Chin. Ann. of Math.
21B: 2(2000),233-236.

THE SECOND EXPONENT SET OF PRIMITIVE DIGRAPHS***

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Abstract

Let $D = (V, E)$ be a primitive digraph. The exponent of D, denoted by $\gamma(D)$, is the least integer k such that for any $u, v \in V$ there is a directed walk of length k from u to v. The local exponent of D at a vertex $u \in V$, denoted by $\exp_D(u)$, is the least integer k such that there is a directed walk of length k from u to v for each $v \in V$. Let $V = \{1, 2, \cdots, n\}$. Following [1], the vertices of V are ordered so that $\exp_D(1) \leq \exp_D(2) \leq \cdots \leq \exp_D(n) = \gamma(D)$. Let $E_n(i) := \{\exp_D(i) \mid D \in PD_n\}$, where PD_n is the set of all primitive digraphs of order n. It is known that $E_n(n) = \{\gamma(D) \mid D \in PD_n\}$ has been completely settled by [7]. In 1998, $E_n(1)$ was characterized by [5]. In this paper, the authors describe $E_n(2)$ for all $n \geq 2$.

Keywords Primitive digraph, Local exponent, Gap
1991 MR Subject Classification 05C20, 05C50, 15A33
Chinese Library Classification O157.5, O151.21 **Document Code** A
Article ID 0252-9599(2000)02-0233-04

§1. Introduction and Notations

Let $D = (V, E)$ be a digraph and $L(D)$ denote the set of cycle lengths of D. For $u \in V$ and integer $i \geq 1$, let $R_i(u)$:= $\{ v \in V \mid$ there exists a directed walk of length i from u to $v\}$. We define $R_0(u) := \{u\}$. Let $u, v \in V$. If $N^+(v) = N^+(v)$ and $N^-(v) = N^-(v)$, then we call v a copy of u.

Let D be a primitive digraph and $\gamma(D)$ denote the exponent of D. In 1950, H. Wielandt[6] found that $\gamma(D) \leq (n-1)^2 + 1$ and showed that there is a unique digraph that attains this bound. In 1964, A. L. Dulmage and N. S. Mendelsohn[2] observed that there are gaps in the exponent set $E_n = \{\gamma(D) \mid D \in PD_n\}$, where PD_n is the set of all primitive digraphs of order n. In 1981, M. Lewin and Y. Vitek[3] found a general method for determining all the gaps between $\lfloor \frac{w_n}{2} \rfloor + 1$ and w_n, and they conjectured that there is no gap in $\{1, 2, \cdots, \lfloor \frac{w_n}{2} \rfloor + 1\}$. In 1985, Shao Jiayu[4] proved that the conjecture is true when n is sufficiently large and gave a counterexample to show that the conjecture is not true in the case when $n = 11$. In 1987, Zhang Kemin[7] proved that the conjecture is true except 48 for $n = 11$. Therefore, the problem of determining the exponent set is completely solved.

The local exponent of D at vertex $u \in V$, denoted by $\exp_D(u)$, is the least integer k such that there is a directed walk of length k from u to v for each $v \in V$. Let $V = \{1, 2, \cdots, n\}$.

Manuscript received September 30, 1998. Revised October 25, 1999.

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***Project supported by the National Natural Science Foundation of China and the Jiangsu Provincial Natural Science Foundation of China.

Following [1], we order the vertices of V so that $\exp_D(1) \le \exp_D(2) \le \cdots \le \exp_D(n) = \gamma(D)$. Let $E_n(i) := \{\exp_D(i) \mid D \in PD_n\}$ and $L(n) = \{(p,q) \mid 2 \le p < q \le n, p+q > n, \gcd(p,q) = 1\}$. Clearly, $E_n(n) = E_n$. In 1998, Shen Jian and S. Neufeld[5] obtained $ES_n(1)$. In this paper, we proved the following

Main Theorem. $E_n(2) = \{1, 2, \cdots, \frac{n^2-3n+6}{2}\} \cup \bigcup_{(p,q)\in L(n), q<n} \{(p-1)(q-1), (p-1)(q-1)+1, \cdots, p(q-2)+n-q+2\} \cup \bigcup_{(p,n)\in L(n)} \{(p-1)(n-1)+1, \cdots, p(n-2)+2\}$ *for all* $n \ge 2$.

§2. Determination of $E_n(2)$

Lemma 2.1.[1] *Suppose* $D \in PD_n$. *Then* $\exp_D(k) \le \exp_D(k-1)+1$ *for all* $2 \le k \le n$.

Lemma 2.2.[5] *Suppose* $D \in PD_n$, *and* $\{p,q\} \subset L(D)$ *with* $p+q \le n$. *Then* $\exp_D(1) \le \frac{n^2-3n+4}{2}$.

Lemma 2.3.[5] *Suppose* $D \in PD_n$ *with* $| L(D) | \ge 3$. *Then* $\exp_D(1) \le \frac{n^2-3n+4}{2}$.

Lemma 2.4.[5] *Let D be a primitive digraph on n vertices. If* $L(D) = \{p,q\}$, *where* $p<q$ *and* $p+q>n$, *then* $\exp_D(1) \ge \max\{(p-1)(q-1), q-1\}$.

Lemma 2.5.[5] *Let D be a primitive digraph on n vertices. Suppose* $L(D) \supset \{p,q\}$, *where* $\gcd(p,q)=1$ *and* $p<q$. *If D contains a p-cycle which intersects a q-cycle, then for all* $1 \le k \le n$, $\exp_D(k) \le p(q-2)+n-q+k$.

Lemma 2.6.[5] $E_n(1) = \{1, 2, \cdots, \frac{n^2-3n+4}{2}\} \cup \bigcup_{(p,q)\in L(n)} \{(p-1)(q-1), (p-1)(q-1)+1, \cdots, p(q-2)+n-q+1\}$.

Lemma 2.7. *For any* $2 \le k \le n$, $E_{n-1}(k-1) \subset E_n(k)$.

Proof. Suppose $m \in E_{n-1}(k-1)$. Then there exists a digraph $D_1 \in PD_{n-1}$ such that $\exp_{D_1}(k-1) = m$. Let $V(D) = \{v_1, v_2, \cdots, v_{n-1}, v_n\}$. And let $D[V(D)\backslash v_n] \cong D_1$. Further let v_n be a copy of vertex u which has exponent $\exp_{D_1}(1)$, i.e. $\exp_{D_1}(u) = \exp_{D_1}(1)$. Then $D \in PD_n$ and $\exp_D(k) = \exp_{D_1}(k-1)$. Thus $m \in E_n(k)$.

Lemma 2.8. *Let n be odd with* $n \ge 5$ *and* $\frac{n+1}{2} \le i \le n-2$. *If D is a digraph with a Hamilton cycle* $(v_1, v_2, \cdots, v_n, v_1)$ *and two additional arcs* (v_n, v_2), (v_i, v_{i+2}), *then* $\exp_D(2) = \frac{(n-2)(n-3)}{2} + n - i + 1$.

Proof. Since $R_1(v_j) = \{v_{j+1}\}$ for $1 \le j \le i-1$ or $i+1 \le j \le n-1$,

$$\exp_D(v_{i+1}) > \exp_D(v_{i+2}) > \cdots > \exp_D(v_{n-1}) > \exp_D(v_n),$$
$$\exp_D(v_1) > \exp_D(v_2) > \cdots > \exp_D(v_i).$$

So $\exp_D(v_1) = \min\{\exp_D(v_n), \exp_D(v_i)\}$. It is easy to check that

$$| R_{(n-2)j+i}(v_n) | = \min\{n, 3+2j\} \text{ and } | R_{(n-2)j+i-1}(v_n) | = \min\{n, 2+2j\}.$$

Thus $\exp_D(v_n) = \frac{(n-2)(n-3)}{2} + i$. Similarly we get $\exp_D(v_i) = \frac{(n-2)(n-3)}{2} + n - i$. Since $i \ge \frac{n+1}{2}$, $\exp_D(v_n) > \exp_D(v_i)$. Thus $\exp_D(1) = \exp_D(v_i) = \frac{(n-2)(n-3)}{2} + n - i$ and $\exp_D(v_j) > \exp_D(v_i)$ for $j \ne i$. Hence $\exp_D(2) = \frac{(n-2)(n-3)}{2} + n - i + 1$.

Theorem 2.1. *Let positive integers p, q and n be given such that* $(p,q) \in L(n)$. *Then*

$$\bigcup_{(p,q)\in L(n), q<n} \{(p-1)(q-1), (p-1)(q-1)+1, \cdots, p(q-2)+n-q+2\}$$
$$\bigcup \bigcup_{(p,n)\in L(n)} \{(p-1)(n-1)+1, \cdots, p(n-2)+2\} \subset E_n(2).$$

Proof. We denote by C_q the cycle of the form $(v_1, v_2, \cdots, v_q, v_1)$. Let $x = (p-1)(q-1)+a$, where $0 \le a \le n-p$.

Case 1. $p = 2$. Then q is odd.

Subcase 1.1. $q = n$. Let $D = C_n \cup \{(v_i, v_{i-1}) \mid a+2 \le i \le q = n\}$. Then $\exp_D(1) = \exp_D(v_n) = (n-1)+a$ and $\exp_D(2) = \exp_D(v_{n-1}) = (n-1)+a+1$. Thus $\{n, n+1, \cdots, 2n-2\} \subset E_n(2)$.

Subcase 1.2. $q = n-1$. Let $D = (V, E)$ with $V = \{v_1, v_2, \cdots, v_{n-1}, v_n\}$. For $0 \le a < n-2$, let $D[V \backslash v_n]$ be exactly like the digraph in Subcase 1.1. And let v_n be a copy of v_{n-1}, then $\exp_D(2) = n-2+a$. If let v_n be a copy of v_{n-2}, then $\exp_D(2) = n-1+a$. For $a = n-2$, let D consist of C_{n-1} and the arcs $(v_1, v_n), (v_n, v_1)$. Then $\exp_D(2) = 2n-3$. Thus $\{n-2, n-1, \cdots, 2n-3\} \subset E_n(2)$.

Case 2. $p \ge 3$.

Subcase 2.1. $0 \le a < q-p$. Consider C_q with additional arcs $\{(v_{p+i}, v_{1+i}) \mid 0 \le i \le q-p-a\}$. For vertex $v_j (q < j \le n)$, further arcs (v_1, v_j), (v_j, v_3) and (v_{p+1}, v_j) are added. In the proof of [5,Theorem 4], it is showed that $\exp_D(v_{q-a}) = x$ and $R_l(v_i) = R_{l-1}(v_{i+1})$ for all $l \ge r_i + 2$, where $r_i = \lfloor \frac{i-p}{p-1} \rfloor$ for $p \le i \le q-a-1$. Let $r = \max\{r_i \mid p \le i \le q-a-1\} = \lfloor \frac{q-a-p-1}{p-1} \rfloor$. Since $x > r+1$,

$$R_{p(q-2)}(v_p) = R_{p(q-2)-1}(v_{p+1}) = \cdots = R_x(v_{q-a-1}) = R_{x-1}(v_{q-a}) \ne V(D),$$
$$R_{p(q-2)+1}(v_p) = R_{p(q-2)}(v_{p+1}) = \cdots = R_{x+1}(v_{q-a-1}) = R_x(v_{q-a}) = V(D).$$

Thus $\exp_D(v_i) > \exp_D(v_{q-a})$ for $p \le i \le q-a-1$. On the other hand, $R_1(v_i) = \{v_{i+1}\}$ for $2 \le i \le p-1$ and $q-a+1 \le i \le q-1$ and $R_1(v_q) = \{v_1\}$, $R_1(v_1) = \{v_2, v_{q+1}, v_{q+2}, \cdots, v_n\}$, $R_1(v_i) = \{v_2\}$ for $q+1 \le i \le n$. Then $\exp_D(v_i) > \exp_D(v_p) > \exp_D(v_{q-a})$ for $1 \le i \le p-1$ and $q+1 \le i \le n$. Hence $\exp_D(1) = \exp_D(v_{q-a}) = x$ and $\exp_D(2) = x+1$. Therefore

$$\{(p-1)(q-1)+1, (p-1)(q-1)+2, \cdots, p(q-2)+1\} \subset E_n(2).$$

Subcase 2.2. $q-p \le a \le n-p$. Let D consist of C_q and the walks

$$\{(v_q, v_{q+1}, \cdots, v_{a+p-1}, v_{a+p}, v_{a+1})\} \cup \{(v_q, v_j, v_2) \mid a+p+1 \le j \le n\}.$$

It is easy to check that $\exp_D(v_q) = x$. Since $R_1(v_j) = \{v_{j+1}\}$ for $1 \le j \le q-1$ or $q+1 \le j \le a+p-1$, $R_1(v_q) = \{v_{q+1}, v_1, v_{a+p+1}, \cdots, v_n\}$, $R_1(v_{a+p}) = \{v_{a+1}\}$ and $R_1(v_j) = \{v_2\}$ for $a+p+1 \le j \le n$,

$$\exp_D(v_1) > \exp_D(v_2) > \cdots > \exp_D(v_a) > \exp_D(v_{a+1}) > \cdots > \exp_D(v_{q-1}) > \exp_D(v_q),$$
$$\exp_D(v_{q+1}) > \exp_D(v_{q+2}) > \cdots > \exp_D(v_{a+p}) > \exp_D(v_{a+1})$$

and $\exp_D(v_j) = \exp_D(v_1)$ for $a+p+1 \le j \le n$. Thus $\exp_D(1) = \exp_D(v_q) = x$ and $\exp_D(2) = \exp_D(v_{q-1}) = x+1$. Hence

$$\{p(q-2)+2, p(q-2)+3, \cdots, p(q-2)+n-q+2\} \subset E_n(2).$$

Subcase 2.3. $q < n$. Consider C_q with additional arcs $\{(v_{p+i}, v_{1+i}) \mid 0 \le i \le q-p\}$. For vertices $v_j (q+1 \le j \le n)$, further arcs (v_{q-1}, v_j), (v_j, v_1) and (v_j, v_{q-p+1}) are added. It is analogous to the Subcase 2.1 that we can get $\exp_D(1) = \exp_D(v_q) = \exp_D(v_n)$. Thus $\exp_D(2) = (p-1)(q-1)$.

Combining Cases 1 and 2, the proof of the theorem is completeed.

Theorem 2.2. $\{1, 2, \cdots, \frac{n^2-3n+6}{2}\} \subset E_n(2)$ *for* $n \ge 4$.

Proof. By Lemmas 2.6 and 2.7, $\{1, 2, \cdots, \frac{n^2-5n+8}{2}\} \subset E_n(2)$.

Case 1. n is odd.

Let $p = \frac{n-1}{2}, q = n$. Then $\{\frac{n^2-4n+5}{2}, \frac{n^2-4n+7}{2}, \cdots, \frac{n^2-3n+6}{2}\} \subset E_n(2)$ by Theorem 2.1. Let $p = \frac{n-1}{2}, q = n-2$. Then $\{\frac{n^2-6n+9}{2}, \frac{n^2-6n+11}{2}, \cdots, \frac{n^2-5n+12}{2}\} \subset E_n(2)$ by Theorem 2.1. Let i take over all numbers in $\{\frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-2\}$ in Lemma 2.8, we get

$$\Big\{\frac{n^2-5n+12}{2}, \frac{n^2-5n+14}{2}, \cdots, \frac{n^2-4n+7}{2}\Big\} \subset E_n(2) \quad \text{for} \quad n \geq 5.$$

Case 2. n is even.

Let $p = \frac{n}{2}, q = n-1$. Then $\{\frac{n^2-4n+4}{2}, \frac{n^2-4n+6}{2}, \cdots, \frac{n^2-3n+6}{2}\} \subset E_n(2)$ by Theorem 2.1.

Subcase 2.1. $n \equiv 0 \pmod 4$. Let $p = \frac{n-2}{2}, q = n$. Then

$$\Big\{\frac{n^2-5n+6}{2},\ \frac{n^2-5n+8}{2},\ \cdots,\ \frac{n^2-4n+8}{2}\Big\} \subset E_n(2)$$

by Theorem 2.1.

Subcase 2.2. $n \equiv 2 \pmod 4$. Let $p = \frac{n}{2}, q = n-2$. Then

$$\Big\{\frac{n^2-5n+6}{2}, \frac{n^2-5n+8}{2}, \cdots, \frac{n^2-4n+8}{2}\Big\} \subset E_n(2)$$

by Theorem 2.1.

Combining Cases 1 and 2, the proof is completed.

Theorem 2.3. *Let $(p,n) \in L(n)$. If $p \geq 3$, then there is no $D \in PD(n)$ such that $L(D) = \{p,n\}$ and $\exp_D(2) = (p-1)(n-1)$.*

Proof. Let D_0 be the digraph on n vertices with $L(D_0) = \{p,n\}$ such that the number of arcs in D_0 is as much as possible. Then

$$D_0 \cong C_n \cup \{(v_{p+i}, v_{1+i}) \mid 0 \leq i \leq n-p\}.$$

Since $p \geq 3$, it is easy to verify that $\exp_{D_0}(2) = (p-1)(n-1)+1$. Suppose D be any digraph on n vertices with $L(D) = \{p,n\}$. Then D is a subdigraph of D_0. Thus

$$\exp_D(2) \geq \exp_{D_0}(2) = (p-1)(n-1)+1.$$

Remark. Theorem 2.3 is not true for $p = 2$. To see this, we consider the digraph $D = (v_1, v_2, \cdots, v_n, v_1) \cup (v_n, v_{n-1}, \cdots, v_1, v_n)$. It is obvious that $L(D) = \{2,n\}$ and $\exp_D(2) = n-1$.

Proof of Main Theorem. Combining Lemmas 2.1–2.5, Theorems 2.1–2.3 and $n-1 \leq \frac{n^2-3n+6}{2}$, the Main Theorem is true for all $n \geq 4$. On the other hand, it is easy to verify that $E_2(2) = \{1,2\}$ and $E_3(2) = \{1,2,3,4\}$. The Main Theorem is also true for $n = 2,3$. So the proof of the Main Theorem is completed.

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