(\mu) = \beta(\varepsilon,\ \bar{v}_o)$ *for* $\Delta = 1$, $0 < s \ll 1$, $0 < \varepsilon \ll 1$, $0 < w_{12} < 1$, *where*

$$\beta(\varepsilon,\ \bar{v}_o) = \delta(1+\varepsilon)^{-1-1/\varepsilon}(w_{12})^{1/\varepsilon} + \text{h.o.t.} \tag{5.6}$$

and $\partial\beta(\varepsilon,\ \bar{v}_o)/\partial\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. Clearly, the necessary and sufficient conditions for $L(s)$ to be tangent to $N(s)$ at some point $s_0 (0 < s_0 \ll 1)$ are $L(s_0) = N(s_0)$ and $L'(s_0) = N'(s_0)$, that is ,

$$s_0^{1+\varepsilon} = w_{12}[s_0 - \varepsilon\delta^{-1}M_1(\mu)] + \text{h.o.t.}, \quad (1+\varepsilon)s_0^{\varepsilon} = w_{12}.$$

By some calculation, we get

$$s_0 = (1+\varepsilon)\delta^{-1}M_1(\mu) + \text{h.o.t.}, \quad M_1(\mu) = \delta(1+\varepsilon)^{-1-1/\varepsilon}(w_{12})^{1/\varepsilon} + \text{h.o.t.},$$
$$\partial\beta(\varepsilon, \bar{v}_o)/\partial\varepsilon = \delta(1+\varepsilon)^{-1-1/\varepsilon}(w_{12})^{1/\varepsilon}\Big(-\frac{1}{2} - \varepsilon^{-2}\ln w_{12}\Big) + \text{h.o.t.} \tag{5.7}$$

Then, it is easy to see that the proposition is valid.

Denote $M(s, \bar{v}_0, \varepsilon) := -\frac{1}{\varepsilon}\delta(w_{12}^{-1}s^{1+\varepsilon} - s) + \text{h.o.t.}$ defined by (5.3), and $M_* := M(0, \bar{v}_0, \varepsilon)$, $M^* := \beta(\varepsilon, \ \bar{v}_o)$.

Theorem 5.1. *Suppose that* (H1)–(H4) *and* (5.1) *hold, and* $\Delta = 1$, $\varepsilon > 0$, $0 < w_{12} < 1$. *Then the following conclusions are true:*

(1) *If there is a* $\mu = \bar{\mu}$ *such that* $(M_1(\bar{\mu}), M_2(\bar{\mu})) = (0, 0)$, $\text{rank}(\partial(M_1, M_2)/\partial\mu|_{\mu=\bar{\mu}}) = 2$, *then, in the neighborhood of* $\bar{\mu}$, *there is a* $(k-1)$*-dimensional surface* Σ_0 *and two* $(k-2)$*-dimensional surfaces* $\Sigma_1 = \Sigma_1(\bar{v}_0, \varepsilon) \subset \Sigma_0$ *and* $\Sigma_2 = \Sigma_2(\bar{v}_0, \varepsilon) \subset \Sigma_0$ *such that, for fixed* $|v_0|$ *small enough,*

(i) *System* (1.1) *has a unique and* 2*-fold* 1*-periodic orbit near* Γ *if and only if* $\mu \in \Sigma_1$ (*corresponding to* $M_1(\mu) = M^*$);

(ii) *System* (1.1) *has no* 1*-homoclinic orbit or* 1*-periodic orbit near* Γ *if and only if* $\mu \bar{\in} \Sigma_0$ *or* $\mu \in \Sigma_0$ *is situated in the region corresponding to* $M_1(\mu) > M^*$*;*

(iii) *System* (1.1) *has exactly two* 1*-periodic orbits near* Γ *if and only if* $\mu \in \Sigma_0$ *is situated in the region bounded by* Σ_1 *and* Σ_2 (*corresponding to* $M_* < M_1(\mu) < M^*$);

(iv) *System* (1.1) *has exactly one* 1*-homoclinic orbit and one* 1*-periodic orbit near* Γ *if and only if* $\mu \in \Sigma_2$ (*corresponding to* $M_1(\mu) = M_*$);

(v) *System* (1.1) *has exactly one* 1*-periodic orbit near* Γ *if and only if* $\mu \in \Sigma_0$ *is situated in the region corresponding to* $-1 \ll M_1(\mu) < M_*$.

(2) *If there is a* $\mu = \bar{\mu}$ *such that* $M_1(\bar{\mu}) \neq 0$, $M_2(\bar{\mu}) = 0$ *and* $w_{14} \neq 0$, *then System* (1.1) *has no* 1*-homoclinic orbit or* 1*-periodic orbit near* Γ.

(3) *If there is a* $\mu = \bar{\mu}$ *such that* $M_1(\bar{\mu}) = 0$, $M_2(\bar{\mu}) \neq 0$, *then System* (1.1) *has no* 1*-homoclinic orbit or* 1*-periodic orbit near* Γ.

Proof. (1) Consider Equation (5.4). Since $s \to 0$ and $\bar{v}_0 \to 0$ as $\varepsilon \to 0$, and $\partial s/\partial\varepsilon$ and $\partial\bar{v}_0/\partial\varepsilon$ exist for $0 \le \varepsilon \ll 1$, we can set $s \to \varepsilon s$, $v_0 \to \varepsilon v_0$, so that (5.4) reads as

$$M_2(\mu) = w_{24}^{-1}w_{14}w_{12}^{-1}\delta\varepsilon^{\varepsilon}s^{1+\varepsilon} + \text{h.o.t.} \tag{5.8}$$

Applying the implicit function theorem at $(\mu, \varepsilon, s, \bar{v}_0) = (\bar{\mu}, 0, 0, 0)$, we see there exists a $(k-1)$-dimensional surface $\Sigma_0 = \Sigma_0(s, \bar{v}_0, \varepsilon)$ near $\bar{\mu}$ for $(s, \bar{v}_0, \varepsilon)$ near $(0, 0, 0)$ such that (5.4) becomes an identity as $\mu \in \Sigma_0$. Now we solve (5.3) for $\mu \in \Sigma_0$. It follows from Proposition 5.1 and its proof that (5.3) has a 2-fold small solution $s_1 = s_2 = s_o > 0$ if and only if $M_1(\mu) = M^*$, where $\mu \in \Sigma_0(\varepsilon^{-1}s, \varepsilon^{-1}\bar{v}_0, \varepsilon)$ and $s = s_0$ is given by (5.7). By the implicit function theorem, it defines a $(k-2)$-dimensional surface $\Sigma_1 = \Sigma_1(\bar{v}_0, \varepsilon)$.

For $\mu \in \Sigma_0(\varepsilon^{-1}s, \varepsilon^{-1}\bar{v}_0, \varepsilon)$, it follows from $\partial L(s)/\partial M_1 < 0$ that the following are true.

(a) If $M_1(\mu) > M^*$, then (5.3) has no small solution.

(b) If $M_* < M_1(\mu) < M^*$, then (5.3) has exactly two nonnegative small solutions $s_1 > 0$ and $s_2 > 0$.

(c) If $M_1(\mu) = M_*$, then (5.3) has exactly two nonnegative small solutions $s_1 = 0$ and $s_2 > 0$. The equation $M_1(\mu) = M_*$ defines a $(k-2)$-dimensional surface $\Sigma_2 = \Sigma_2(\bar{v}_0, \varepsilon)$.

(d) If $-1 << M_1(\mu) < M_*$, then (5.3) has a unique nonnegative small solution $s_1 > 0$.

(2) Due to (5.4) and $w_{14} \neq 0$, we have $s^{1+\varepsilon} = O(\varepsilon|\mu - \bar{\mu}|)$. Substituting it into (5.3), we get $M_1(\mu) = O(|\mu - \bar{\mu}|)$, which means $M_1(\bar{\mu}) = 0$, a contradiction to the hypothesis $M_1(\bar{\mu}) \neq 0$.

(3) The proof is similar to that of (2). The proof is complete.

Remark 5.1. We call $\Sigma_1(\bar{v}_0, \varepsilon)$ the 2-fold periodic orbit bifurcation surface, and $\Sigma_2(\bar{v}_0, \varepsilon)$ the homoclinic bifurcation surface.

Remark 5.2. If $w_{12} > 1$, we can consider the case $0 < -\varepsilon \ll 1$ in a similar way and obtain a similar result.

Remark 5.3. If $\Delta = -1$, then we can consider the 2-homoclinic and the 2-periodic orbit bifurcation near Γ.

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