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THE STABILITY OF THE PERIODIC SOLUTIONS OF SECOND ORDER HAMILTONIAN SYSTEMS

LIU Chungen*

Abstract

This paper studies the stability of the periodic solutions of the second order Hamiltonian systems with even superquadratic or subquadratic potentials. The author proves that in the subquadratic case, there exist infinite geometrically distinct elliptic periodic solutions, and in the superquadratic case, there exist infinite geometrically distinct periodic solutions with at most one instability direction if they are half period non-degenerate, otherwise they are elliptic.

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§1. Introduction and Main Results

In this paper, we consider the stability of the periodic solutions of the following second order Hamiltonian systems

$$\ddot{x} + V_x'(t,x) = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where n is a positive integer. $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function and for $\tau > 0$ it is τ-periodic for the variable t. V_x' denotes its gradient with respect to x. We now state the main results of this paper.

For the superquadratic case, we have the following two theorems

Theorem 1.1. *Suppose V satisfies the following conditions*:

(V1) $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *and* $V(t+\tau, x) = V(t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$.

(V2) *There exist constants* $\mu > 2$ *and* $r_0 > 0$ *such that*

$$0 < \mu V(t,x) \le V_x'(t,x) \cdot x,\ \forall |x| \ge r_0, \quad t \in \mathbb{R}.$$

(V3) $V(t,x) \ge 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$.

(V4) $V(t,x) = o(|x|^2)$, *at* $x = 0$.

(V5) $V(t,-x) = V(t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$.

(V6) $V_{xx}''(t,x) > 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$.

Then Equation (1.1) *has* $2j\tau$*-periodic solution* x_{2j} *for* $j \in \mathbb{N}$. *Moreover,* x_{2j} *has at least* $2(n-1)$ *Floquet multipliers lying on the unit circle in the complex plane if it is half period*$(j\tau)$*non-degenerate,* x_{2j} *is elliptic* (*i.e., all* $2n$ *Floquet multipliers of* x_{2j} *lying on the unit circle in the complex plane*) *if it is half period*$(j\tau)$ *degenerate, and the sequence* $\{x_{2j}\}$ *has a geometrically*

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*Department of Mathematics, Nankai University, Tianjin 300071, China.

distinct subsequence $\{x_{2^k}\}_{k\in\mathbb{N}}$. *In particular, if* $V(t,x)\equiv V(x)$, *i.e. Equation* (1.1) *is autonomous, then for any* $\tau>0$, *there exists a* τ*-periodic elliptic solution* x.

Theorem 1.2. *In Theorem* 1.1, *if we replace the condition* (V2) *by the following condition*

(V7) *There exist constants* $\delta>0$ *and* $r_1>0$ *such that* $V(t,x)\le\frac{\delta}{2}|x|^2,\ \ \forall|x|\le r_1,\quad t\in\mathbb{R}$,

then Equation (1.1) *has* $2j\tau$*-periodic solution* x_{2j} *for* $1\le j<\frac{\pi}{\tau\sqrt{\delta}}$ *which has at least* $2(n-1)$ *Floquet multipliers lying on the unit circle in the complex plane if it is half period*$(j\tau)$ *non-degenerate,* x_{2j} *is elliptic if it is half period*$(j\tau)$ *degenerate, and the finite sequence* $\{x_{2j}\}$ *has a geometrically distinct subsequence* $\{x_{2^k}\}$ *for* $k\in\{s\in\mathbb{N}\cup\{0\}|2^s<2\pi/\sqrt{\delta}\}$. *In particular, in the autonomous case, for any* $0<\tau<2\pi/\sqrt{\delta}$, *there exists a* τ*-periodic elliptic solution* x.

For the subquadratic case, we have the following two theorems.

Theorem 1.3. *Suppose* V *satisfies* (V1), (V3), (V5)–(V6) *and the following*

(V8) *There exists constant* $1<\alpha<2$ *such that* $V_x'(t,x)\cdot x\le\alpha V(t,x),\ \ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^n$.

(V9) *There holds* $V_x'(t,0)\ne0$ *for* $t\in\mathbb{R}$.

Then Equation (1.1) *has* $2j\tau$*-periodic elliptic solution* x_{2j} *for* $j\in\mathbb{N}$, *and the sequence* $\{x_{2j}\}$ *has a geometrically distinct subsequence* $\{x_{2^k}\}_{k\in\mathbb{N}}$. *In particular, in the autonomous case, for any* $\tau>0$, *there exists a* τ*-periodic elliptic solution* x.

Theorem 1.4. *In Theorem* 1.3, *if we replace the condition* (V8) *by the following condition*

(V10) *There exist positive constants* m *and* b *such that* $\tau<\pi/\sqrt{m}$ *and*

$$V(t,x)\le\frac{m}{2}|x|^2+b,\ \ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^n,$$

then Equation (1.1) *has* $2j\tau$*-periodic elliptic solution* x_{2j} *for* $j<\pi/\tau\sqrt{m}$, *and the sequence* $\{x_{2j}\}$ *has a geometrically distinct subsequence* $\{x_{2^k}\}$ *for* $k\in\{s\in\mathbb{N}\cup\{0\}|2^s<2\pi/\tau\sqrt{m}\}$. *In particular, in the autonomous case, for any* $\tau<\frac{2\pi}{\sqrt{m}}$, *there exists a* τ*-periodic elliptic solution* x.

Our results should be compared with those results in [3–6], especially Theorems I-IV of [4] (also the results in Section 7 of [3]). In Theorems I-IV of [4], they did not assume V satisfies the condition (V5), but in their conclusions there was a condition for the period $\tau\in J_+(A)$ or $\tau\in J_-(A)$. In the autonomous case, in [5], they obtained an elliptic periodic solution for the first order subquadratic Hamiltonian systems. In [8], the author of this paper and Long studied the stability of characteristics on star-shaped surfaces. The elliptic solutions are first obtained here for the superquadratic case. All papers mentioned above only obtained the elliptic solutions for the subquadratic case. The methods in this paper are related with that in [7].

§2. The ω-Index Theory of Second Order Hamiltonian Systems

In this section, for $\tau>0$ and $\omega\in\mathbf{U}=\{z\in\mathbb{C}|\,|z|=1\}$, we consider the following boundary valued problem

$$\begin{cases}\ddot q(t)+M(t)q(t)=0, & q(t)\in\mathbb{C}^n,\\ q(\tau)=\omega q(0), & \dot q(\tau)=\omega q(0),\end{cases}\tag{2.1}$$

where $M\in C(S_\tau,\mathcal{L}_s(\mathbb{R}^n))$ the τ-periodic symmetric $n\times n$ continuous matrix function, $S_\tau=\mathbb{R}/(\tau\mathbb{Z})$.

Denote the usual norm and inner product in $\mathbb{C}^n$ $|q|$ and $p\cdot q$ for p and $q\in\mathbb{C}^n$, and denote $H_1(n)=W^{1,2}(S_\tau,\mathbb{C}^n)$, $E(n)=W^{1,2}([0,\tau],\mathbb{C}^n)$. Then we have

$$H_1(n)=\{v\in E(n)\,|\,v(\tau)=v(0),\ \dot v(\tau)=\dot v(0)\}.$$

For $\omega \in \mathbf{U}$, we also denote $H_\omega(n) = \{v \in E(n) \mid v(\tau) = \omega v(0),\ \dot{v}(\tau) = \omega \dot{v}(0)\}$, and define an operator from $H_1(n)$ into $E(n)$ by $\varphi_\omega : H_1(n) \to E(n)$, $\varphi_\omega(v)(t) = \omega^{t/\tau} v(t)$. Then we have $H_\omega(n) = \varphi_\omega(H_1(n))$. We consider the following quadratic form defined in the space $H_\omega(n)$,

$$\phi_\omega(q) = \int_0^\tau \{|\dot{q}|^2 - M(t)q(t) \cdot q(t)\}\, dt. \tag{2.2}$$

The space $H_\omega(n)$ has an orthogonal decomposition $H_\omega(n) = H_\omega^+(n) \oplus H_\omega^0(n) \oplus H_\omega^-(n)$ according as ϕ_ω is positive, null, or negative definite respectively. Define a linear operator M on $H_\omega(n)$ by the following formula

$$\langle Mq(t), p(t)\rangle_{E(n)} = \int_0^\tau (q(t) + M(t)q(t)) \cdot p(t)\, dt.$$

Then M is self-adjoint and compact, and there holds

$$\phi_\omega(q) = ((I - M)q, q)_{E(n)} \ \text{ for all } \ q \in H_\omega(n).$$

Thus there hold $H_\omega^0(n) = \ker(I - M)$, $\dim H_\omega^0(n) \le 2n$ and $\dim H_\omega^-(n) < +\infty$. We define the Morse index $j(M, \tau, \omega)$ and the nullity $\nu(M, \tau, \omega)$ of (2.2) by

$$j(M, \tau, \omega) = \dim H_\omega^-(n), \quad \nu(M, \tau, \omega) = \dim H_\omega^0(n).$$

Let $p = \dot{q}$ and $x = (p, q)$. For all $t \in \mathbb{R}$ define $B(t) = \begin{pmatrix} I & 0 \\ 0 & M(t) \end{pmatrix}$. Then $B \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$ and the first equation of the system (2.1) is transformed into the following first order linear Hamiltonian system

$$\dot{x}(t) = JB(t)x(t), \tag{2.3}$$

where $\mathcal{L}_s(\mathbb{R}^{2n})$ is the set of $2n \times 2n$ symmetric matrices in $\mathbb{R}$ and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbb{R}^{2n}$. Let γ be the fundamental solution of (2.3). Then γ is a continuous path starting from the identity matrix I_{2n} in the symplectic group $\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T JM = J\}$. The ω-index theory for such paths in $\mathrm{Sp}(2n)$ is defined in [10]. This index theory assignes to the system (2.3) or the same to the matrix B through γ a pair of integers denoted by $(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) := (i_{\tau,\omega}(B), \nu_{\tau,\omega}(B)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\}$. The following theorem is the main result of this section.

Theorem 2.1. *For $\tau > 0$, $M \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$ and $B \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$ defined above, then there hold*

$$j(M, \tau, \omega) = i_{\tau,\omega}(B), \quad \nu(M, \tau, \omega) = \nu_{\tau,\omega}(B). \tag{2.4}$$

We know that the ω-index $(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega})$ for a symplectic path γ is defined by geometric or algebraic methods, but the Morse index $(j(M, \tau, \omega), \nu(M, \tau, \omega)$ is defined by analytic methods. Theorem 2.1 tells us that these two concepts are essentially the same. We note that (2.4) can be regarded as a consequence of Theorem 6.1 of [10]. We now follow the ideas in [1] to prove it.

We define $L_\tau = L^2([0, \tau], \mathbb{C}^{2n})$, the Hilbert space with the usual L^2 inner product

$$\langle x, y\rangle_{L^2} = \int_0^\tau \langle x(t), y(t)\rangle\, dt, \quad \forall x, y \in L_\tau, \tag{2.5}$$

and $E_\tau = W^{1,2}([0, \tau], \mathbb{C}^{2n})$ as a subspace of L_τ, where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{C}^{2n}$. For $\omega = \exp(i\theta) \in \mathbf{U}$, where $i = \sqrt{-1}$, define E_τ^ω to be the subspace $\{y \in E_\tau \mid y(\tau) = \omega y(0)\}$ of

L_τ. Any $y \in E^\omega_\tau$ has the form $y(t) = \sum_{k\in\mathbb{Z}} e^{i(\theta+2k\pi)t/\tau}\xi_k$, $\xi_k \in \mathbb{C}^{2n}$ with

$$\sum_{k\in\mathbb{Z}}((\theta+2k\pi)^2+1)|\xi_k|^2 < +\infty.$$

Define $E^\omega_{\tau,k} = E^{\omega,+}_{\tau,k} \oplus E^{\omega,-}_{\tau,k}$ with $E^{\omega,\pm}_{\tau,k} = e^{i(\theta+2k\pi)t/\tau}(J \pm iI)\mathbb{R}^{2n}$. Then there holds

$$E^\omega_\tau = \oplus E^\omega_{\tau,k}. \tag{2.6}$$

Define $A = -J\frac{d}{dt} : E^\omega_\tau \to L_\tau$. Then A is continuous and symmetric, i.e., it satisfies

$$\langle Ax, y\rangle_{L^2} = \langle x, Ay\rangle_{L^2}, \quad \forall x, y \in E^\omega_\tau.$$

Viewing A as from the subspace E^ω_τ of L_τ to L_τ, we have

$$\sigma(A) = \{\lambda^\pm_k \mid k \in \mathbb{Z}\}, \quad \lambda^\pm_k = \pm\frac{\theta+2k\pi}{\tau},$$

where each eigenvalue $\lambda^\pm_k$ of A has multiplicity $2n$ and the corresponding eigenspace is $E^{\omega,\pm}_{\tau,k}$. For a given $B(t) \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$, it induces a symmetric operator on L_τ by

$$\langle Bx, y\rangle_{L^2} = \int_0^\tau \langle B(t)x(t), y(t)\rangle\, dt, \quad \forall x, y \in L_\tau. \tag{2.7}$$

The system (2.1) is equivalent to the following linear Hamiltonian system

$$\dot{y} = JB(t)y \;\text{ for }\; y \in E^\omega_\tau. \tag{2.8}$$

The solutions of (2.8) are one to one correspondent to the critical points of the functional

$$f_{\tau,\omega}(y) = \frac{1}{2}\langle (A-B)y, y\rangle_{L^2}, \quad \forall y \in E^\omega_\tau \subset L_\tau. \tag{2.9}$$

Then $f_{\tau,\omega} : E^\omega_\tau \to \mathbb{C}$ is smooth in the topology of L_τ.

Using the saddle point reduction method (cf.[2]), we obtain a subspace

$$Z^\omega_{\tau,k_0} = \bigoplus_{|k|\le k_0} E^\omega_{\tau,k} \tag{2.10}$$

with a sufficiently large $k_0 \in \mathbb{N}$ (see [2, Section 4.2.1]), an injection map $u_{\tau,\omega,k_0} \in C^\infty(Z^\omega_{\tau,k_0}, L_\tau)$ and a smooth functional $a_{\tau,\omega,k_0} \in C^\infty(Z^\omega_{\tau,k_0}, \mathbb{C})$ defined by

$$a_{\tau,\omega,k_0}(z) = f_{\tau,\omega}(u_{\tau,\omega,k_0}(z)), \; \forall z \in Z^\omega_{\tau,k_0}. \tag{2.11}$$

Note that the origin of Z^ω_{τ,k_0} as a critical point of a_{τ,ω,k_0} corresponds to the origin of E^ω_τ as a critical point of $f_{\tau,\omega}$. Denote by $m^*_{k_0}$ for $* = +, 0$, and $-$, the positive, null, and negative Morse indices of the functional a_{τ,ω,k_0} at the origin respectively.

We denote $2d_{\tau,\omega,k_0} = \dim_{\mathbb{C}} Z^\omega_{\tau,k_0}$. Then by Theorem 3.1 of [10], there hold

$$m^-_{k_0} = d_{\tau,\omega,k_0} + i_{\tau,\omega}(B), \tag{2.12}$$

$$m^0_{k_0} = \nu_{\tau,\omega}(B), \tag{2.13}$$

$$m^+_{k_0} = d_{\tau,\omega,k_0} - i_{\tau,\omega}(B) - \nu_{\tau,\omega}(B). \tag{2.14}$$

For the above given $B(t) = \begin{pmatrix} I & 0 \\ 0 & M(t) \end{pmatrix}$ and $y = (p, q) \in E^\omega_\tau$ the functional $f_{\tau,\omega}$ defined by (2.9) becomes

$$f_{\tau,\omega}(y) = f_{\tau,\omega}(p, q) = \frac{1}{2}\int_0^\tau (\dot{q}\cdot p - \dot{p}\cdot q - |p|^2 - M(t)q\cdot q)\, dt. \tag{2.15}$$

The bilinear form Φ of $f_{\tau,\omega}$ is defined by

$$\begin{aligned}\Phi(x,y) &= \frac{1}{2}(f_{\tau,\omega}(x+y)-f_{\tau,\omega}(x)-f_{\tau,\omega}(y)) \\ &= \mathrm{Re}\int_0^\tau (\dot{x}_2\cdot y_1+\dot{y}_2\cdot x_1 - x_1\cdot y_1 - M(t)x_2\cdot y_2)\,dt\end{aligned} \tag{2.16}$$

for all $x=(x_1,x_2)$ and $y=(y_1,y_2)\in E_\tau^\omega$. Let X be the dense subspace of E_τ^ω defined by $X=H_\omega(n)\times H_2$, where $H_2=\{v\in W^{2,2}([0,\tau],\mathbb{C}^n)\,|\,v(\tau)=\omega v(0),\ \dot{v}(\tau)=\omega\dot{v}(0)\}$. Define two subspaces of X by $W_1=\{(p,q)\in X\,|\,q=0\}$, $W_2=\{(p,q)\in X\,|\,p=\dot{q}\}$.

Lemma 2.1. *W_1 and W_2 are Φ-orthogonal and $X=W_1\oplus W_2$.*

Proof. For any $x=(p,0)\in W_1$ and $y=(\dot{q},q)\in W_2$, by direct computation and (2.16), we obtain $\Phi(x,y)=0$. On the other hand, for any $x=(p,q)\in X$, we have $x=(p-\dot{q},0)+(\dot{q},q)\in W_1\oplus W_2$. This proves the lemma.

Define $Z_{\tau,k_0}^{1,\omega}=Z_{\tau,k_0}^\omega\cap W_1$ and $Z_{\tau,k_0}^{2,\omega}=Z_{\tau,k_0}^\omega\cap W_2$. Then $Z_{\tau,k_0}^\omega=Z_{\tau,k_0}^{1,\omega}\oplus Z_{\tau,k_0}^{2,\omega}$.

Lemma 2.2. *The injection $u_{\tau,\omega,k_0}:Z_{\tau,k_0}^\omega\to E_\tau^\omega$ is linear, maps $Z_{\tau,k_0}^{1,\omega}$ and $Z_{\tau,k_0}^{2,\omega}$ into W_1 and W_2 respectively, and there holds $u_{\tau,\omega,k_0}(Z_{\tau,k_0}^\omega)\subset X$.*

Proof. The proof is a simple modification of the proof of Lemma 4 of [1].

Lemma 2.3. *For a sufficiently large $k_0\notin\sigma(A)$, the Morse index of a_{τ,ω,k_0} on Z_{τ,k_0}^ω is the sum of the Morse indexes of $a_{\tau,\omega,k_0}|_{Z_{\tau,k_0}^{1,\omega}}$ and $a_{\tau,\omega,k_0}|_{Z_{\tau,k_0}^{2,\omega}}$.*

Proof. Denote by Ψ_{k_0} the bilinear form of a_{τ,ω,k_0}. Then for $Z_l\in Z_{\tau,k_0}^{l,\omega}$ with $l=1,2$, we obtain

$$\begin{aligned}\Psi_{k_0}(z_1,z_2) &= \frac{1}{2}(a_{\tau,\omega,k_0}(z_1+z_2)-a_{\tau,\omega,k_0}(z_1)-a_{\tau,\omega,k_0}(z_2)) \\ &= \frac{1}{2}(f_{\tau,\omega}(u_{\tau,\omega,k_0}(z_1)+u_{\tau,\omega,k_0}(z_2))-f_{\tau,\omega}(u_{\tau,\omega,k_0}(z_1))-f_{\tau,\omega}(u_{\tau,\omega,k_0}(z_2))) \\ &= \Phi(u_{\tau,\omega,k_0}(z_1),u_{\tau,\omega,k_0}(z_2))=0.\end{aligned}$$

Here we have used Lemmas 2.1 and 2.2.

Lemma 2.4. *For sufficiently large $k_0\notin\sigma(A)$, the Morse index $\hat{m}^-$ of $a_{\tau,\omega,k_0}|_{Z_{\tau,k_0}^{2,\omega}}$ satisfies $\hat{m}^-=i_{\tau,\omega}(B)$.*

Proof. Note that $\dim Z_{\tau,k_0}^\omega=2d_{\tau,\omega,k_0}$. From direct computations on eigenvectors of A, we obtain $\dim Z_{\tau,k_0}^{1,\omega}=d_{\tau,\omega,k_0}$. For any $z=(z_1,0)\in Z_{\tau,k_0}^{1,\omega}\setminus\{0\}$, there holds $\Psi_{k_0}(z,z)=a_{\tau,\omega,k_0}(z)=f_{\tau,\omega}(z)<0$. So Ψ_{k_0} is negative definite on $Z_{\tau,k_0}^{1,\omega}$ and the Morse index of $a_{\tau,\omega,k_0}|_{Z_{\tau,k_0}^{1,\omega}}$ is equal to d_{τ,ω,k_0}. By (2.12) and Lemma 2.3, we obtain $\hat{m}^-=i_{\tau,\omega}(B)$.

Lemma 2.5. *Denote by $\tilde{m}^-$ the Morse index of $f_{\tau,\omega}|_{W_2}$. Then $\tilde{m}^-=j(M,\tau,\omega)$.*

Proof. Note that $f_{\tau,\omega}(\dot{q},q)=\phi_\omega(q)$ for all $(\dot{q},q)\in W_2$. Since H_2 is dense in $H_\omega(n)$ and there holds $\dim W_2=\dim H_2$, we obtain $\tilde{m}^-=j(M,\tau,\omega)$.

Proof of Theorem 2.1. By the equivalent of the systems (2.1) and (2.8) for $B(t)=\begin{pmatrix} I & 0\\ 0 & M(t)\end{pmatrix}$, we obtain $\nu(M,\tau,\omega)=\nu_\omega(B,\tau)$.

For the first equality in (2.4), by Lemmas 2.4 and 2.5, it suffices to prove $\tilde{m}^-=\hat{m}^-$ for k_0 large enough. Since $u_{\tau,\omega,k_0}:Z_{\tau,k_0}^{2,\omega}\to W_2$ is a linear injection and $a_{\tau,\omega,k_0}(z)=f_{\tau,\omega}(u_{\tau,\omega,k_0}(z))$ for $z\in Z_{\tau,k_0}^{2,\omega}$, we obtain $\hat{m}^-\le\tilde{m}^-$ for large k_0. On the other hand, suppose $f_{\tau,\omega}$ is negative definite on a subspace W_2^- of W_2. Since W_2^- is finite dimensional, there exists a constant $\delta>0$ such that $f_{\tau,\omega}(x)\le-\delta<0$, $\forall x\in W_2^-$ with $\|x\|=1$. We denote by

$P_{k_0} : E_\tau^\omega \to Z_{\tau,k_0}^\omega$ the projection. Then by the Lemma 2 of [1] or Lemma 2.4 of [11], it is clear that $\|u_{\tau,\omega,k_0}(P_{k_0}x) - x\|_{W^{1,2}} \to 0$ as $k_0 \to +\infty$. Thus for k_0 large enough, there holds

$$a_{\tau,\omega,k_0}(P_{k_0}x) = f_{\tau,\omega}(u_{\tau,\omega,k_0}(P_{k_0}x)) \le -\frac{1}{2}\delta < 0, \quad \forall x \in W_2^- \text{ with } \|x\| = 1.$$

Therefore a_{τ,ω,k_0} is negative definite on $P_{k_0}(W_2^-)$ by its 2-homogeneity. Since W_2^- is a finite dimensional space, we can choose k_0 large enough so that $\dim P_{k_0}(W_2^-) = \dim W_2^-$. This proves that $\hat{m}^- \ge \tilde{m}^-$ and the Theorem 2.1.

From Corollary 4.12 of [10], we have the following consequence.

Corollary 2.1. *The locally constant function $j(M,\tau,\omega)$ is not continuous only at the Floquet multipliers of M lying on the unit circle, and we have*

$$\lim_{\varepsilon\to 0^+}(j(M,\tau,e^{\varepsilon\sqrt{-1}\omega}) - j(M,\tau,e^{-\varepsilon\sqrt{-1}\omega})) = p - q, \tag{2.17}$$

where (p,q) is the Krein type of $\omega \in \mathbf{U}$.

Proof. By Corollary 4.12 of [10], there holds $\lim\limits_{\varepsilon\to 0^+}(i_\tau(e^{\varepsilon\sqrt{-1}\omega}) - i_\tau(e^{-\varepsilon\sqrt{-1}\omega})) = p - q$. So we obtain (2.17).

§3. The Stability of the Periodic Solutions of Second Order Hamiltonian Systems

In this section, we give a proof of Theorems 1.1–1.4 stated in the first section. In his pioneering work[12], Rabinowitz introduced the following variation formulation for the system (1.1),

$$\phi(x) = \int_0^{2\tau} \Big(\frac{1}{2}|\dot{x}|^2 - V(t,x)\Big)\,dt, \quad \forall x \in W^{1,2}(S_{2\tau},\mathbb{R}^n), \tag{3.1}$$

where $E_{2\tau} = W^{1,2}(S_{2\tau},\mathbb{R}^n)$ is a Hilbert space with the usual norm

$$\|x\|_{2\tau} = \Big(\int_0^{2\tau}(|\dot{x}|^2 + |x|^2)\,dt\Big)^{1/2}, \quad \forall x \in E_{2\tau}.$$

By the condition (V1), it is well known that $\phi \in C^2(E_{2\tau},\mathbb{R})$, i.e. ϕ is continuously 2-times Fréchet differentiable on $E_{2\tau}$, and there hold

$$(\phi'(x),y)_{2\tau} = \int_0^{2\tau}(\dot{x}\cdot\dot{y} - V_x'(t,x)\cdot y)\,dt, \quad \forall x,\ y \in E_{2\tau}, \tag{3.2}$$

$$(\phi''(x)y,z)_{2\tau} = \int_0^{2\tau}(\dot{y}\cdot\dot{z} - V_{xx}''(t,x)y\cdot z)\,dt, \quad \forall x,\ y,\ z \in E_{2\tau}. \tag{3.3}$$

By the conditions (V1) and (V2), ϕ satisfies the (PS) condition on $E_{2\tau}$. So finding the periodic solutions is equivalent to finding the critical points of the functional ϕ on $E_{2\tau}$. By the condition (V5), one can restrict ϕ on the closed subspace of $E_{2\tau}$, $E_{2\tau,\mathrm{odd}} = \{v \in E_{2\tau} |\ v(\tau) = -v(0)\}$ and denote its restriction by ϕ_{odd}. It is easy to check that any critical point of ϕ_{odd} on $E_{2\tau,\mathrm{odd}}$ is a critical point of ϕ on $E_{2\tau}$. This critical point satisfies $x(\tau+t) = -x(t)$.

Proposition 3.1. *Suppose V satisfies the conditions* (V1)–(V5). *Then there exists a critical point x_2 of ϕ_{odd} on $E_{2\tau,\mathrm{odd}}$ with its Morse index $m^-(x_2) \le 1$.*

Proof. It is easy to check that $E_{2\tau,\mathrm{odd}}$ is orthogonal to $\mathbb{R}^n$, and $\phi_{\mathrm{odd}}(0) \le 0$. For any $x \in E_{2\tau,\mathrm{odd}}$, we have $\int_0^{2\tau} x(t)\,dt = 0$, so we can take $\|x\| = \big(\int_0^{2\tau}|\dot{x}|^2\,dt\big)^{1/2}$. By the condition (V4), one can choose a small number $\rho > 0$ such that $\phi_{\mathrm{odd}}(x) \ge \eta$ for any $x \in E_{2\tau,\mathrm{odd}}$ with

$\|x\| = \rho$, where $\eta > 0$ is a constant. By the condition (V2), one can choose an element $e \in E_{2\tau,\mathrm{odd}}$ with $\|e\| > \rho$ such that $\phi_{\mathrm{odd}}(e) \le 0$. Since ϕ_{odd} also satisfies the (PS) condition on $E_{2\tau,\mathrm{odd}}$, by the Mountain pass lemma there exists a critical point x_2 of ϕ_{odd} on $E_{2\tau,\mathrm{odd}}$ with its Morse index $m^-(x_2) \le 1$.

By the condition (V5), one can check that

$$\phi_{\mathrm{odd}}(x) = 2\int_0^\tau \Big(\frac{1}{2}|\dot{x}|^2 - V(t,x)\Big)\,dt, \quad \forall x \in E_{2\tau,\mathrm{odd}}.$$

We now consider the following quadratic form

$$Q_{\mathrm{odd}}(v) = \int_0^\tau \big(|\dot{v}|^2 - (V''_{xx}(t,x_2(t))v,v)\big)\,dt, \quad \forall v \in H_\omega(n)$$

and denote its Morse index by $j(x_2,\tau,\omega)$. By Proposition 3.1 and the arguments of the previous section, we obtain

$$j(x_2,\tau,-1) \le 1. \tag{3.4}$$

Proposition 3.2. *Suppose V satisfies* (V1)–(V6). *Then there holds*

$$j(x_2,\tau,\omega) \ge n, \quad \omega = e^{\sqrt{-1}\theta}, \quad \theta \to 0. \tag{3.5}$$

Proof. Since the set $[0,\tau]$ is compact and on this interval $V''_{xx}(x_2(t)) > 0$, we can suppose that $V''_{xx}(x_2(t)) \ge \lambda > 0$ for small λ. On $H_\omega(n)$ we define the following quadratic form

$$Q(g) = \int_0^\tau \big(|\dot{g}|^2 - \lambda|g|^2\big)\,dt.$$

Any $g \in H_\omega(n)$ has the form

$$g(t) = \sum_{k=-\infty}^{+\infty} \exp\Big(\frac{(2k\pi+\theta)\sqrt{-1}t}{\tau}\Big)a_k, \quad a_k \in \mathbb{C}^n.$$

Thus there holds $Q(g) = \sum\limits_{k=-\infty}^{+\infty}\Big[\Big(\frac{2k\pi+\theta}{\tau}\Big)^2 - \lambda\Big]|a_k|^2$. Since $\theta \to 0$, we can see that $\Big(\frac{2k\pi+\theta}{\tau}\Big)^2 - \lambda < 0$ for $k = 0$. Thus the dimension of the negative space of $Q(g)$ is at least n. It is clear that $Q_{\mathrm{odd}}(g) \le Q(g)$ for all $g \in H_\omega(n)$. So we obtain (3.6).

Proof of Theorem 1.1. If $\omega \in \mathbf{U}$ is a Floquet multiplier of x_2 with its Krein type (p,q), then we say ω has positive multiplicity p and negative multiplicity q. By Corollary 2.1, Propositions 3.1 and 3.2, there holds

$$n + n^+ - n^- - d_{-1} \le j(x_2,\tau,\omega) + n^+ - n^- + j^-(x_2,\tau,-1) = j(x_2,\tau,-1) \le 1,$$

where n^+ and n^- are the total positive and negative multiplicity of the Floquet multipliers of x_2 lying on the upper unit circle respectively, $2d_{-1}$ is the multiplicity of -1 as the Floquet multipliers of x_2, and $j^-(x_2,\tau,-1) = \lim\limits_{\varepsilon\to 0^+}[j(x_2,\tau,-1) - j(x_2,\tau,e^{(\pi-\varepsilon)\sqrt{-1}})]$. Thus we get $n^- + d_{-1} \ge n-1$. Since the Floquet multipliers lying on the unit circle are symmetric, we obtain at least $2(n-1)$ Floquet multipliers lying on the unit circle if 1 is not a Floquet multiplier, i.e., x_2 is half period non-degenerate, otherwise x_2 has $2n$ Floquet multipliers lying on the unit circle, i.e., it is elliptic. We note that all solutions of autonomous systems are degenerate, so in this case the above obtained solution is elliptic. Since $V(t,x)$ is also a $j\tau$-periodic function about variable t for $j \in \mathbb{N}$, by the above arguments, we can obtain x_{2j} for $j \in \mathbb{N}$ as x_2 satisfies the stability condition. That every two elements of the subsequence $\{x_{2^k}\}_{k\in\mathbb{N}}$ are geometrically distinct follows from the fact that one is periodic for some period but another is anti-periodic.

Proof of Theorem 1.2. If the condition (V4) is replaced by (V7), then in Proposition 3.1, if $2\tau < 2\pi/\sqrt{\delta}$, we can still obtain a critical point x_2 of ϕ_{odd} on $E_{2\tau,\text{odd}}$ with its Morse index $m^-(x_2) \le 1$. In fact, under the condition (V7), by the Wirtinger's inequality, for any $x \in E_{2\tau,\text{odd}}$ with $\|x\|$ small (we note $E_{2\tau}$ can be embedded in $C(S_{2\tau}, \mathbb{R}^n)$), there holds

$$\phi_{\text{odd}}(x) \ge \frac{1}{2}\|x\| - \frac{\delta}{2} \cdot \frac{4\tau^2}{4\pi^2}\|x\| = \frac{1}{2}\Big(1 - \frac{\delta\tau^2}{\pi^2}\Big)\|x\|^2.$$

So if $0 < \tau < \pi/\sqrt{\delta}$, there exists a small positive number ρ such that $\phi_{\text{odd}}(x) \ge \eta > 0$ for any $x \in E_{2\tau,\text{odd}}$ with $\|x\| = \rho$ and some constant η. Thus we can still use the Mountain pass lemma. The remainders are the same as in the proof of Theorem 1.1.

Proof of Theorem 1.3. Under the conditions (V3) and (V8), we can prove that for some constant c there holds $0 \le V(t,x) \le |x|^\alpha + c, \ \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^n$. By the same arguments as in [9], we can prove that ϕ_{odd} is weak lower semi-continuous and coercive on $E_{2\tau,\text{odd}}$, and attains its minimum on $E_{2\tau,\text{odd}}$. By the condition (V9), as a critical point of ϕ_{odd}, the minimun point x_2 is nonconstant. So we have its Morse index $m^-(x_2) = 0$. Thus there holds

$$n + n^+ - n^- - d_{-1} \le j(x_2,\tau,\omega) + n^+ - n^- + j^-(x_2,\tau,-1) = j(x_2,\tau,-1) = 0.$$

From this estimate, we obtain $n^- + d_{-1} \ge n$. Note that $n^- + d_{-1} \le n$, so we have $n^- + d_{-1} = n$. The remainders are the same as in the proof of Theorem 1.1.

Proof of Theorem 1.4. Under the conditions (V1) and (V10), by the Wirtinger's inequality, if $2\tau < 2\pi/\sqrt{m}$, we can still prove that ϕ_{odd} is weak lower semi-continuous and coercive on $E_{2\tau,\text{odd}}$, and attains its minimum on $E_{2\tau,\text{odd}}$. The remainders are the same as in the proof of Theorems 1.1, 1.2 and 1.3.

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