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DOUBLE Φ-FUNCTION INEQUALITY FOR NONNEGATIVE SUBMARTINGALES**

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Abstract

The authors establish a kind of inequalities for nonnegative submartingales which depend on two functions Φ and Ψ, and obtain the equivalent conditions for Φ and Ψ such that this kind of inequalities holds. In the case $\Phi = \Psi \in \Delta_2$, it is proved that this necessary and sufficient condition is equivalent to $q_\Phi > 1$.

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§1. Introduction

Let Φ be a nonnegative nondecreasing continuous function on $[0,\infty)$ with $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$, and (Ω, Σ, μ) be a complete probability space. We denote by $\mathcal{M}$ the set of all Σ-measurable functions and $L^\Phi(\Omega) = \{f \in \mathcal{M},\ \exists\epsilon > 0,\ E\Phi(\epsilon|f|) < \infty\}$, where E stands for the expectation with respect to μ, $L^\Phi(\Omega)$ is called an Orlicz space. In fact, it is a kind of space more extensive than classical Orlicz space. When Φ is convex, we define the norm on it by $\| f \|_\Phi = \inf\{k > 0, E\Phi(\frac{|f|}{k}) \le 1\}$. Let $\Sigma_n (n \ge 1)$ be a nondecreasing sequence of complete sub-σ-fields with $\Sigma = \bigvee\limits_{n=0}^{\infty} \Sigma_n$ and define martingale or submartingale $f = (f_n)_{n\ge0}$ as usual. Denote the maximal function of f by $f^*(\omega) = \sup\limits_{n\ge0} |f_n(\omega)|$. As well known in martingale theory, when Φ is a strictly convex function on $[0,\infty)$, i.e. $q_\Phi = \inf\limits_{t\ge0} \frac{t\varphi(t)}{\Phi(t)} > 1$ (where φ is the right continuous derivative of Φ), the following inequalities hold: $E\Phi(f^*) \le \sup\limits_{n\ge0} E\Phi(cf_n)$, for every nonnegative submartingale $f = (f_n)_{n\ge0}$, where c is a constant only depending on Φ. When Φ is not strictly convex, the situation is very different. To see this, we only need to recall Doob′s inequality in the case $p = 1$,

$$Ef^* \le \frac{e}{e-1}\Big(1 + \sup_{n\ge0} E\,|f_n| \log^+ |f_n|\Big).$$

That is to say, f^* is in L^1 when $f \in L\log^+ L$. This inspect inspires us to consider maximal function inequalities related to two functions Φ and Ψ.

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Suppose Φ, Ψ are two nonnegative nondecreasing continuous functions defined on $[0,\infty)$ with $\Phi(0) = \Psi(0) = 0$, and Ψ is convex , $\lim\limits_{t\to\infty} \Psi(t) = \infty$. Throughout this paper, φ, ψ always stand for the right continuous derivative of Φ and Ψ, respectively. We shall condsider the following condition about φ and ψ,

$$\int_0^t \frac{\varphi(s)}{s} ds \le c_1\psi(c_1 t), \quad \forall t > 0. \tag{1.1}$$

We will prove that (1.1) is equivalent to any one of the following conditions:

(i) There exists $c_2 > 0$ such that $E\Phi(f^*) \le \sup\limits_{n\ge 0} E\Psi(c_2 f_n)$ for every nonnegative sub-martingale $f = (f_n)_{n\ge 0}$.

(ii) There exists $c_1 > 0$ such that $E\Phi\Big(\frac{f}{\alpha}\Big) \le c_1 Eg\psi\Big(\frac{c_1 f}{\beta}\Big)$ for every nonnegative function pair f , g which satisfies

$$\lambda|\{f > \alpha\lambda\}| \le \int_{\{f>\beta\lambda\}} g d\mu, \quad \forall \lambda > 0, \tag{1.2}$$

where $0 < \alpha, \beta < \infty$. Moreover, in the case $\Phi = \Psi \in \Delta_2$ (i.e. there exists a positive constant c such that $\Phi(2t) \le c\Phi(t)$ for all $t > 0$), we proved that Condition (1.1) is equivent to $q_\Phi > 1$.

§2. The Maximal Inequalities

In this paper, $|A|$ means the measure of A with respect to μ.

Lemma 2.1.[1] *Let Ψ be the function mentioned above and $f \in L^\Psi(\Omega) \cup L^1(\Omega)$. Then*

$$\int_{\{|f|>\frac{t}{2}\}} |f(\omega)| d\mu = \frac{t}{2}\Big|\Big\{|f| > \frac{t}{2}\Big\}\Big| + \int_{\frac{t}{2}}^\infty |\{|f| > \lambda\}| d\lambda, \quad \forall t > 0. \tag{2.1}$$

Lemma 2.2. *Let $f = (f_n)_{n\ge 0}$ be a nonnegative submartingale. Then*

$$|\{f_n^* > t\}| \le \frac{2}{t}\int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}| d\lambda, \quad \forall t > 0,\ n \in N, \tag{2.2}$$

$$|\{f_n^* > t\}| \le \frac{2}{t}\int_{\{|f_n|>\frac{t}{2}\}} |f_n(\omega)| d\mu, \quad \forall t > 0,\ n \in N. \tag{2.3}$$

Proof. Here we only prove (2.2), and (2.3) can be obtained easily from (2.1) and (2.2). For $t > 0$, let $g_n = f_n \wedge \frac{t}{2}$, $h_n = f_n - g_n$. Then $f_n = g_n + h_n$ and $g_n^* \le \frac{t}{2}$, $h_n = (f_n - \frac{t}{2}) \vee 0$ for every $n \in N$. It is easy to see that $h = (h_n)_{n\ge 0}$ is a nonnegative sub-martingale. Applying Kolmogorof's inequality, we have

$$\begin{aligned} |f_n^* > t| &\le \Big|\Big\{g_n^* > \frac{t}{2}\Big\}\Big| + \Big|\Big\{h_n^* > \frac{t}{2}\Big\}\Big| \le \frac{2}{t}\int_\Omega h_n d\mu = \frac{2}{t}\int_{f_n>\frac{t}{2}} \Big(f_n - \frac{t}{2}\Big) d\mu \\ &= \frac{2}{t}\int_{\{f_n>\frac{t}{2}\}} f_n d\mu - \Big|\Big\{f_n > \frac{t}{2}\Big\}\Big|. \end{aligned}$$

Hence (2.2) follows from (2.1).

Theorem 2.1. *Suppose Φ and Ψ are the functions mentioned above. Then* (1.1) *is equivalent to any one of the following statements:*

(i) *There exists $c_2 > 0$ such that*

$$E\Phi(f^*) \le \sup_n E\Psi(c_2 f_n) \tag{2.4}$$

for every nonnegative submartingale $f = (f_n)_{n\ge 0}$.

(ii) *There exists* $c_1 > 0$ *such that*

$$E\Phi\Big(\frac{f}{\alpha}\Big) \leq c_1 Eg\psi\Big(\frac{c_1 f}{\beta}\Big) \tag{2.5}$$

for every nonnegative function pair f , g *satisfying* (1.2).

Proof. (i) To prove (1.1) $\Longrightarrow$ (2.4), notice that $(2f_n)_{n\geq 0}$ is a nonnegative submartingale and replace t, f in (2.2) by $2t, 2f$, respectively. Then integrate its both sides with respect to $d\Phi(\lambda)$, and we get

$$\begin{aligned} E\Phi\left(f_n^*\right) &= \int_0^\infty |\{2f_n^* > 2t\}| d\Phi(t) \leq \int_0^\infty \frac{1}{t}\int_t^\infty |\{2f_n > \lambda\}| d\lambda d\Phi(t) \\ &= \int_0^\infty |\{2f_n > \lambda\}| \int_0^\lambda \frac{1}{t} d\Phi(t) d\lambda \leq \int_0^\infty |\{2f_n > \lambda\}| c_1\psi(c_1\lambda) d\lambda = E\Psi(2c_1 f_n). \end{aligned}$$

Letting $n \to \infty$,we get (2.4).

Next we prove (2.4)$\Longrightarrow$ (1.1). Consider the following dyadic martingale on $(0, 1]$: Let $A_k = (1 - \frac{1}{2^k}, 1]$, $\mathcal{F}_k = \sigma\{A_1, A_2, \cdots, A_k\}$, $f = t\chi_{A_n}$,where $t > 0, k, n \in N$, and denote $f = (E(f \mid \mathcal{F}_k))_{k\geq 0}$. Then f is a finite martingale with $f_k = f$ $(k \geq n)$.

It is clear that $|f_k| \leq t$ $(\forall k \geq 0)$, thus $f^* \leq t$. By the convexity of Ψ, we get

$$E\Psi\left(c_2 f\right) = EE(\Psi(c_2 f)|\mathcal{F}_k) \geq E\Psi(E(c_2 f|\mathcal{F}_k)) = E\Psi\left(c_2 f_k\right), \quad k \geq 0$$

and then $\sup E\Psi\left(c_2 f_m\right) = E\Psi\left(c_2 f\right)$. From (2.4) we have $E\Phi\left(f^*\right) \leq E\Psi\left(c_2 f\right)$, i.e.

$$\int_0^\infty |\{f^* > s\}|\varphi(s) ds \leq \int_0^\infty |\{c_2 f > s\}|\psi(s) ds.$$

Notice that when $s \in (\frac{t}{2^{n-k}}, \frac{t}{2^{n-k-1}})$ $(0 \leq k \leq n-1)$, we have $|\{f^* > s\}| = \frac{1}{2^{k+1}}$ and $\frac{1}{s} < \frac{2^{n-k}}{t}$, so $\frac{1}{s} < \frac{2^{n+1}}{t} \cdot \frac{1}{2^{k+1}} = \frac{2^{n+1}}{t} \cdot |\{f^* > s\}|$, $\forall s \in (\frac{t}{2^n}, t)$. Therefore

$$\int_{\frac{t}{2^n}}^t \frac{\varphi(s)}{s} ds \leq \frac{2^{n+1}}{t}\int_{\frac{t}{2^n}}^t |\{f^* > s\}|\varphi(s) ds \leq \frac{2^{n+1}}{t}\int_0^\infty |\{c_2 f > s\}|\psi(s) ds.$$

Now from $f \leq t$ we get

$$\int_{\frac{t}{2^n}}^t \frac{\varphi(s)}{s} ds \leq \frac{2^{n+1}}{t}\int_0^{c_2 t} |\{c_2 f > s\}|\psi(s) ds \leq \frac{2^{n+1}}{t}\int_0^{c_2 t} \frac{1}{2^n}\psi(s) ds \leq 2c_2\psi(c_2 t).$$

Letting $n \to \infty$, we obtain $\int_0^t \frac{\varphi(s)}{s} ds \leq 2c_2\psi(c_2 t)$, $\forall t > 0$. This proves (1.1).

(ii) To prove (1.1)$\Longrightarrow$ (2.5), we integrate both sides of (1.2) with respect to $d\Phi(\lambda)$, and get

$$\begin{aligned} E\Phi\Big(\frac{f}{\alpha}\Big) &= \int_0^\infty |\{f > \alpha\lambda\}|\, d\Phi(\lambda) \leq \int_0^\infty \frac{1}{\lambda}\int_{\{f>\beta\lambda\}} g d\mu\; d\Phi(\lambda) \\ &= \int_\Omega g\int_0^{\frac{f}{\beta}} \frac{d\Phi(\lambda)}{\lambda}\; d\mu \leq \int_\Omega g\psi\Big(c_1\frac{f}{\beta}\Big)\; d\mu. \end{aligned}$$

This is (2.5).

To prove (2.5)$\Longrightarrow$ (1.1), let $f = (f_n)_{n\geq 0}$ be the dyadic finite martingale as in the proof of (i). Then the nonnegative function pair f^*, f_n satisfies (1.2) with $\alpha = 1, \beta = 1$. Hence (2.5) holds, that is to say $E\Phi\left(f_n^*\right) \leq c_1 Ef_n\psi(c_1 f_n^*)$. From this and the proof of (i) we get

$$\begin{aligned} \int_{\frac{t}{2^n}}^t \frac{\varphi(s)}{s} ds &\leq \frac{2^{n+1}}{t} E\Phi(f_n^*) \leq \frac{2^{n+1}}{t} c_1 Ef_n\psi(c_1 f_n^*) \leq \frac{2^{n+1}}{t} c_1\psi(c_1 t) Ef_n \\ &= \frac{2^{n+1}}{t} c_1\psi(c_1 t)\frac{t}{2^n} = 2c_1\psi(c_1 t) \leq c\psi(ct). \end{aligned}$$

Letting $n \to \infty$, we have $\int_0^t \frac{\varphi(s)}{s} ds \le c\psi(ct)$. Then Theorem 2.1 follows.

Theorem 2.2. *Suppose* Φ *and* Ψ *are the functions mentioned above. Then the condition that* φ, ψ *satisfy*

$$\int_0^{s_0} \frac{\varphi(s)}{s} ds \ = L < \infty \text{ and } \int_{s_0}^t \frac{\varphi(s)}{s} ds \le c_1\psi(c_1 t) \tag{2.6}$$

for some $s_0, c_1 > 0$, *is equivalent to any one of the following statements*:

(i) *There exist* $c, c_2 > 0$ *such that*

$$E\Phi(f^*) \le \sup_n [cEf_n + E\Psi\left(c_2 f_n\right)] \tag{2.7}$$

for every nonnegative submartingale $f = (f_n)_{n\ge 0}$.

(ii) *There exist constants* $c, c_1 > 0$ *such that*

$$E\Phi\Big(\frac{f}{\alpha}\Big) \le cEg + c_1 Eg\psi\Big(\frac{c_1 f}{\beta}\Big) \tag{2.8}$$

for every pair of nonnegative function f, g *satisfying* (1.2).

Proof. (i) To prove (2.6) $\Longrightarrow$ (2.7), from Fubini theorem, we get

$$\begin{aligned} E\Phi\left(f_n^*\right) &= \int_{s_0}^\infty |\{2f_n^* > 2t\}| d\Phi(t) + \int_0^{s_0} |\{2f_n^* > 2t\}| d\Phi(t) \\ &\le \int_{s_0}^\infty \frac{1}{t} \int_t^\infty |\{2f_n > \lambda\}| d\lambda d\Phi(t) + \int_0^{s_0} \frac{1}{t} \int_t^\infty |\{2f_n > \lambda\}| d\lambda d\Phi(t) \\ &\le \int_{s_0}^\infty |\{2f_n > \lambda\}| \int_{s_0}^\lambda \frac{1}{t} d\Phi(t) d\lambda + \int_0^{s_0} |\{2f_n > \lambda\}| \int_0^\lambda \frac{1}{t} d\Phi(t) d\lambda \\ &\le \int_0^\infty |\{2f_n > \lambda\}| c_1\psi(c_1\lambda) d\lambda + L \int_0^{s_0} |\{2f_n > \lambda\}| d\lambda \\ &\le E\Psi\left(2c_1 f_n\right) + 2LEf_n. \end{aligned}$$

Let $n \to \infty$, then (2.7) follows.

To prove (2.7) $\Longrightarrow$ (2.6), by the discussion similar to the proof of Theorem 2.1, we can get $\lim\limits_{n\to\infty} \int_{\frac{t}{2^n}}^t \frac{\varphi(s)}{s} ds \le \frac{2^{n+1}}{t} E\Phi(f_n^*) \le 2c + 2c_2\psi(2c_2 t)$. Denote $s_0 = 1 + \inf\{t > 0, \psi(2c_2 t) > 0\}$. Then $\forall t > s_0$,

$$\begin{aligned} \int_{s_0}^t \frac{\varphi(s)}{s} ds &\le 2c + 2c_2\psi(2c_2 t) = \frac{2c}{\psi(2c_2 s_0)} \psi(2c_2 s_0) + 2c_2\psi(2c_2 t) \\ &\le \Big(\frac{2c}{\psi(2c_2 s_0)} + 2c_2\Big)\psi(2c_2 t) \le c_1\psi(c_1 t), \\ \int_0^{s_0} \frac{\varphi(s)}{s} ds &= \lim_{n\to\infty} \int_{\frac{s_0}{2^n}}^{s_0} \frac{\varphi(s)}{s} ds \le 2c + 2c_2\psi(2c_2 s_0), \end{aligned}$$

which is desired.

(ii) By an argument similar to (ii) of Theorem 2.1, we get that (2.6) is equivalent to (2.8).

Eventually we get the following result.

Corollary 2.1. *If the functions* Φ *and* Ψ *are as above, then the condition* (2.6) *is equivalent to the statement that there exists* $c_\delta > 0$ *for any* $\delta > 0$, *such that* $E\Phi(f^*) \le \sup\limits_n [\delta Ef_n + E\Psi\left(c_\delta f_n\right)]$ *for every nonnegative submartingale* $f = (f_n)_{n\ge 0}$.

§3. The Discussion About Condition (1.1) and Some Examples

In the case $\Phi = \Psi$, Theorems 2.1 and 2.2 become the classical maximal function inequalities for nonnegative submartingale. The following theorem gives an exact result under

$\Phi \in \Delta_2$.

Theorem 3.1. *Let $\Phi \in \Delta_2$ be a nondecreasing convex function on $[0,\infty)$. Then $q_\Phi > 1$ if and only if there exists $c_1 > 0$, such that $\int_0^t \frac{\varphi(s)}{s}ds \le c_1\varphi(c_1 t)$, $\forall t > 0$.*

Proof. (i) We first prove the necessity. Notice that the condition $\int_0^t \frac{\varphi(s)}{s}ds \le c_1\varphi(c_1 t)$ implies $\varphi(s) \downarrow 0$ (as $s \to 0$) and $\int_0^t \frac{\varphi(s)}{s}ds < \infty$, and from $\Phi(t) = \int_0^t \varphi(s)\,ds \le t\varphi(t)$ we get $\inf\limits_{t>0} \frac{t\varphi(t)}{\Phi(t)} \ge 1$. Denote

$$a_k = 2^k\varphi(2^k) - \Phi(2^k) - [2^{k-1}\varphi(2^{k-1}) - \Phi(2^{k-1})] \quad (-\infty < k < +\infty). \tag{3.1}$$

Then

$$2^{k-1}[\varphi(2^k) - \varphi(2^{k-1})] \le a_k \le 2^k[\varphi(2^k) - \varphi(2^{k-1}), \tag{3.2}$$

$$\sum_{k=-\infty}^{m} 2^{-k}a_k \le \varphi(2^m) - \varphi(0) \le \sum_{k=-\infty}^{m} 2^{-k+1}a_k. \tag{3.3}$$

Therefore

$$\begin{aligned}\Phi(2^m) &= \int_0^{2^m}\varphi(s)ds = \sum_{k=-\infty}^{m-1}\int_{2^k}^{2^{k+1}}\varphi(s)ds \le \sum_{k=-\infty}^{m-1}2^k\varphi(2^{k+1}) \\ &\le \sum_{k=-\infty}^{m-1}2^k\sum_{i=-\infty}^{k+1}2^{-i+1}a_i \le \sum_{i=-\infty}^{m}2^{m-i+1}a_i.\end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned}\int_0^{2^m}\frac{\varphi(s)}{s}ds &= \sum_{k=-\infty}^{m-1}\int_{2^k}^{2^{k+1}}\frac{\varphi(s)}{s}ds \ge \sum_{k=-\infty}^{m-1}2^k\frac{\varphi(2^k)}{2^{k+1}} \\ &\ge \frac{1}{2}\sum_{k=-\infty}^{m-1}\sum_{i=-\infty}^{k}2^{-i}a_i = \frac{1}{2}\sum_{i=-\infty}^{m-1}2^{-i}(m-i)a_i.\end{aligned} \tag{3.5}$$

Notice that $\inf\limits_{t>0}\frac{t\varphi(t)-\Phi(t)}{\Phi\left(\frac{t}{2}\right)} = 0$ if $\inf\limits_{t>0}\frac{t\varphi(t)}{\Phi(t)} = 1$, and (3.1), (3.4) imply the fact: $\forall j > 1$, $\exists t_j \in (2^{n_j}, 2^{n_j+1}]$ such that

$$\Big(\sum_{k=-\infty}^{n_j}2^{n_j-k+1}a_k\Big)^{-1}\sum_{k=-\infty}^{n_j}a_k \le \frac{2^{n_j}\varphi(2^{n_j}) - \Phi(2^{n_j})}{\Phi(2^{n_j})} \le \frac{t_j\varphi(t_j) - \Phi(t_j)}{\Phi\left(\frac{t_j}{2}\right)} \le \frac{1}{2^j}.$$

Then for $k_0 = n_j - j + 1$,

$$\Big(\sum_{k=k_0+1}^{n_j}2^{n_j-k+1}a_k\Big)^{-1}\sum_{k=-\infty}^{n_j}a_k \ge \frac{1}{2^{j-1}}, \quad \Big(\sum_{k=-\infty}^{k_0}2^{n_j-k+1}a_k\Big)^{-1}\sum_{k=-\infty}^{n_j}a_k \le \frac{1}{2^{j-1}}.$$

Thus $\sum\limits_{k=-\infty}^{n_j}2^{-k}a_k \le 2\sum\limits_{k=-\infty}^{k_0}2^{-k}a_k$, and

$$\frac{\varphi\left(\frac{t_j}{2}\right)}{\int_0^{2t_j}\frac{\varphi(s)}{s}ds} \le \Big(\frac{1}{2}\sum_{k=-\infty}^{n_j}(n_j-k+1)2^{-k}a_k\Big)^{-1}\sum_{k=-\infty}^{n_j}2^{-k+1}a_k \le \frac{8}{(n_j-k_0+1)} = \frac{8}{j}.$$

Since $\Phi \in \Delta_2$, for every $c > 0$, we can find a $c_1 > 0$ such that $\varphi(4ct) \le c_1\varphi(t), \forall t > 0$. Thus for every $c, j > 1$, $\exists t_j > 0$ such that $\frac{\varphi(c2t_j)}{\int_0^{2t_j}\frac{\varphi(s)}{s}ds} \le \frac{8c_1}{j}$, which contradicts $\int_0^t \frac{\varphi(s)}{s}ds < c\varphi(ct)$. The necessity follows.

(ii) To prove sufficiency, using the same method in (i) we get

$$\sum_{k=-\infty}^{m} 2^{-k}a_k \le \varphi(2^m) - \varphi(0) \le \sum_{k=-\infty}^{m} 2^{-k+1}a_k,$$

$$\Phi(2^m) \ge \sum_{k=-\infty}^{m-1} 2^{m-k-1}a_k,$$

$$\int_0^{2^m} \frac{\varphi(s)}{s}ds \le \frac{1}{2}\sum_{i=-\infty}^{m-1} 2^{-i+1}(m-i+1)a_i,$$

and by using Δ_2 condition the sufficiency follows.

The following examples show that the double Φ function inequalities in this paper is very extensive. Here we write $f(t) \sim g(t)$ if there exist positive constants a, b such that $af(t) \le g(t) \le bf(t)$ for all $t > 0$.

Example 3.1. Suppose $1 < p < \infty$ and $\Phi(t) = \Psi(t) = \frac{1}{p}t^p,\ t > 0$. In this case

$$\psi(t) = \varphi(t) = t^{p-1}, t > 0.$$

Example 3.2.

$$\Phi = \begin{cases} 0, & 0 \le t \le 1, \\ t-1, & t > 1, \end{cases} \qquad \Psi(t) = \begin{cases} 0, & 0 \le t \le 1, \\ t\log t, & t > 1, \end{cases}$$

$$\varphi(t) \le \begin{cases} 0, & 0 \le t \le 1, \\ 1, & t > 1, \end{cases} \qquad \psi(t) = \begin{cases} 0, & 0 \le t \le 1, \\ 1+\log t, & t > 1. \end{cases}$$

Example 3.3.

$$\Phi(t) = \begin{cases} \frac{t^{\frac{3}{2}}}{\sqrt{e}}, & 0 \le t < e, \\ \frac{t}{\log^t}, & t \ge e, \end{cases} \qquad \Psi(t) = \begin{cases} t, & 0 \le t < e, \\ t(1+\log\log t), & t \ge e, \end{cases}$$

$$\varphi \le \begin{cases} \frac{3\sqrt{t}}{2\sqrt{e}}, & 0 \le t < e, \\ \frac{1}{\log t}, & t \ge e, \end{cases} \qquad \psi \ge \begin{cases} 1, & 0 \le t < e, \\ 1+\log\log t, & t \ge e. \end{cases}$$

Example 3.4. Suppose $0 < \epsilon < 1$.

$$\Phi(t) = \begin{cases} \frac{t}{(1-\log t)^{1+\epsilon}}, & 0 < t \le 1, \\ t, & t > 1, \end{cases} \qquad \Psi(t) = \begin{cases} \frac{t}{\epsilon(1-\log t)^{\epsilon}}, & 0 < t \le 1, \\ \frac{t}{\epsilon}(1+\log t) & t > 1, \end{cases}$$

$$\varphi(t) \sim \begin{cases} \frac{1}{(1-\log t)^{1+\epsilon}}, & 0 < t \le 1, \\ 1, & t > 1, \end{cases} \qquad \psi(t) \sim \begin{cases} \frac{1}{\epsilon(1-\log t)^{\epsilon}}, & 0 < t \le 1, \\ \frac{1}{\epsilon}(1+\log t) & t > 1. \end{cases}$$

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