Chin. Ann. of Math.
21B: 2(2000),217-224.

ON THE EXISTENCE OF SOLUTIONS TO A TYPE OF NONLINEAR DIFFERENTIAL-ITERATIVE EQUATIONS***

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Abstract

Under the assumption that $h(z)$ is strictly monotone the existence of solutions to a type of nonlinear differential-iterative equations in the form of $x'(t) = g(x(t)) - h(x(x(t)))$ is discussed according to the behavior of the quasi-isoclinic curve C: $x = h^{-1}(g(t))$

Keywords Differential-iterative equation, Quasi-isoclinic curve, Fixed point
1991 MR Subject Classification 34K15
Chinese Library Classification O175.12, O175.14 **Document Code** A
Article ID 0252-9599(2000)02-0217-08

§1. Introduction

Differential-iterative equation is a kind of special functional differential equations which were discussed recently[1−6]. Most of the researches[1−3] now available focused on the existence of autonomous differential-iterative equation in the form of

$$x'(t) = f(x^{\langle n\rangle}(t)),$$

where $x^{\langle 2\rangle} = x(x(t))$ and $x^{\langle k\rangle} = x(x^{\langle k-1\rangle}(t))$, $k = 3, 4, \cdots, n$, and some in them[4−5] dealt with the equation $x'(t) = (a^2 - x^2(t))f(x^{\langle n\rangle}(t))$. We proposed a transformation theorem in [6] and used it to discuss the existence and behavior of solutions to the equation $x'(t) = ax(t) - bx(x(t))$, where $0 < b < a$. In this paper we study a more general equation of the form

$$x'(t) = g(x(t)) - h(x(x(t))), \tag{1.1}$$

where h is strictly monotone on $\Re$ and $h(\Re) = \Re$. Without loss of generality we suppose h is strictly increasing since we can change it into the case by use of the transformation $y = -x$ and $\tau = -t$ when h is decreasing. Then we have

$$\lim_{z\to\pm\infty} h(z) = \pm\infty.$$

Manuscript received January 27, 1999. Revised December 10, 1999.
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***Project supported by the National Natural Science Foundation of China and the Doctoral Program Foundation of the Ministry of Education of China. Research was completed during the first author's visit to Institute of Applied Mathematics, Academia Sinica.

Definition 1.1. *A function $x(t)$ defined on an interval I is said to be a solution of* (1.1) *on I iff $x(t)$ satisfies* (1.1) *on I and $x(I) \subset I$.*

Definition 1.2. *The sets $\Gamma = \{(t, x) \in \Re^2 : x = h^{-1}(g(t))\}$ and $L = \{(t, x) \in \Re^2 : x = t\}$ are said to be a quasi-isoclinic curve and a base line to Equation* (1.1), *respectively.*

We suppose in this paper

(H) $g, h \in C^0(\Re.\Re)$ are locally Lipschitz, h is strictly increasing and

$$\lim_{z\to\pm\infty} h(z) = \pm\infty,$$

$$\lim_{x\to+\infty} \frac{\max\{0, g(x) - h(x)\}}{x^2} = 0.$$

Let $m = \inf\{t \in \Re : g(t) - h(t) = 0\}$, $M = \sup\{t \in \Re : g(t) - h(t) = 0\}$, where $m = -\infty$ and $M = +\infty$ are included. And set

$$A_0 = \{t \in \Re : g(t) - h(t) = 0\},$$
$$A_+ = \{t \in \Re : g(t) - h(t) > 0\},$$
$$A_- = \{t \in \Re : g(t) - h(t) < 0\}.$$

Obviously under hypothesis (H) A_0 consists of only one point when g is decreasing. It is easy to show (see [4]).

Proposition 1.1. *All the solutions of* (1.1) *are monotone.*

Proposition 1.2. *For $\eta \in A_0$ and $\xi \in \Re, x(t) \equiv \eta$ is a constant solution satisfying $x(\xi) = \eta$.*

Proposition 1.3. *Suppose $f(u, v)$ is continuous and locally Lipschitz with respect to (u, v) on $\Re^2$ and $x = x(t)$ is a solution of $x'(t) = f(x(t), x(x(t)))$ on I. If there is $t_0 \in I$ such that $x'(t_0) = 0$, then $x(t) \equiv x(t_0)$ on I.*

Proof. Let $F(y) = f(y, x(y))$. Then F is locally Lipschitz with respect to y for $y \in I$ since x is differentiable on the interval. The condition $x'(t_0) = 0$ implies $f(x(t_0), x(x(t_0))) = 0$. Consider

$$\begin{cases} y' = f(y, x(y)), \\ y(t_0) = x(t_0). \end{cases} \tag{1.2}$$

Obviously both $y = x(t)$ and $y \equiv x(t_0)$ are solutions of (1.2) on I. The Picard's theorem implies $x(t) \equiv x(t_0)$ on I.

Proposition 1.4. *Every decreasing solution of* (1.1) *has one and only one common point with $L : x = t$.*

§2. Existence and Behavior of Increasing Solutions

Under hypothsis (H) the assumption that g is increasing implies that $h^{-1}(g(t))$ is increasing and the sets $\{(t, x) \in \Gamma : t \in A_+\}, \{(t, x) \in \Gamma : t \in A_-\}$ are above and below the line $x = t$ respectively.

For $\eta \in A_+$, let $u_1 = \sup\{t \in A_0 : t \le \eta\}, u_2 = \inf\{t \in A_0 : t > \eta\}$.

Theorem 2.1. *Suppose* (H) *holds. If $\eta \in A_+, m < \eta < M$, then* (1.1) *has a strictly increasing solution $x(t)$ satisfying $x(\xi) = \eta$ for arbitray $\xi \in \Re$. And*

(i) *$x = x(t)$ intersects Γ at a point $(c, h^{-1}(g(c))), c \in (u_1, u_2)$ and such a solution can be extended to the left to $-\infty$ with $\lim_{t\to-\infty} x(t) = c$ and*

(ii) *$x(t)$ can be extended to the right in one of the three ways.*

(a) *$x = x(t)$ meets Γ at a point $(\sigma_1, h^{-1}(g(\sigma_1))), \sigma_1 > u_2, \sigma_1 > \inf\{t \geq \eta : t \in A_0\}$ and then $x(t)$ can be extended to $+\infty$ with $x(t) < \sigma_1, \lim_{t\to+\infty} x(t) = \sigma_1$;*

(b) *$x = x(t)$ does not meet Γ but has a common point with L at $t_0 > \inf\{t \geq \eta : t \in A_0\}$ with $x'(t) \geq 1$;*

(c) *$x(t)$ has no common point with $L \cup \Gamma$ for $t > \inf\{t > \eta : t \in A_0\}$ and then $x(t)$ can be extended to $+\infty$ with $x(+\infty) = +\infty$.*

Proof. Let $u_1 = \max\{t \leq \eta : t \in A_0\}, u_2 = \min\{t \geq \eta : t \in A_0\}$, and $I = [v_1, v_2]$, where $v_1 = \min\{u_1, \xi\}, v_2 = \max\{u_2, \xi\}$. Set $M = \max_{(x,y)\in[u_1,u_2]} | g(x) - h(y) |> 0$. Take in consideration a set

$$G = \{z \in C^0(I, [u_1, u_2]) : z(\xi) = \eta, 0 \leq z(t_2) - z(t_1) \leq M(t_2 - t_1) \ \text{ for } \ t_1 \leq t_2\}$$

and a map $T : G \to C^0(I, [u_1, u_2])$ defined by

$$(Tz)(t) = \Big[\eta + \int_\xi^t \max\{0, g(z(s)) - h(z(z(s)))\}ds\Big]_{u_1}^{u_2}, \tag{2.1}$$

where $[A]_{u_1}^{u^2}$ stands for $\min\{u_2, \max\{u_1, A\}\}$. Obviously

$$| [A]_{u_1}^{u_2} - [B]_{u_1}^{u_2} |\leq| A - B | .$$

It is easy to show that G is convex and compact and T continuous. Since $(Tz)(\xi) = \eta, u_1 \leq (Tz)(t) \leq u_2$ and for $t_1, t_2 \in I, t_1 \leq t_2$,

$$0 \leq (Tz)(t_2) - (Tz)(t_1) \leq \int_{t_1}^{t_2} \max\{0, g(z(s)) - h(z(z(s)))\}ds \leq M(t_2 - t_1),$$

we have $TG \subset G$. The Schauder's fixed point theorem implies the existence of $x \in G$ such that $x = Tx$. If

$$u_1 < x(t) < u_2, \quad \text{when} \quad v_1 < t < v_2, \tag{2.2}$$

then

$$x(t) = \eta + \int_\xi^t \max\{0, g(z(s)) - h(z(z(s)))\}ds \tag{2.3}$$

holds on I. We show at first $\{t \in I : x(t) = u_2\} = \Phi$.

Suppose the contrary that $\{t \in I : x(t) = u_2\} \neq \Phi$ and let $\inf\{t \in I : x(t) = u_2\} = t_0$. In this case $u_1 < x(\xi) = \eta < u_2$ implies $\xi < t_0 \leq v_2$. Then $v_2 = u_2$ and $t_0 \leq u_2$. Consequently $x(t)$ satisfies (2.2) on $[\xi, t_0]$. When $t \in [t_0, u_2]$, we have $g(x(s)) - h(x(x(t))) = 0$ since $x(t) = u_2$ and $x(x(t)) = u_2$. Then (2.2) holds on $[\xi, u_2]$. That is to say, $x(t)$ satisfies

$$\begin{cases} y' = \max\{0, g(y) - h(x(y))\}, \\ y(t_0) = u_2, \end{cases} \tag{2.4}$$

on $[\xi, u_2]$. At the same time, $y \equiv u_2$ is also a solution to (2.3). It is easy to see that $F(\cdot) = \max\{0, g(\cdot) - h(x(\cdot))\}$ is locally Lipschitz and then we have $x(t) \equiv u_2$ on $[\xi, u_2]$. However, $x(\xi) = \eta \neq u_2$, a contradiction. Therefore $\{t \in I : x(t) = u_2\} = \Phi$. Similarly $\{t \in I : x(t) = u_1\} = \Phi$. Hence

$$u_1 < x(t) < u_2, \quad t \in I. \tag{2.5}$$

It follows from (2.2) that

$$x'(t) = \max\{0, g(x(t)) - h(x(x(t)))\}. \tag{2.6}$$

We show $x'(t) > 0$ for $t \in I$. Otherwise there is a $t_0 \in I$ such that $x'(t_0) = 0$.

Because $f(u,v) = \max\{0, g(u) - h(v)\}$ is locally Lipschitz, Proposition 1.3 implies $x(t) \equiv x(t_0)$ and then $\eta = x(\xi) = x(t_0)$. It follows that

$$g(x(\xi)) - h(x(x(\xi))) = g(\eta) - h(\eta) \leq 0.$$

However $\eta \in A_+$ implies $g(\eta) - h(\eta) > 0$, a contradiction.

Obviously $x'(t) > 0$ means $g(x(t)) - h(x(x(t))) > 0$ and then (2.6) becomes

$$x'(t) = g(x(t)) - h(x(x(t))), \qquad t \in I.$$

That is to say, $x(t)$ is a solution of (1.1) on I with $x(\xi) = \eta$.

Finally we consider the continuation of $x(t)$.

It follows from that $x(v_2) < u_2 \leq v_2$. Consider

$$\begin{cases} y' = g(y) - h(x(y)), \\ y(v_2) = x(v_2), \qquad\qquad x(v_2) \leq y \leq v_2. \end{cases} \tag{2.7}$$

The solution $y(t)$ of (2.7) is uniquely determined by

$$\int_x^y \frac{ds}{g(s) - h(x(s))} = t - \xi. \tag{2.8}$$

(a) If there is a $\sigma \in [x(v_2), v_2]$ such that $g(\sigma) - h(x(\sigma)) = 0$, then we claim that

$$\lim_{y \to \sigma} \int_{x(v_2)}^y \frac{ds}{g(s) - h(x(s))} = +\infty.$$

Otherwise suppose the limit is $\tau < +\infty$. Then $y(\tau + \xi) = \sigma$,

$$y'(\tau + \xi) = g(\sigma) - h(x(\sigma)) = 0.$$

Proposition 1.3 implies $y(t) \equiv \sigma$ for $t \in [v_2, v_2 + \tau]$, a contradiction.

Therefore y is defined on $[v_2, +\infty)$ with $y(v_2) = u_2$ and $\lim_{t \to \infty} y(t) = \sigma$. If $g(y) - h(x(y)) > 0$ for $y \in (u_2, v_2]$, then

$$\lim_{y \to v_2} \int_\xi^y \frac{ds}{g(s) - h(x(s))} = d_1 = t - \xi,$$

and $v_2 = y(v_4)$ where $v_4 := v_2 + d_1 > v_2$. So $y(t)$ is defined on $[v_2, v_4]$ and $y(v_4) = v_2 < v_4$. Let

$$\tilde{x} = \begin{cases} x(t), & t \in [v_1, v_2], \\ y(t), & t \in [v_2, v_4], \end{cases}$$

and $x(t)$ stands for $\tilde{x}(t)$ on $[v_1, v_4]$. Continue the above process with v_2 replaced consequently by $v_4, v_6, \cdots$.

Suppose there is $\sigma \in (v_{2k}, v_{2k+2}]$ such that $g(\sigma) - h(x(\sigma)) = 0$. Then we can extend $x(t)$ to $+\infty$ with $\lim_{t \to +\infty} x(t) = \sigma$. Otherwise we have an infinite series $\{v_{2k}\}, k = 1, 2, \cdots$, such that $g(t) - h(x(t)) \neq 0$ for $t \in \{v_{2k}, v_{2k+2}\}$.

(b) When $\{v_{2k}\}$ is bounded and $\lim_{k \to +\infty} v_{2k} = t_0$, we claim that $x(t_0) = t_0$ since otherwise $x(t)$ can be further extended to the right. It follows from $x(t) < t, t \in [\xi, t_0)$, that $x'(t_0) \geq 1$.

(c) If $\{v_{2k}\}$ is unbounded, then $x(t)$ is defined on $[u_1, +\infty)$. We show that $\lim_{t \to +\infty} x(t) = \infty$. Otherwise suppose $\lim_{t \to +\infty} x(t) = d < \infty$ and hence

$$\lim_{t \to +\infty} x'(t) = \lim_{t \to +\infty} [g(x(t)) - h(x(x(t)))] = g(d) - h(x(d)) = 0.$$

Obviously $d > \eta$. It follows that $x = x(t)$ meets Γ at a point d, a contradiction to the process.

The last part of the proof is to show $x(t)$ can be extended to the left to $-\infty$ with $\lim_{t\to-\infty} x(t) = c > u_1$.

We have shown in (2.5) that $x(u_1) > u_1 = h^{-1}(g(u_1))$ and $x(u_2) < u_2 = h^{-1}(g(u_2))$. It follows that $\{t \in (u_1, u_2) : x(t) = h^{-1}(g(t))\} \neq \Phi$. Set $c = \max\{t \in (u_1, u_2) : x(t) = h^{-1}(g(t))\}$ and $x(c) > c$. Then the unique solution of

$$\begin{cases} y' = g(y) - h(x(y)), \\ y(v_1) = x(v_1), \qquad c < y \le x(c) \end{cases} \tag{2.9}$$

is determined by

$$\int_{x(v_1)}^{y} \frac{ds}{g(s) - h(x(s))} = t - v_1.$$

A similar discussion as above leads to the conclusion that

$$\lim_{y\to C} \int_{x(v_1)}^{y} \frac{ds}{g(s) - h(x(s))} = -\infty,$$

and then $y(t)$ is defined on $(-\infty, v_1)$ and $\lim_{t\to-\infty} y(t) = C$. The proof is completed by setting

$$\tilde{x}(t) = \begin{cases} x(t), & t \in [v_1, v_2], \\ y(t), & t \le v_1 \end{cases}$$

and replacing symbol $\tilde{x}(t)$ by $x(t)$ again.

Lemma 2.1. *Given a differential-iterative equation*

$$x'(t) = f(x(t), x(x(t))), \tag{2.10}$$

where $f \in C^0(\Re^2, \Re)$*, such that for a* $\sigma \in \Re$ *and a small* $\varepsilon > 0$ *satisfying* $f(\sigma, \sigma) = 1, f(t,t) < 0$*, for* $t \in (\sigma, \sigma + \varepsilon)$*,* (2.10) *has an increasing solution* $x(t)$ *on* $I = [\sigma, \sigma + \varepsilon]$ *such that* $x(\sigma) = \sigma, x(t) < t$*, for* $t \in (\sigma, \sigma + \varepsilon]$*.*

Proof. Let $I = [\sigma, \sigma + \varepsilon]$ and $M = \max_{x\in I, \sigma\le y\le x} \{f(x, y)\}$. Take a set

$$G = \{z \in C^0(I, I) : z(\sigma) = \sigma, 0 \le z(t_2) - z(t_1) \le M(t_2 - t_1) \quad \text{for} \quad t_1 \le t_2\}$$

and a map $T : G \to C^0(I, I)$ defined by

$$(Tg)(t) = \left[\sigma + \int_{\sigma}^{t} f(z(s), z(z(s)))ds\right]^t, \qquad t \in I.$$

By a normal discussion shown in Theorem 2.1, we can show that a fixed point x of T in G is the needed solution of (2.10) on I.

Theorem 2.2. *Suppose* (H) *holds. If* $\eta \in A_+, \eta > M$*, then* (1.1) *has a strictly increasing solution* $x(t)$ *satisfying* $x(\xi) = \eta$ *for arbitrary* $\xi \in \Re$*. Such a solution* $x(t)$ *can be extended to the left to* $-\infty$ *with* $\lim_{t\to-\infty} x(t) = c > M$ *and to the right in one of the two ways.*

(a) $x(t)$ *meets* L *at a point* $t_0 > M$ *with* $x'(t_0) \ge 1$*;*

(b) $x(t)$ *has no common point with* L *and then* $x(t)$ *can be extended to* $+\infty$ *with* $x(+\infty) = +\infty$*.*

Proof. The condition that $\eta \in A_+$ and $\eta > M$ implies Γ is above L when $t > M$.

For $M < \eta \le \xi$, the proof is similar to that of Theorem 2.1 except that $x(t)$ can never meet Γ below line L. So we prove the theorem only for the case $\eta \ge \xi$. (When $\eta = \xi, g(t) - h(t) > 1, t \in (\eta, \eta + \delta)$ is required for a certain $\delta > 0$ small enough.)

Let $y(\tau) = \frac{1}{x(t)-(\xi-1)}$ and $\tau = \frac{1}{t-(\xi-1)}$. Then (1.1) is transformed into

$$\frac{dy(\tau)}{d\tau} = f(\tau, y(\tau), y(y(\tau))), \tag{2.11}$$

where $y(\tau, y(\tau), y(y(\tau))) = \frac{y^2(\tau)}{\tau^2}\big[g\big(\frac{1}{y(\tau)} + (\xi-1)\big) - h\big(\frac{1}{y(y(\tau))} + (\xi-1)\big)\big]$.

It is easy to see that if $x(t)$ is a solution of (1.1), satisfying $x(\xi) = \eta \geq \xi$ and $x(t) \geq t$ for $t \geq \xi$, then it becomes an appropriate solution $y(\tau)$ of (2.10) satisfying

$$y(1) = \frac{1}{\eta - \xi + 1} \leq 1 \quad \text{and} \quad 0 < y(\tau) \leq \tau \quad \text{for} \quad \tau < 1.$$

Let

$$f_n(\tau, y, z) = \begin{cases} \frac{y^2}{\tau^2+\varepsilon_n} \max\{0, g(\frac{1}{y} + (\xi-1)) - h(\frac{1}{z} + (\xi-1))\}, & y \geq z > 0, \\ 0, & z = 0, \end{cases}$$

where $\varepsilon_n = \frac{\varepsilon}{2^n} > 0$. Obviously f_n is continuous on the set

$$D = \{(\tau, y, z) \in \Re^3 : 0 \leq z \leq y \leq 1, 0 \leq \tau \leq 1\}$$

and locally Lipschtiz with y and z when $z \neq 0$. Condition (H) implies

$$0 \leq \lim_{y\to 0} f_n(\tau, y, z) \leq \lim_{y \to 0} \frac{y^2}{\varepsilon^n} \max\Big\{0, g\Big(\frac{1}{y} + (\xi-1)\Big) - h\Big(\frac{1}{z} + (\xi-1)\Big)\Big\} = 0,$$

and then there is $M > 0$ such that

$$f_n(\tau, y, z) \leq M, \quad 0 \leq \max\Big\{0, g\Big(\frac{1}{y} + (\xi-1)\Big) - h\Big(\frac{1}{z} + (\xi-1)\Big)\Big\} \leq M, \quad (\tau, y, z) \in D.$$

Take a close and convex subset of $C([0,1],[0,1])$,

$$H = \Big\{z \in C([0,1],[0,1]) : z(1) = \frac{1}{\eta+\xi+1},\ 0 \leq z(s) \leq s,$$
$$0 \leq z(s_2) - z(s_1) \leq \frac{M(s_2 - s_1)}{\varepsilon_n},\ \ \forall s_1 < s_2\Big\},$$

and a map $P : H \to C([0,1],[0,1])$ defined by

$$(Pz)(\tau) = \Big[\frac{1}{\eta+\xi+1} + \int_0^\tau f_n(s, z(s, z(z(s)))ds\Big]_0^\tau, \quad 0 \leq \tau \leq 1.$$

Obviously H is compact, P continuous and $Ph \subset H$. By the Schauder's theorem of fixed points, we have $y_n \in H$ such that

$$y_n(\tau) = \Big[\frac{1}{\eta - \xi + 1} + \int_1^\tau f_n(s, y_n(s), y_n(y_n(s)))ds\Big]_0^\tau, \qquad 0 \leq \tau \leq 1.$$

A similar discussion as in Theoerm 1.1 shows that

$$0 < y_n(\tau) \leq \frac{1}{\eta - \xi + 1}. \tag{2.12}$$

Then

$$y_n(\tau) = \Big[\frac{1}{\eta - \xi + 1} + \int_1^\tau f_n(s, y_n(s), y_n(y_n(s)))ds\Big]^\tau, \quad 0 \leq \tau \leq 1, \quad 0 \leq y_n(\tau) \leq 1,$$

and for any $\sigma \in (0,1)$, clearly

$$0 \leq f_n(\tau, y_n(\tau), y_n(y_n(\tau))) \leq \frac{1}{\sigma^2} M, \quad \sigma \leq \tau \leq 1.$$

Then $\{y_n(\tau)\}$ is uniformly bounded and uniformly continuous. This implies $\{y_n(\tau)\}$ has a subseries which tends to a function $y(t) > 0$,

$$y(\tau) = \Big[\frac{1}{\eta - \xi + 1} + \int_1^\tau f(s, y(s), y(y(s)))ds\Big]^\tau, \quad \tau \in (0,1].$$

Obviously $\eta = y(\tau)$ must meet $L : y = \tau$ at a point $\tau_0 \in [0, 1]$. Then

$$y(\tau) = \frac{1}{\eta - \xi + 1} + \int_1^\tau f(s, y(s), y(y(s)))ds.$$

It follows from $y(t) > 0$ that $\eta(s) \neq 0$ implies $y(y(s)) \neq 0$ for $t \in [\tau_0, 1]$. The function $f(\tau, y, z)$ is locally Lipschitz when $y, z \neq 0$ and Proposition 1.3 implies $y'(\tau) > 0$ for $y(\tau) \neq 0$. Therefore

$$y(\tau) = \frac{1}{\eta - \xi + 1} \int_1^\tau \frac{y^2(s)}{s^2}\Big[g\Big(\frac{1}{y(s)} + (\xi - 1)\Big) - h\Big(\frac{1}{y(y(s))} + (\xi - 1)\Big)\Big]ds, \quad \tau \in (\tau_0, 1)$$

with $y(\tau_0) = \tau_0$. That is to say, $y(\tau)$ is a solution to (2.10) with $y(1) = \frac{1}{\eta - \xi + 1}$.

After making the inverse transformation of (2.10), we obtain the solution to (1.1), $x(t) = \frac{1}{y(1/(t-\xi+1))} + (\xi - 1)$ defined on $[\xi, \frac{1}{\tau_0} - \xi + 1]$ satisfying $x(\xi) = \eta$ and $x(1/\tau_0 - \xi + 1) = 1/\tau_0 - \xi + 1$. Clearly when $\tau_0 = 0$, $x(t)$ is defined on $[\xi, +\infty)$ with $\lim\limits_{t\to+\infty} x(t) = +\infty$.

The proof for the assertion that $x(t)$ can be extended to $-\infty$ is similar to that in Theorem 2.1.

Theorem 2.3. *Suppose* (H) *holds. If* $\eta \in A_+, \eta < m$, *then* (1.1) *has a strictly increasing solution* $x(t)$ *satisfying* $x(\xi) = \eta$ *for any* $\xi \in \Re$. *Such a solution can be extended to the whole* $\Re$, *and if* $x(t)$ *meets* Γ *at a point* $t_1 > m$, *then* $\lim\limits_{t\to+\infty} x(t) = t_1$. *Otherwise* $\lim\limits_{t\to+\infty} x(t) = +\infty$. *On the other hand, if* $x(t)$ *meets* Γ *at a point* $t_0 < m$, *then* $\lim\limits_{t\to-\infty} x(t) = t_0$. *Otherwise* $\lim\limits_{t\to-\infty} x(t) = -\infty$.

Proof. (i) Set $I = [\xi, m]$ when $\xi < \eta$ or $\xi = \eta$ with $g(t) - h(t) < 1, t \in [\eta - \delta, \eta)$, required for $\delta < 0$. The same proof as in Theorem 2.1 shows the existence of solution $x(t)$, which can be extended both to the right and to the left in three ways. The theorem is valid except the case that $x(t)$ meets L at a point $\tilde{t}, \tilde{t} < \xi$ or $\tilde{t} > m$.

(ii) Applying a transformation in the form $y = \frac{1}{x-(\xi+1)}, \tau = \frac{1}{t-(\xi+1)}$ when $\xi > \eta$ or $\xi = \eta$ with $g(t) - h(t) > 1, t \in [\eta - \delta, \eta)$, required for a certain $\delta > 0$, the same proof as in Theorem 1.2 gives the same results as in (i).

(iii) $\xi = \eta, g(\eta) - h(\eta) = 1$. Set $s = \inf\{t \leq \eta : f(s) - h(s) = 1$ for all $t \leq s \leq \eta\}$. The existence and continuation to the right of $x(t)$ can be proved in the way as showed in Theorem 2.1. If $s = -\infty$, then $x(t) = t$ for $t \leq \eta$. If $s > -\infty$, then there is $\delta > 0$ such that for $t \in [s - \delta, s)$, one of the inequalities $g(t) - h(t) < 1$ and $g(t) - h(t) > 1$ is valid. Then the continuation of solutions is changed into the case (i) or (ii).

In order to prove our assertion it suffices to discuss the case that $x(t)$ meets L at a point $t_1 > m$ with $x(t_1) = t_1$. Obviously $x'(t_1) \geq 1$ implies $g(t_1) - h(t_1) \geq 1 > 0$ and then $t_1 \in A_+, t_1 > m_1$. Taking (t_1, t_1) as (ξ, η) discussed in Theorems 2.1 or 2.2, we can extend $x(t)$ to the right furthermore. Continue the process and then get a set $\{t_n\}$, $t_{n-1} < t_n, x(t_n) = t_n$. If $\{t_n\}$ is finite or infinite but bounded, the continuation to the right is proved as in using Theorems 2.1 or 2.2. If $\{t_n\}$ is infinte and unbounded, then $\lim\limits_{t\to+\infty} x(t) = +\infty$ since $x(t)$ is increasing.

The continuation to the left can be proved in a similar way.

When the same argument is applied to the case discussed in Theorem 2.1 or 2.2 we have

Corollary 2.1. *Suppose* (H) *holds. If* $\eta \in A_+$, *then* (1.1) *has a solution* $x(t)$ *defined on* $\Re$ *satisfying* $x(\xi) = \eta$ *for any* $\xi \in \Re$. *If* $x(t)$ *has a common point* $t_1 > \eta[t_1 < \eta]$ *with the line*

$L: x = t$, then $\lim_{t\to+\infty} x(t) = t_2$ $[\lim_{t\to-\infty} x(t) = t_1]$. *Otherwise*

$$\lim_{t\to+\infty} x(t) = +\infty\ [\lim_{t\to-\infty} x(t) = -\infty].$$

Corollary 2.2. *Suppose* (H) *holds. If $g(t)$ is bounded in the positive direction, then all increasing solutions are bounded in the same direction.*

§3. Existence and Behavior of Decreasing Solutions

Theorem 3.1. *Suppose* (H) *holds. If $\eta \in A_-$,* (1.1) *has a decreasing solution $x(t)$ defined on $\Re$ satisfying $x(\xi) = \eta$ for any $\xi \in \Re$. Let σ be the unique common point of $x = x(t)$ and $x = t$ and $S_1 = \{t \le \sigma : x(t) = h^{-1}(g(t))\}, S_2 = \{t > \sigma : x(t) = h^{-1}(g(t))\}$,$t_1 = \max S_1, t_2 = \min S_2$ (If $S_1 = \Phi, [S_2 = \Phi]$, then $t_1[t_2]$ is defined to be $-\infty[+\infty]$). Then $\lim_{t\to-\infty} x(t) = t_2$ and $\lim_{t\to+\infty} x(t) = t_1$.*

Proof. (i) If $m < \eta < M$, then let

$$u_1 = \max\{t \le \eta : t \in A_0\}, \quad u_2 = \min\{t \ge \eta : t \in A_0\}$$

and $I = [v_1, v_2]$, where $v_1 = \min\{u_1, \xi\}, v_2 = \min\{u_2, \xi\}$. Set $M = \max_{(x,y)\in I^2} \mid g(x) - h(y) \mid > 0$. Take in $C^0(I, [u_1, u_2])$ a subset

$$H = \{z \in C^0(I, [u_1, u_2] : z(\xi) = \eta, \quad 0 \le z(t_2) - z(t_1) \le M(t_2 - t_1), \quad t_1 \le t_2\}$$

and a map $P : H \to C^0(I, [u_1, u_2])$ defined by

$$(Tz)(t) = \Big[\eta + \int_\xi^t \min\{0, g(z(s)) - h(z(z(s)))\}ds\Big]_{u_1}^{u_2}, \quad t \in I. \tag{3.1}$$

A similar discussion as that in Theorem 2.1 leads to the conclusion.

(i) If $\eta < m$, then the proof is almost the same as the case (i) except that $I = [\min\{\xi, \eta\}, \max\{\xi, \eta\}]$ this time and u_1 and u_2 are replaced by $\min\{\xi, \eta\}$ and m respectively.

(ii) If $\eta > M$, then $\overline{\lim}_{x\to+\infty} h^{-1}(g(x)) = +\infty$. Let $\tau = \min\{t > M : h^{-1}(g(t)) = \eta\}$. Obviously $\tau > M$ since $h^{-1}(g(M)) = M < \eta$ and $\overline{\lim}_{t\to+\infty} h^{-1}(g(t)) = +\infty$. Take a set and a map similar to those in (i) except $I = [\min\{\xi, M\}, \max\{\tau, \xi\}], u_1 = \min\{\xi, M\}, u_2 = \tau$. Then applying the same discussion as the proof of Theorem 2.1 leads to our assertion.

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