Chin. Ann. of Math.
21B: 2(2000),237-248.

THE ($\mathcal{U}$+$\mathcal{K}$)-ORBIT OF ESSENTIALLY NORMAL OPERATORS AND COMPACT PERTURBATIONS OF STRONGLY IRREDUCIBLE OPERATORS

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Abstract

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, $T \in \mathcal{L}(\mathcal{H})$. $(\mathcal{U}+\mathcal{K})(T)$ denotes the $(\mathcal{U}+\mathcal{K})$-orbit of T, i.e., $(\mathcal{U}+\mathcal{K})(T) = \{R^{-1}TR :\ R$ is invertible and of the form unitary plus compact$\}$. Let Ω be an analytic and simply connected Cauchy domain in $\mathbb{C}$ and $n \in \mathbb{N}$. $\mathcal{A}(\Omega, n)$ denotes the class of operators, each of which satisfies

(i) T is essentially normal; (ii) $\sigma(T) = \overline{\Omega}$, $\rho_F(T) \cap \sigma(T) = \Omega$;

(iii) $\operatorname{ind}(\lambda - T) = -n$, $\operatorname{nul}(\lambda - T) = 0$ $(\lambda \in \Omega)$.

It is proved that given $T_1, T_2 \in \mathcal{A}(\Omega, n)$ and $\epsilon > 0$, there exists a compact operator K with $\|K\| < \epsilon$ such that $T_1 + K \in (\mathcal{U}+\mathcal{K})(T_2)$. This result generalizes a result of P. S. Guinand and L. Marcoux[6,15]. Furthermore, the authors give a character of the norm closure of $(\mathcal{U}+\mathcal{K})(T)$, and prove that for each $T \in \mathcal{A}(\Omega, n)$, there exists a compact (SI) perturbation of T whose norm can be arbitrarily small.

Keywords Essentially normal, $(\mathcal{U}+\mathcal{K})$–orbit, Compact perturbation, Spectrum, Strongly irreducible operator
1991 MR Subject Classification 47A55, 47B, 47C
Chinese Library Classification O177.1 **Document Code** A
Article ID 0252-9599(2000)02-0237-12

§1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space. Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebra of bounded linear operators and, respectively, the ideal of compact operators acting on $\mathcal{H}$. We will call $(\mathcal{U}+\mathcal{K})(T) = \{R^{-1}TR :\ R \in (\mathcal{U}+\mathcal{K})(\mathcal{H})\}$ the $(\mathcal{U}+\mathcal{K})$-orbit of T, where

$$(\mathcal{U}+\mathcal{K})(\mathcal{H}) = \left\{ R \in \mathcal{L}(\mathcal{H}) : \begin{array}{l} R \text{ is invertible and of the form unitary} \\ \text{operator plus compact operator} \end{array} \right\}.$$

$A \sim_{\mathcal{U}+\mathcal{K}} T$ and $T \rightarrow_{\mathcal{U}+\mathcal{K}} B$ imply $A \in (\mathcal{U}+\mathcal{K})(T)$ and, respectively, $B \in (\mathcal{U}+\mathcal{K})(T)^-$, the norm closure of $(\mathcal{U}+\mathcal{K})(T)$. While $\sim_{\mathcal{U}+\mathcal{K}}$ defines an equivalence relation, $\rightarrow_{\mathcal{U}+\mathcal{K}}$ does not. An operator is strongly irreducible, or briefly, $T \in (SI)$, if it does not commute with any nontrivial idempotent. An operator is essentially normal if $[T, T^*] := T^*T - TT^* \in \mathcal{K}(\mathcal{H})$. An operator T is said to be shift-like if T is essentially normal with $\sigma(T) = \overline{D} = \{z \in \mathbb{C} :$

Manuscript received September 4, 1998. Revised October 10, 1999.

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$|z| \leq 1\}$ and $\sigma_e(T) = \partial D$ with $\operatorname{ind}(\lambda - T) = -1$ and $\operatorname{nul}(\lambda - T) = 0$ for all $\lambda \in D$. P. S. Guinand and L. Marcoux[6,15] proved the following

Theorem G-M. *Let T_1, T_2 be shift-like and let ϵ be a positive number. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T_1 + K \sim_{\mathcal{U}+\mathcal{K}} T_2$.*

In this paper we will strengthen the above theorem.

Let Ω be an analytic and simply connected Cauchy domain in $\mathbb{C}$ and $n \in \mathbb{N}$. Then $\mathcal{A}(\Omega, n)$ will denote the class of operators, each of which satisfies

(i) T is essentially normal; (ii) $\sigma(T) = \overline{\Omega}$, $\rho_F(T) \cap \sigma(T) = \Omega$;

(iii) $\operatorname{ind}(\lambda - T) = -n$, $\operatorname{nul}(\lambda - T) = 0$ for all $\lambda \in \Omega$.

The next three results are our main results.

Theorem 1.1. *Given $T_1, T_2 \in \mathcal{A}(\Omega, n)$ and $\epsilon > 0$, there exists a compact operator K with $\|K\| < \epsilon$ such that $T_1 + K \sim_{\mathcal{U}+\mathcal{K}} T_2$.*

Theorem 1.2. *Let T be in $\mathcal{A}(\Omega, n)$. Then $(\mathcal{U} + \mathcal{K})(T)^-$ consists of all operators A satisfying*

(i) *A is essentially normal;* (ii) *$\sigma(A) = \overline{\Omega}$, $\rho_F(A) \cap \sigma(A) = \Omega$;*

(iii) *$\operatorname{ind}(\lambda - A) = -n$ for all $\lambda \in \Omega$.*

Theorem 1.3. *Let $T \in \mathcal{A}(\Omega, n)$ and $\epsilon > 0$. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K \in (SI)$.*

Theorem 1.3 partially answers an interesting question posed by D. A. Herrero: Given an essentially normal operator T with connected spectrum and $\epsilon > 0$, is there a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$, such that $T + K \in (SI)$?

§2. Proof of the Main Theorems

It follows from the Riemann mapping theorem that there is an analytic function ϕ satisfying

(i) $\phi(0) = z_0$, where z_0 is a fixed point in Ω; (ii) ϕ is injective; (iii) $\phi(D) = \Omega$.

By Schwarz reflection principle ϕ has an analytic continuation on $\overline{D}$ such that $\phi(\partial D) = \partial\Omega$. Let T_ϕ be the Toeplitz operator with symbol ϕ. Then $T_\phi^* \in \mathcal{B}_1(\Omega^*)$ and $\sigma(T_\phi^*) = \overline{\Omega}^*$, where $\mathcal{B}_n(\Omega)$ denotes the set of Cowen-Douglas operators of index n, i.e., $\mathcal{B}_n(\Omega)$ consists of operators B satisfying

(i) $\sigma(B) \supset \Omega$; (ii) $\operatorname{ran}(\lambda - B) = \mathcal{H}$ for all $\lambda \in \Omega$;

(iii) $\operatorname{nul}(\lambda - B) = n$ for all $\lambda \in \Omega$; (iv) $\bigvee \{ \ker(\lambda - B) : \lambda \in \Omega \} = \mathcal{H}$.

(iv) can be replaced by (iv)$'$.

(iv)$'$ $\bigvee\{\ker(\lambda_0 - B)^k : k = 1, 2, \cdots\} = \mathcal{H}$ for a fixed $\lambda_0 \in \Omega$ (see [3]).

In order to prove Theorems 1.1 and 1.2, we need several lemmas.

Lemma 2.1. *Let $\mathcal{M} = (\ker(T_\phi - \lambda)^*)^\perp$. Then $T_\phi|_{\mathcal{M}}$ is unitarily equivalent to T_ϕ, where $\lambda \in \Omega$.*

Proof. Since $\mathcal{M}$ is a hyperinvariant subspace of T_ϕ, $\mathcal{M}$ is an invariant subspace of T_z. By Beurling theorem, $\mathcal{M} = T_g H^2$, where g is an inner function and H^2 is the Hardy space. Since $T_\phi^* \in \mathcal{B}_1(\Omega^*)$, $g = (z - a)/(1 - a^* z)$, $(0 \leq |a| < 1)$. Thus T_g can be considered as a unitary operator from H^2 to $\mathcal{M}$ and $T_g^*(T_g|_{\mathcal{M}})T_g f = T_\phi f$ for $f \in H^2$, i.e., $T_g^*(T_\phi|_{\mathcal{M}})T_g = T_\phi$.

Lemma 2.2. *Let $\mathcal{H} = \mathbb{C} \oplus H^2$ and $\lambda \in \Omega$ and let $T = \begin{bmatrix} \lambda & 0 \\ E & T_\phi \end{bmatrix} \in \mathcal{L}(\mathcal{H})$, where $E = \alpha e_0 \otimes 1$, $e_0 \in \ker(\lambda - T_\phi)^*$ and $\|e_0\| = 1$, $\alpha \in \mathbb{C}$ and $\alpha \neq 0$. Then $T \sim_{\mathcal{U}+\mathcal{K}} T_\phi$.*

Proof. Note that $e_0 \in \operatorname{ran}(T_\phi - \lambda)^\perp$. By Lemma 2.1,

$$T_\phi = \begin{bmatrix} \lambda & 0 \\ g_1 & T_\phi|_{\mathcal{M}} \end{bmatrix} \begin{matrix} e_0 \\ \mathcal{M} = [\ker(T_\phi - \lambda)^*]^\perp \end{matrix} \simeq \begin{bmatrix} \lambda & 0 \\ g_2 & T_\phi \end{bmatrix} \begin{matrix} \mathbb{C} \\ H^2 \end{matrix},$$

where g_1, g_2 are rank-1 operators and we assume that $g_2 = f \otimes 1$, $f \in H^2$. Since $e_0 \in \operatorname{ran}(T_\phi - \lambda)^\perp$, $f = \beta e_0 + g_3$, where $\beta \in \mathbb{C}$ and $g_3 \in \operatorname{ran}(T_\phi - \lambda)$. If $\beta = 0$, let $g_3 = (T_\phi - \lambda)x$, then computation shows that

$$\begin{bmatrix} 1 & 0 \\ x \otimes 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ g_2 & T_\phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x \otimes 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & T_\phi \end{bmatrix}.$$

This is contradictory to $T_\phi \in (SI)$. Thus $\beta \neq 0$, and $X \begin{bmatrix} \lambda & 0 \\ g_2 & T_\phi \end{bmatrix} X^{-1} = \begin{bmatrix} \lambda & 0 \\ \beta e_0 \otimes 1 & T_\phi \end{bmatrix}$, where $X = \begin{bmatrix} 1 & 0 \\ x \otimes 1 & 1 \end{bmatrix} \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$. Since $\begin{bmatrix} \lambda & 0 \\ \beta e_0 \otimes 1 & T_\phi \end{bmatrix} \sim_{\mathcal{U}+\mathcal{K}} \begin{bmatrix} \lambda & 0 \\ E & T_\phi \end{bmatrix}$, the proof of Lemma 2.2 is complete.

By the same argument of Lemma 2.2, we can prove

Lemma 2.3. *Let* $\mathcal{H} = \mathbb{C} \oplus H^2$, $g \notin \operatorname{ran}(T_\phi - \lambda)$, $\lambda \in \Omega$ *and let* $T = \begin{bmatrix} \lambda & 0 \\ g \otimes 1 & T_\phi \end{bmatrix} \in \mathcal{L}(\mathcal{H})$. *Then* $T \sim_{\mathcal{U}+\mathcal{K}} T_\phi$.

Lemma 2.4. *Let* $T \in \mathcal{L}(\mathbb{C}^n \oplus H^2)$ *and* $T = \begin{bmatrix} F & 0 \\ C_n & T_\phi \end{bmatrix} \begin{matrix} \mathbb{C}^n \\ H^2 \end{matrix}$, *where* $F \in \mathcal{L}(\mathbb{C}^n)$ *with* $\sigma(F) \subset \Omega$, $C_n \in \mathcal{L}(\mathbb{C}^n, H^2)$ *is a nonzero operator. Then for each* $\epsilon > 0$, *there exists a compact operator* K, $\|K\| < \epsilon$, *such that* $T + K \sim_{\mathcal{U}+\mathcal{K}} T_\phi$.

Proof. We will prove the lemma by induction on n.

If $n = 1$, $T = \begin{bmatrix} \lambda & 0 \\ C_1 & T_\phi \end{bmatrix}$, where $C_1 = f \otimes 1$ is a rank-1 operator. Choose $g \in H^2$, $\|g\| < \epsilon$, such that $f + g \notin \operatorname{ran}(T_\phi - \lambda)$. Set $K = \begin{bmatrix} 0 & 0 \\ g \otimes 1 & 0 \end{bmatrix}$. Then $\|K\| < \epsilon$ and by Lemma 2.3, $T + K \sim_{\mathcal{U}+\mathcal{K}} T_\phi$.

Assume that the conclusion of the lemma is true when $n \leq k - 1$, and let $n = k$. It is obvious that there is a unitary operator U such that

$$UTU^* = \begin{bmatrix} \lambda & 0 & 0 \\ * & F_{n-1} & 0 \\ * & C_{n-1} & T_\phi \end{bmatrix} \begin{matrix} \mathbb{C} \\ \mathbb{C}^{n-1} \\ H^2 \end{matrix},$$

where $F_{n-1} \in \mathcal{L}(\mathbb{C}^{n-1})$, $\lambda \in \sigma(F)$. By the inductive assumption, we can find a compact operator K_1 with $\|K_1\| < \epsilon/2$ and $X_1 \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$ such that

$$X_1(UTU^* + K_1)X_1^{-1} = \begin{bmatrix} \lambda & 0 \\ C_1 & T_\phi \end{bmatrix}.$$

By the assumption again, we can find a compact operator K_2 such that $\|K_2\| < \frac{\epsilon}{(2\|X_1\|\|X_1^{-1}\|)}$ and $X_2 \left(\begin{bmatrix} \lambda & 0 \\ C_1 & T_\phi \end{bmatrix} + K_2 \right) X_2^{-1} = T_\phi$ for some $X_2 \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$. Thus $K = U^* K_1 U + U^* X_1^{-1} K_2 X_1 U$ satisfies the requirement of the lemma.

Lemma 2.5. *Given* A, $B \in \mathcal{L}(\mathcal{H})$, *let* τ_{AB} *denote the Rosenblum operator given by* $\tau_{AB}(X) = AX - XB$, $X \in \mathcal{L}(\mathcal{H})$. *Let* $\tau = \tau_{AB}|_{\mathcal{K}(\mathcal{H})}$. *Then* $\tau^* = -\tau_{BA}|_{\mathcal{C}^1(\mathcal{H})}$ *and* $(\tau^*)^* = \tau_{AB}$, *where* τ^* *is the dual of* τ *and* $\mathcal{C}^1(\mathcal{H})$ *is the set of trace class operators.*

Proof. Recall that $C^1(\mathcal{H})$ is isometrically isomorphic to the dual $\mathcal{K}(\mathcal{H})^*$ of $\mathcal{K}(\mathcal{H})$. This isomorphism is defined by $C^1(\mathcal{H}) \ni K \leftrightarrow \phi_K \in \mathcal{K}(\mathcal{H})^*$, where $\phi_K(X) = \operatorname{tr}(KX)$, $X \in \mathcal{K}(\mathcal{H})$.

Thus

$$\phi_{\tau^*(K)}(X) = \phi_K(\tau(X)) = \mathrm{tr}\,[K(AX - XB)] = \mathrm{tr}\,(KAX) - \mathrm{tr}\,(KXB)$$
$$= \mathrm{tr}\,(KAX) - \mathrm{tr}\,(BKX) = \mathrm{tr}\,(-\tau_{BA}(K)X) = \phi_{-\tau_{BA}(K)}(X).$$

Therefore $\tau^* = -\tau_{BA}\big|_{C^1(\mathcal{H})}$.

Since $\mathcal{L}(\mathcal{H})$ is isometrically isomorphic to the dual $C^1(\mathcal{H})^*$ of $C^1(\mathcal{H})$, by the similar arguments we can prove that $(\tau^*)^* = \tau_{AB}$.

Lemma 2.6. *Given $T \in \mathcal{A}(\Omega, n)$ and $\epsilon > 0$, then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K \sim_{\mathcal{U}+\mathcal{K}} T_\phi^{(n)}$.*

Proof. Note that T_ϕ admits a lower triangular matrix representation with respect to the $ONB\,\{e^{ik\theta}\}_{k=0}^{\infty}$ of H^2. Set $\mathcal{M}_k = \bigvee\{e^{ij\theta} : j = 0, 1, \cdots, k\}$ and denote by P_k the orthogonal projection onto $\bigoplus\limits_{k=1}^{n} \mathcal{M}_k$. By Brown-Douglas-Fillmore theorem $UTU^* = T_\phi^{(n)} + K$, where U is a unitary operator and K is compact. Set $K_1 = P_m K P_m - K$, and m will be determined later. By Lemma 2.1, we can find a unitary operator U_1 such that

$$U_1(UTU^* + K_1)U_1^* = \begin{bmatrix} F & 0 & \dots & 0 \\ C_1 & T_\phi & & \\ \vdots & & \ddots & 0 \\ C_n & & 0 & T_\phi \end{bmatrix},$$

where $F \in \mathcal{L}(\mathbb{C}^{nm})$, C_k is a finite rank operator $(k = 1, \cdots, n)$. Fix m so that $\|K_1\| < \epsilon/8$ and $\sigma(F) \subset \Omega_{\epsilon/8}$. Thus we can find an operator $C \in \mathcal{L}(\mathbb{C}^{nm})$ such that $\|C\| < \epsilon/4$ and $\sigma(F') \subset \Omega$, where $F' = F + C$. Therefore there exists a compact operator K_2 with $\|K_2\| < \epsilon/4$ such that

$$U_1(UTU^* + K_1)U_1^* + K_2 = \begin{bmatrix} F' & 0 & \dots & 0 \\ C_1 & T_\phi & & \\ \vdots & & \ddots & 0 \\ C_n & & 0 & T_\phi \end{bmatrix} = A_1.$$

By Lemma 2.4, there is an $X_1' \in (\mathcal{U} + \mathcal{K})(\mathbb{C}^{nm} \oplus H^2)$ such that $X_1' \begin{bmatrix} F' & 0 \\ C_1 & T_\phi \end{bmatrix} X_1'^{-1} = T_\phi$. Thus we can find $X_1 \in (\mathcal{U} + \mathcal{K})(\mathbb{C}^{nm} \oplus (H^2)^{(n)})$ such that

$$X_1 A_1 X_1^{-1} = \begin{bmatrix} T_\phi & 0 & \dots & 0 \\ C_1' & T_\phi & & \\ \vdots & & \ddots & 0 \\ C_{n-1}' & & 0 & T_\phi \end{bmatrix} = A_2,$$

where C_k' $(k = 1, 2, \cdots, n-1)$ is a finite rank operator. Since $\mathcal{A}'(T_\phi)$ does not contain any compact operators, $\ker \tau_{T_\phi T_\phi}|_{\mathcal{K}(H^2)} = \{0\}$. By Lemma 2.5 $[\mathrm{ran}\,\tau_{T_\phi T_\phi}|_{C^1(H^2)}]^- = C^1(H^2)$. Thus for each k $(1 \le k \le n-1)$ we can find compact operators D_k and E_k satisfying $T_\phi E_k - E_k T_\phi = -C_k' - D_k$ and $\|D_k\| < \epsilon/(\|X_1\|\|X_1^{-1}\|8^k)$.

Set

$$K_3 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ D_1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ D_{n-1} & 0 & \dots & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ E_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ E_{n-1} & 0 & \dots & 1 \end{bmatrix}.$$

Then K_3 is compact, $\|K_3\| < \epsilon/(8\|X_1\|\|X_1^{-1}\|)$ and $X_2 \in (\mathcal{U}+\mathcal{K})((H^2)^{(n)})$. Calculation indicates that $X_2(A_2+K_3)X_2^{-1} = T_\phi^{(n)}$. Thus, $X(T+K)X^{-1} = T_\phi^{(n)}$, where $K = U^*K_1U + U^*U_1^*K_2U_1U + X_1^{-1}K_3X_1$ is a compact operator with $\|K\| < \epsilon$, $X = X_2X_1U_1U$ is invertible and of the form unitary plus compact.

Lemma 2.6 implies that $\mathcal{A}(\Omega, n) \subset (\mathcal{U}+\mathcal{K})(T_\phi^{(n)})^-$.

Lemma 2.7. *Let $A, B \in \mathcal{L}(\mathcal{H})$. Assume that $\mathcal{H} = \bigvee\{\ker(\lambda - B)^k : \lambda \in \Gamma, k \geq 1\}$ for a certain subset Γ of the point spectrum $\sigma_p(B)$ of B, and $\sigma_p(A) \cap \Gamma = \varnothing$; then τ_{AB} is injective.*

Proof. Let p be a monic polynomial with zeros in Γ and let $x \in \mathcal{H}$ be any vector such that $p(B)x = 0$; then $AX = XB$ implies $p(A)Xx = Xp(B)x = 0$.

Since $p(A)$ is injective, we infer $Xx = 0$. It readily follows that $\ker X \supset \bigvee\{\ker(\lambda - B)^k : \lambda \in \Gamma, k \geq 1\} = \mathcal{H}$. Hence, $X = 0$.

Lemma 2.8.[1,Theorem 4.15] Suppose that $T \in \mathcal{L}(\mathcal{H})$ is essentially normal and $\sigma(T) = \sigma_e(T) \cup \sigma_0(T)$. Assume, moreover, that $C(\sigma(T)) = \mathrm{Rat}(\sigma(T))^-$. Then $T \to_{\mathcal{U}+\mathcal{K}} N$, where N is a normal operator such that $\sigma(N) = \sigma(T)$, $\sigma_0(T) = \sigma_0(N)$, and $\mathrm{nul}\,(\lambda - N) = \dim \mathcal{H}(\lambda; T)$ for all $\lambda \in \sigma_0(N)$, $\mathrm{Rat}\,(\sigma(T))^-$ is the uniform closure of rational functions with poles outside $\sigma(T)$.

An operator is almost normal if it has the form of normal + compact.

Lemma 2.9. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is almost normal, $\sigma(T)$ is a perfect set with Lebesgue measure* 0, *N is normal with $\sigma(N) = \sigma(T)$; then for each $\epsilon > 0$, there exists $X \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$ such that*

(i) $XTX^{-1} - N \in \mathcal{K}(\mathcal{H})$; (ii) $\|XTX^{-1} - N\| < \epsilon$.

Proof. Since $m(\sigma(T)) = 0$, it follows from [8] that $\mathrm{Rat}(\sigma(T))^- = C(\sigma(T))$. By Lemma 2.8, for each $\delta > 0$ there is an $X_1 \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$ such that $\|X_1TX_1^{-1} - N\| < \delta$. Since T is almost normal, $X_1TX_1^{-1}$ is also almost normal. Therefore there are normal operator M and compact operator K such that $X_1TX_1^{-1} = M + K$. Since $\|M + K - N\| < \delta$, $\|[(M+K)^*, (M+K)]\| < 4\|N\|\delta + 2\delta^2$. I. D. Berg and K. R. Davidson[2] asserts that there exists a positive valued continuous function f on $[0, \infty)$, $f(0) = 0$, such that for each almost normal operator Q, there is a compact operator $K(Q)$ with $\|K(Q)\| \leq f\big(\|Q^*Q - QQ^*\|^{1/2}\big)$ and $Q + K(Q)$ is normal. According to this theorem, we can find a δ and a compact operator K_1 such that

(i) $\|K_1\| \leq \epsilon/4$; (ii) $X_1TX_1^{-1} + K_1 = M + K + K_1$ is normal;
(iii) $\sigma(M + K + K_1) \subset \sigma(T)_{\epsilon/4}$.

Since T is almost normal and $\sigma(T)$ is a perfect set with $m(\sigma(T)) = 0$, $\sigma(T) = \sigma_e(T)$. Thus $\sigma(M) = \sigma(T) \cup \sigma_0(M)$. Since $M+K+K_1$ is normal, $\sigma(M+K+K_1) = \sigma(T) \cup \sigma_0(M+K+K_1)$. Thus we can find a compact operator K_2, $\|K_2\| < \epsilon/2$, such that $\sigma(M + K + K_1 + K_2) = \sigma(T) = \sigma(N)$ and $M + K + K_1 + K_2$ is normal. It follows from Voiculescu theorem that there is a unitary operator U and a compact operator K_3 with $\|K_3\| < \epsilon/2$ such that

$$U(X_1TX_1^{-1} + K_1 + K_2)U^* = U(M + K + K_1 + K_2)U^* = N + K_3.$$

Thus $UX_1TX_1^{-1}U^* - N \in \mathcal{K}(\mathcal{H})$, $\|UX_1TX_1^{-1}U^* - N\| < \epsilon$, and $X := UX_1$ satisfies the requirements of the lemma.

Let $T \in \mathcal{L}(\mathcal{H})$. λ is an approximate normal eigenvalue of T if for each $\epsilon > 0$, there is a unit vector e_ϵ such that $\|(\lambda - T)e_\epsilon\| < \epsilon$ and $\|(\lambda - T)^*e_\epsilon\| < \epsilon$. If $\lambda \in \sigma_{lre}(T) \subset \sigma_{le}(T)$, by Apostol-Foiaş-Voiculescu Theorem[9], there exist a compact operator K and an infinite dimensional subspace $\mathcal{H}_1$ such that $T = \begin{bmatrix} \lambda & E \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix} + K$. Let $\{e_n\}_{n=1}^\infty$ be an ONB of $\mathcal{H}_1$. Since $e_n \to 0$ weakly, $\|Ke_n\| \to 0$ $(n \to \infty)$, i.e. $\|(\lambda - T)e_n\| \to 0$. If T is a hyponormal operator, then $\|(\lambda - T)e_n\|^2 - \|(\lambda - T)^* e_n\|^2 \to 0$ $(n \to \infty)$. Thus λ is an approximate normal eigenvalue. Thus we get the following proposition.

Proposition 2.1. *Assume that T is essentially normal. Then $\sigma_{lre}(T)$ is contained in the set of approximate normal eigenvalues of T.*

Using Propostion 2.1, we get

Theorem 2.1. *Given T, $N \in \mathcal{L}(\mathcal{H})$, such that $\sigma_{lre}(T)$ is contained in the set of approximate normal eigenvalues of T, N is a normal operator with $\sigma(N) = \sigma_e(N) = \Gamma$, $\Gamma \subset \sigma_{lre}(T)$ and given $\epsilon > 0$, there exists a compact operator K with $\|K\| < \epsilon$ such that*

$$T - K \simeq \begin{bmatrix} N & 0 \\ 0 & T \end{bmatrix}.$$

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers such that $\{\lambda_n\}^- = \Gamma$ and $\operatorname{card}\{m : \lambda_m = \lambda_n\} = \infty$ for each $n = 1, 2, \cdots$. By the definition of approximate normal eigenvalue, there exists a unit vector e_1 such that $\|(\lambda_1 - T)e_1\| < \epsilon/16$ and $\|(\lambda_1 - T)^* e_1\| < \epsilon/16$. Thus, we have

$$\|P_{e_1}(\lambda_1 - T)P_{e_1}\| < \epsilon/16, \quad \|P_{e_1}(\lambda_1 - T)P_{e_1^\perp}\| = \|P_{e_1^\perp}(\lambda_1 - T)^* P_{e_1}\| < \epsilon/16,$$

and $\|P_{e_1^\perp}(\lambda_1 - T)P_{e_1}\| < \epsilon/16$, where P_{e_1} and $P_{e_1^\perp}$ denote the orthogonal projections onto $\bigvee\{e_1\}$ and, respectively, $\left[\bigvee\{e_1\}\right]^\perp$. Under the decomposition $\mathcal{H} = \bigvee\{e_1\} \bigoplus \left[\bigvee\{e_1\}\right]^\perp$, T admits the representation $T = \begin{bmatrix} \lambda_1 + t_{11} & T_{12} \\ T_{21} & T_1 \end{bmatrix}$. Set $K_1 = \begin{bmatrix} t_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}$; then $K_1 \in \mathcal{K}(\mathcal{H})$, $\|K_1\| < \epsilon/8$ and $T - K_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & T_1 \end{bmatrix}$. Clearly, $\sigma_{lre}(T_1) = \sigma_{lre}(T)$. Repeat the argument, we can get $K_2 \in \mathcal{K}(\mathcal{H})$, $\|K_2\| < \epsilon/2^4$ such that

$$T - K_1 - K_2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & T_2 \end{bmatrix} \begin{matrix} \bigvee\{e_1\} \\ \bigvee\{e_2\} \\ \left[\bigvee\{e_1, e_2\}\right]^\perp \end{matrix}.$$

By induction, we can find an orthonormal sequence $e_1, e_2, \cdots, e_n, \cdots$ in $\mathcal{H}$ and a sequence $K_1, K_2, \cdots, K_n, \cdots$ in $\mathcal{K}(\mathcal{H})$ such that $\|K_n\| < \epsilon/2^{n+2}$ $(n = 1, 2, \cdots)$ and

$$T - \sum_{n=1}^m K_n = \begin{bmatrix} \sum\limits_{n=1}^m \lambda_n e_n \otimes e_n & 0 \\ 0 & T_{m+1} \end{bmatrix} \begin{matrix} \bigvee\{e_1, \cdots, e_m\} \\ \left[\bigvee\{e_1, \cdots, e_m\}\right]^\perp \end{matrix} \qquad m = 1, 2, \cdots.$$

Set $C_1 = \sum\limits_{n=1}^\infty K_n$, $N_1 = \sum\limits_{n=1}^\infty \lambda_n e_n \otimes e_n$; then $T - C_1 = \begin{bmatrix} N_1 & 0 \\ 0 & T_\infty \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = \bigvee\{e_n : 1 \le n < \infty\} \bigoplus \left[\bigvee\{e_n\}\right]^\perp$, and $C_1 \in \mathcal{K}(\mathcal{H})$, $\|C_1\| < \epsilon/4$. Applying Voiculescu's non-commutative Weyl-von Neumann Theorem to N_1, we can find compact operator C_2', $\|C_2'\| < \epsilon/8$, such that $N_1 + C_2' \simeq N \oplus N_1$. Therefore,

$$T - C_1 - C_2 \simeq \begin{bmatrix} N & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & T_\infty \end{bmatrix}$$

where C_2 is compact and $\|C_2\| < \epsilon/8$. Finally, we can find a compact operator C_3 with

$\|C_3\| < \epsilon/4$, such that $T - C_1 - C_2 - C_3 \simeq \begin{bmatrix} N & 0 \\ 0 & T \end{bmatrix}$. Set $K = C_1 + C_2 + C_3$. Then K satisfies all requirements of the theorem.

Lemma 2.10. *Let $T \in \mathcal{A}(D,n)$, $\epsilon > 0$. Then there exist an operator $T' \in \mathcal{A}(D,n)$ and a compact operator K such that $\|K\| < \epsilon$, T' admits a lower triangular matrix representation with respect to some ONB of $\mathcal{H}$ and $T' + K \sim_{\mathcal{U}+\mathcal{K}} T$.*

Proof. By Apostol's triangular representation theorem $T = \begin{bmatrix} T_0 & B \\ 0 & T_l \end{bmatrix} \begin{matrix} \mathcal{H}_0(T) \\ \mathcal{H}_l(T) \end{matrix}$ and

$$\mathcal{H}_0 \text{ is a hyperinvariant subspace of } T, \tag{2.1}$$

$$T_l \text{ admits a lower triangular matrix representation,} \tag{2.2}$$

$$\rho_{s-F}(T) \subset \rho(T_0). \tag{2.3}$$

By Brown-Douglas-Fillmore theorem $T \simeq T_z^{(n)} + K$, K is compact, and $\pi(T)$ is a unitary element in the Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, i.e., $\pi(T^*)\pi(T) - \pi(1) = 0$. Thus

$$\begin{bmatrix} \pi(T_0^*T_0) - \pi(1) & \pi(T_0^*B) \\ \pi(B^*T_0) & \pi(B^*B + T_l^*T_l) - \pi(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $D \subset \rho(T_0) \subset \rho(\pi(T_0))$, $\pi(T_0)$ is invertible and $\pi(T_0)^{-1} = \pi(T_0)^*$. Thus $\pi(T_0)$ is a unitary element in $\mathcal{A}(\mathcal{H}_0(T))$. Since T_0 is invertible, by Brown-Douglas-Fillmore theorem $T_0 \simeq U_0 + K_0$, where U_0 is unitary and K_0 is compact. By (2.3) $\sigma(U_0 + K_0) \subset \partial D$. Since $\pi(T)$ is a unitary element in $\mathcal{A}(\mathcal{H})$, $TT^* - 1 \in \mathcal{K}(\mathcal{H})$, i.e., $\begin{bmatrix} T_0T_0^* + BB^* - 1 & BT_l^* \\ T_lB^* & T_lT_l^* - 1 \end{bmatrix}$ is compact. Since $\pi(T_0)$ is a unitary element, $T_0T_0^* - 1$ is compact, thus BB^* and B are compact. Since $T_l^* \in \mathcal{B}_n(D)$ and $D \cap \sigma(T_0^*) = \varnothing$, by Proposition 2.1 $\ker \tau_{T_0^*T_l^*} = \{0\}$ and therefore $\ker \tau_{T_lT_0} = \{0\}$. By Lemma 2.5, $\operatorname{ran} \tau_{T_0T_l}$ is dense in $\mathcal{K}(\mathcal{H}_l, \mathcal{H}_0)$. Thus for $\delta > 0$, there exist compact operators E, $G \in \mathcal{L}(\mathcal{H}_l(T), \mathcal{H}_0(T))$ such that $T_0G - GT_l = B + E$ and $\|E\| < \delta$.

Set $X_1 = \begin{bmatrix} 1 & -G \\ 0 & 1 \end{bmatrix} \begin{matrix} \mathcal{H}_0(T) \\ \mathcal{H}_l(T) \end{matrix}$ and $K_1 = \begin{bmatrix} 0 & -E \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_0(T) \\ \mathcal{H}_l(T) \end{matrix}$. Then $X_1((T_0 \oplus T_l) + K_1)X_1^{-1} = T$, where $X_1 \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$, K_1 is compact and $\|K_1\| < \delta/2$. Since $\sigma(T_0) \subset \partial D$, Lemma 2.9 indicates that there are compact operator K_2 and $X_2 \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$ such that $\|K_2\| < \delta$ and $X_2((N_0 \oplus T_l) + K_2)X_2^{-1} = T_0 \oplus T_l$, where N_0 is a diagonal normal operator with $\sigma(N_0) = \sigma(T_0)$. By Theorem 2.1 we can find a compact operator K_3 and a unitary operator U such that $\|K_3\| < \delta$ and $U((N_0 \oplus T_l) + K_3)U^* = T_l$, i.e., $U^*(T_l - UK_3U^*)U = N_0 \oplus T_l$. Thus let $\delta = \epsilon/(4\|X_2\|\|X_2^{-1}\|)$, $K = UK_2U^* - U^2K_3U^{*2} + UX_2^{-1}K_1X_2U^*$, $T' = T_l$ satisfy the requirement of the lemma.

Lemma 2.11. *Let $T \in \mathcal{A}(D,n)$, $0 < \epsilon < 1/10$ and $\|T^*T - 1\| < \epsilon$. Then there exists a unitary operator W such that $W^*TW - T_z^{(n)}$ is compact and $\|W^*TW - T_z^{(n)}\| < \epsilon$.*

Proof. Consider the polar decomposition $T = U|T|$ of T, where U is a partial isometry, $\operatorname{ran} U = \operatorname{ran} T$. By the Wold decomposition theorem (see [17]), $U \simeq T_z^{(n)} \oplus V$, where V is a unitary operator. By the assumption $\||T|^2 - 1\| < \epsilon$, $|z^2 - 1| < \epsilon$ for all $z \in \sigma(|T|)$. Since $\epsilon < 1/10$, $z > 1/2$ and $\epsilon > |z^2 - 1| = |z + 1||z - 1| > 3|z - 1|/2$. Thus $\sigma(|T|) \subset \{x \in \mathbb{R} : |x - 1| < 2\epsilon/3\}$. Since $|T|$ is self-adjoint, $\||T| - 1\| < 2\epsilon/3$ and therefore $\|T - U\| = \|U|T| - U\| = \|U(|T| - 1)\| \le \|U\|\||T| - 1\| < 2\epsilon/3$. Note that $T \simeq T_z^{(n)} + K$, where K is compact. Thus $|T|^2 - 1 = T^*T - 1 \simeq (T_z^{(n)} + K)^*(T_z^{(n)} + K) - 1$, and $|T|^2 = 1 + K_1$ and $|T| = 1 + K_2$ for some K_1, $K_2 \in \mathcal{K}(\mathcal{H})$.

Since V is unitary and $U \simeq T_z^{(n)} \oplus V$, by Theorem 2.1 there is a unitary operator W such that

$$\|W^*UW - T_z^{(n)}\| < \epsilon/3, \tag{2.4}$$

$$K_3 := W^*UW - T_z^{(n)} \text{ is compact.} \tag{2.5}$$

Thus $W^*TW - T_z^{(n)} = W^*(T-U)W + W^*UW - T_z^{(n)} = W^*[U(|T|-1)]W + K_3 = W^*(UK_2)W + K_3$. It is compact, and

$$\|W^*TW - T_z^{(n)}\| \le \|W^*(T-U)W\| + \|K_3\| \le \|T-U\| + \frac{\epsilon}{3} < \epsilon.$$

Proposition 2.2. *Let $T \in \mathcal{A}(D,n)$ admit a lower triangular matrix representation $T = \begin{bmatrix} F & 0 \\ X & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$, where, F is a diagonal operator on a finite dimensional Hilbert space $\mathcal{H}_1$, $S \in \mathcal{A}(D,n)$ and $\|S^*S - 1\| < \delta < 1/10$. Then there exists $Q \in (U+K)(T)^-$ satisfying* $\operatorname{dist}(Q, \mathcal{U}(T_z^{(n)})) < 4\delta$ *and $Q - W^*(T_z^{(n)})W \in \mathcal{K}(\mathcal{H})$, for some unitary operator W.*

Proof. Case 1. Assume that $\sigma(F) \subset \{\lambda : |\lambda| < 1/(1+3\delta)\}$. We will proceed by induction on the dimension m of $\mathcal{H}_1$ to prove that there exists $Q \in (\mathcal{U}+\mathcal{K})(T)$ such that $\operatorname{dist}(Q, \mathcal{U}(T_z^{(n)})) < \delta$ and $Q - W^*T_z^{(n)}W$ is compact for some unitary W.

When $m = 1$, $T = \begin{bmatrix} \lambda & 0 \\ x \otimes e & S \end{bmatrix} \begin{matrix} \bigvee\{e\} = \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$, where $x \in \mathcal{H}_2$ and $x \neq 0$ ($x = 0$ is contradictory to $T \in \mathcal{A}(D,n)$). If $x \in \operatorname{ran}(S-\lambda)$, computation shows that $T \sim \lambda \oplus S$. This is also contradictory to $T \in \mathcal{A}(D,n)$. Thus $x \notin \operatorname{ran}(S-\lambda)$. For $z \otimes e \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$,

$$T \sim_{\mathcal{U}+\mathcal{K}} \begin{bmatrix} 1 & 0 \\ z \otimes e & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ x \otimes e & S \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z \otimes e & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ y \otimes e & S \end{bmatrix},$$

where $y = x + (\lambda - S)z$. Since $\lambda - S$ is a Fredholm operator, we always can choose z so that $y \in \ker(\lambda - S)^*$ and $y \neq 0$. For $\alpha \in \mathbb{C}$ ($\alpha \neq 0$) and $\omega \in \mathcal{H}_2$, we have

$$T \sim_{\mathcal{U}+\mathcal{K}} \begin{bmatrix} 1 & 0 \\ \omega \otimes e & 1 \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ y \otimes e & S \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\omega \otimes e & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ v \otimes e & S \end{bmatrix},$$

where $v = \alpha y + (\lambda - S)\omega$.

Set $Q_1 = \begin{bmatrix} \lambda & 0 \\ v \otimes e & S \end{bmatrix}$. We will choose adequate α and ω so that $\|Q_1^*Q_1 - 1\| < \delta$. Note that

$$Q_1^*Q_1 = \begin{bmatrix} |\lambda|^2 + (e \otimes v)(v \otimes e) & e \otimes (S^*v) \\ (S^*v) \otimes e & S^*S \end{bmatrix}.$$

We intend to choose α and ω so that

$$|\lambda|^2 + \|v\|^2 - 1 = 0 \quad \text{or} \quad \|v\| = \sqrt{1 - |\lambda|^2}, \tag{2.6}$$

$$S^*v = 0 \tag{2.7}$$

and $\alpha \neq 0$.

Since S^* is a Fredholm operator and $\operatorname{nul} S = 0$, S^*S is invertible. Set $S^*S = A$. Thus

$$S^*v = S^*[\alpha y + (\lambda - S)\omega] = \alpha S^*y + (\lambda S^* - A)\omega = \alpha S^*y + A(\lambda A^{-1}S^* - 1)\omega.$$

Since $A > 0$, $\sigma(A) \subset (1-\delta, 1+\delta)$. Therefore $\|A^{-1}\| < 1/(1-\delta)$. It follows from $\|S^*\| \le 1+\delta$ that

$$\|A^{-1}S^*\| \le \|A^{-1}\|\|S^*\| < (1+\delta)/(1-\delta) < 1+3\delta.$$

By assumption, $\|\lambda A^{-1}S^*\| < 1$. Thus $(\lambda A^{-1}S^* - 1)$ and $(\lambda S^* - A)$ are invertible. Choose $\omega = (-\alpha\lambda^*)(\lambda S^* - A)^{-1}y$, we have

$$S^*v = \alpha S^*y + (\lambda S^* - A)(-\alpha\lambda^*)(\lambda S^* - A)^{-1}y = \alpha\lambda^*y - \alpha\lambda^*y = 0.$$

Since

$$\begin{aligned}\|v\| &= \|\alpha y + (\lambda - S)\omega\| = \big[\|\alpha y\|^2 + \|(\lambda - S)(-\alpha\lambda^*)(\lambda S^* - A)^{-1}y\|^2\big]^{1/2}\\ &= |\alpha|\big[\|y\|^2 + \|\lambda^*(\lambda - S)(\lambda S^* - A)^{-1}y\|^2\big]^{1/2}\end{aligned}$$

and since λ and y are fixed, we can choose $\alpha \neq 0$ so that $\|v\| = (1 - |\lambda|^2)^{1/2}$. Thus for the chosen v, $Q_1^*Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & S^*S \end{bmatrix}$ and $\|Q_1^*Q_1 - 1\| < \delta$. Since $Q_1 \in (\mathcal{U} + \mathcal{K})(T)$, $Q_1 \in \mathcal{A}(D, n)$. By Lemma 2.11 $\operatorname{dist}(Q_1, \mathcal{U}(T_z^{(n)})) < \delta$ and $Q_1 - W^*T_z^{(n)}W$ is compact for some unitary operator W.

To complete the induction step, we now assume that the result is true when F is an $(m-1) \times (m-1)$ diagonal matrix.

Let $T = \begin{bmatrix} F & 0 \\ x & S \end{bmatrix}$ satisfy the condition of the lemma, where F is an $m \times m$ matrix. Then

$$T \simeq \begin{bmatrix} F' & 0 & 0 \\ 0 & \lambda & 0 \\ X_1 & x_2 & S \end{bmatrix} = \begin{bmatrix} F' & 0 \\ X_1' & T_1 \end{bmatrix},$$

where F' is an $(m-1) \times (m-1)$ diagonal matrix, $X_1' = \begin{bmatrix} 0 \\ X_1 \end{bmatrix}$ and $T_1 = \begin{bmatrix} \lambda & 0 \\ x_2 & S \end{bmatrix}$. But T_1 is precisely of the form handled when $m = 1$, and so we can find $R \in (\mathcal{U} + \mathcal{K})(\mathcal{H}_2')$ such that if $R^{-1}T_1R := Q_1$, then $Q_1 \in \mathcal{A}(D, n)$, $\|Q_1^*Q_1 - 1\| < \delta$, $\|R^{-1}T_1R - T_z^{(n)}\| < \delta$ and $R^{-1}T_1R - T_z^{(n)}$ is compact, where T_1 is acting on $\mathcal{H}_2'$. Then

$$T \sim_{\mathcal{U}+\mathcal{K}} \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} F' & 0 \\ X_1' & T_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} F' & 0 \\ R^{-1}X_1' & R^{-1}T_1R \end{bmatrix} := T'.$$

Thus $T' \in (\mathcal{U} + \mathcal{K})(T)$ and therefore $T' \in \mathcal{A}(D, n)$ and satisfies the same condition as T does. By the inductive assumption, we can find $Q \in (\mathcal{U} + \mathcal{K})(T') = (\mathcal{U} + \mathcal{K})(T)$ such that $\operatorname{dist}(Q, u(T_z^{(n)})) < \delta$ and $Q - W^*T_z^{(n)}W \in \mathcal{K}(\mathcal{H})$ for some unitary operator W. This completes the proof of Case 1. Note that our distance estimate is actually δ as opposed to 4δ in this case.

Case 2. $\sigma(F) \cap \{\lambda : |\lambda| \geq 1/(1 + 3\delta)\} \neq \varnothing$.

By an appropriate choice of basis, we can assume that the eigenvalues $\{\lambda_1, \cdots, \lambda_m\}$ of F are listed in nonincreasing order of absolute value. Thus

$$T \simeq \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & 0 & & \\ & & \lambda_r & & & & 0 \\ & & & \lambda_{r+1} & & & \\ & 0 & & & \ddots & & \\ & & & & & \lambda_m & \\ x_1 & \ldots & x_r & x_{r+1} & \ldots & x_m & S \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bigoplus\limits_1^r \lambda_i & 0 \\ G & T_0 \end{bmatrix} \end{array} \begin{array}{l} \mathcal{H}_1' \\ \mathcal{H}_2' \end{array}$$

where $|\lambda_i| \in \big[1/(1+3\delta), 1\big)$ if and only if $1 \le i \le r$, $G = (x_1, \cdots, x_r) \in \mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$,

$$T_0 = \begin{bmatrix} \lambda_{r+1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_m & \\ x_{r+1} & \dots & x_m & S \end{bmatrix} \in \mathcal{L}(\mathcal{H}_2').$$

Set $Y_k = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{matrix} \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$. Then $Y_k T Y_k^{-1} = \begin{bmatrix} \bigoplus\limits_1^r \lambda_i & 0 \\ \frac{1}{k} G & T_0 \end{bmatrix} \to \Big(\bigoplus\limits_{i=1}^r \lambda_i\Big) \bigoplus T_0$ $(k \to \infty)$, i.e., $T \to_{\mathcal{U}+\mathcal{K}} \Big(\bigoplus\limits_{i=1}^r \lambda_i\Big) \bigoplus T_0$. Moreover, it is not difficult to check that T_0 satisfies all the conditions of Case 1. Because of this, we can conclude that $\Big(\bigoplus\limits_{i=1}^r \lambda_i\Big) \bigoplus T_0 \sim_{\mathcal{U}+\mathcal{K}} \Big(\bigoplus\limits_{i=1}^r \lambda_i\Big) \bigoplus Q_1$, where $Q_1 \in (\mathcal{U}+\mathcal{K})(T_0)$, $\|Q_1 - T_z^{(n)}\| < \delta$ and $Q_1 - T_z^{(n)}$ is compact. Let $Q = \Big(\bigoplus\limits_{i=1}^r \lambda_i\Big) \bigoplus Q_1$, thus $Q \in (\mathcal{U}+\mathcal{K})(T)^-$. Since $||\lambda_i| - 1| < |1/(1+3\delta) - 1| < 3\delta$ $(1 \le i \le r)$, we may easily find $\lambda_i' \in \partial D$ such that $|\lambda_i - \lambda_i'| < 3\delta$ $(1 \le i \le r)$. Since $T_z^{(n)}$ is essentially normal and $\partial D \subset \sigma_e(T_z^{(n)})$, by the arguments of Proposition 2.1 we have $T_z^{(n)} = W\Big[\Big(\bigoplus\limits_{i=1}^r \lambda_i'\Big) \bigoplus T_z^{(n)}\Big] W^* + K$, where K is a compact operator with $\|K\| < \delta$, W is a unitary operator. Thus

$$\begin{aligned} \|Q - W^* T_z^{(n)} W\| &= \Big\| \Big(\bigoplus_{i=1}^r \lambda_i\Big) \bigoplus Q_1 - \Big(\bigoplus_{i=1}^r \lambda_i'\Big) \bigoplus T_z^{(n)} - W^* K W \Big\| \\ &\le \Big\| \Big(\bigoplus_{i=1}^r (\lambda_i - \lambda_i')\Big) \bigoplus (Q_1 - T_z^{(n)}) \Big\| + \|K\| \le 3\delta + \delta = 4\delta. \end{aligned}$$

Moreover, $Q - W^* T_z^{(n)} W$ is compact, since $Q_1 - T_z^{(n)}$ is compact.

Proposition 2.3. *Suppose that $T \in \mathcal{A}(D, n)$. Then given δ, $0 < \delta < 1/10$, we can find a compact operator K with $\|K\| < \delta$ such that $T_z^{(n)} + K \sim_{\mathcal{U}+\mathcal{K}} T$. In particular, $T_z^{(n)} \in (\mathcal{U}+\mathcal{K})(T)^-$, and so $(\mathcal{U}+\mathcal{K})(T_z^{(n)})^- = (\mathcal{U}+\mathcal{K})(T)^-$.*

Proof. By Lemma 2.10, $T \to_{\mathcal{U}+\mathcal{K}} T'$, where $T' \in \mathcal{A}(D, n)$ and is a lower triangular. Thus $T'^* \in \mathcal{B}_n(D)$. From the definition of $\mathcal{B}_n(D)$ operators we may assume that the diagonal entries $\{\lambda_k\}_{k=1}^\infty$ of T' are pairwise distinct and form a dense subset of D. Thus, for each $k \ge 1$, we have

$$T' = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \vdots \\ & * & \lambda_k & 0 \\ Z_{11} & \dots & Z_{1k} & T_k' \end{bmatrix} := \begin{bmatrix} G_k & 0 \\ Z_k & T_k' \end{bmatrix}.$$

Since $T'^* T' - 1 =: K \in \mathcal{K}(\mathcal{H})$, we obtain

$$\begin{bmatrix} G_k^* G_k + Z_k^* Z_k & Z_k^* T_k' \\ T_k' Z_k & T_k'^* T_k' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} K_{1k} & K_{2k} \\ K_{3k} & K_{4k} \end{bmatrix}$$

with respect to these decompositions of the spaces. Since K is compact, we can choose N large enough so that if $k \ge N$, $\|K_{4k}\| < \delta/4$. Let $S = T_N'$. Then $S \in \mathcal{A}(D, n)$ and $\|S^* S - 1\| < \delta/4$. By Lemma 2.11, $d(S, \mathcal{U}(T_z^{(n)})) < \delta/4$. Since the eigenvalues of G_N are all distinct, we can find an invertible matrix R such that $F = R^{-1}G_NR$ is diagonal. Let $X = Z_NR$. Then

$$T' \sim_{\mathcal{U}+\mathcal{K}} \begin{bmatrix} R^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G_N & 0 \\ Z_N & S \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} F & 0 \\ X & S \end{bmatrix} = T_0.$$

By Proposition 2.2 we can find an operator $Q \in (\mathcal{U}+\mathcal{K})(T_0)^- = (\mathcal{U}+\mathcal{K})(T)^-$ and a unitary operator W such that

(i) $Q - W^*T_z^{(n)}W$ is compact,

(ii) $\|Q-W^*T_z^{(n)}W\| < \delta$.

Thus, setting $K = WQW^* - T_z^{(n)}$, we have $\|K\| < \delta$ and $T_z^{(n)} + K \sim_{\mathcal{U}+\mathcal{K}} T$. Therefore $T_z^{(n)} \in (\mathcal{U}+\mathcal{K})(T)^-$. By Lemma 2.6, $(\mathcal{U}+\mathcal{K})(T_z^{(n)})^- = (\mathcal{U}+\mathcal{K})(T)^-$.

Now we are in a position to prove Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. By Lemma 2.6, there are a compact operator K_1 with $\|K_1\| < \epsilon/2$ and an operator $Y \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$ such that $T_1 + K_1 = YT_\phi^{(n)}Y^{-1}$. Let ϕ be the analytic function in Lemmas 2.2–2.3. Then ϕ is analytic in $\overline{D}$ and $\phi^{-1} = \psi$ is analytic in $\overline{\Omega}$. We can find δ_1, $\delta_2 > 0$ such that $\psi(\Omega_{\delta_1}) \supset D_{\delta_2}$. Set $A = \psi(T_2) \in \mathcal{A}(D,n)$, $B = T_z^{(n)}$. Let $m_1 = \max\{|\phi(z)| : |z| = 1+\delta_2\}$ and $m_2 = \max\{\|(z-B)^{-1}\| : |z| = 1+\delta_2\}$. By Proposition 2.2, there are a compact operator K_2 and an operator $X \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$ such that $X(B+K_2)X^{-1} = A$, and $\|K_2\| < \min\{1/[(m+1)m_2], \delta_2\}$, where $m \in \mathbb{N}$ satisfies $[m_1m_2(1+\delta_2)]/m < \epsilon/(2\|Y\|\|Y^{-1}\|)$. This implies that $\sigma(B+K_1) = \overline{D}$. Therefore

$$\phi(X(B+K_1)X^{-1}) = X\,\phi(B+K_1)X^{-1} = \phi(A) = \phi\circ\psi(T) = T_2.$$

Set

$$\begin{aligned} K_3 = \phi(B+K_2) - T_\phi^{(n)} &= \phi(B+K_2) - \phi(B) \\ &= \frac{1}{2\pi i}\int_{|z|=1+\delta_2} \phi(z)\big[(z-B-K_2)^{-1} - (z-B)^{-1}\big]\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=1+\delta_2} \phi(z)(z-B)^{-1}\sum_{n=1}^{\infty}\big[(z-B)^{-1}K_2\big]^n dz. \end{aligned}$$

Thus

$$\begin{aligned} \|K_3\| &\le \frac{2\pi m_1(1+\delta_2)m_2}{2\pi}\sum_{n=1}^{\infty}\|(z-B)^{-1}\|^n\|K_2\|^n \\ &< m_1m_2(1+\delta_2)\sum_{n=1}^{\infty}\frac{1}{(m+1)^n} = \frac{m_1m_2(1+\delta_2)}{m} < \frac{\epsilon}{2\|Y\|\|Y^{-1}\|}. \end{aligned}$$

Since K_2 is compact, K_3 is compact. Thus $T_\phi^{(n)}+K_3 \sim_{\mathcal{U}+\mathcal{K}} T_2$, i.e., there is a $Z \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$ such that $T_\phi^{(n)}+K_2 = ZT_2Z^{-1}$. Thus

$$T_1+K_1+YK_3Y^{-1} = Y(T_\phi^{(n)}+K_3)Y^{-1} = YZT_2Z^{-1},$$

and $K = K_1 + YK_3Y^{-1}$ satisfies the requirements of the theorem.

Proof of Theorem 1.2. If A satisfies (i), (ii) and (iii), then for each $\epsilon > 0$ by [9, Theorem 3.48], there exists a compact operator K, $\|K\| < \epsilon$, such that $A + K \in \mathcal{A}(\Omega, n)$. By Theorem 1.1, $A + K \in (\mathcal{U}+\mathcal{K})(T)^-$. Thus $A \in (\mathcal{U}+\mathcal{K})(T)^-$. If $A \in (\mathcal{U}+\mathcal{K})(T)^-$, then $X_nTX_n^{-1} \to A$ $(n\to\infty)$ for a sequence of invertible operators $\{X_n\} \subset (\mathcal{U}+\mathcal{K})(\mathcal{H})$. Since $\pi(X_n)\,\pi(T)\,\pi(X_n^{-1}) \to \pi(A)$ $(n\to\infty)$ and since each $\pi(X_n)$ is a unitary element,

$\sigma_e(T) = \sigma_e(A) = \partial\Omega$ and $\mathrm{nul}\,(\lambda - T) = \mathrm{nul}\,(\lambda - A)$ for all $\lambda \in [\sigma(T) \cup \sigma(A)]\backslash\sigma_e(T)$. Thus A is essentially normal, $\overline{\Omega} \subset \sigma(A)$ and for $\lambda \in \Omega$, $\mathrm{ind}\,(\lambda - A) = -n$. Conversely, assume that $\lambda \in \sigma(A) \cap \rho(A)$ but $\lambda \notin \overline{\Omega}$. Since $\lambda \in \rho(X_n T X_n^{-1})$, $\mathrm{ind}\,(\lambda - A) = 0 = \mathrm{nul}\,(\lambda - A)$. Thus $\lambda \in \rho(A)$, a contradiction. Thus $\sigma(A) = \overline{\Omega}$.

Proof of Theorem 1.3. By Theorem 1.1, it suffices to prove that there exists a compact operator K, $\|K\| < \epsilon$, such that $T_\phi^{(n)} + K \in (SI)$, or $(T_\phi^*)^{(n)} + K \in (SI)$, where ϕ is an analytic homeomorphism from D onto Ω. Since $T_\phi^* \in \mathcal{B}_1(\Omega)$, by [12, Lemma 2.3], we can find compact operators $K_1, K_2, \cdots, K_n$ such that $\|K_i\| < \epsilon/2$, $A_i := T_\phi^* + K_i \in \mathcal{B}_1(\Omega)$ $(i = 1, 2, \cdots, n)$ and $\ker \tau_{A_i A_j} = \{0\}$ $(i \neq j)$. Since $\sigma_r(A_1) \cap \sigma_l(A_i) \neq \varnothing$ $(i = 2, 3, \cdots, n)$, by [9, Theorem 3.53] there are compact operators $C_2, C_3, \cdots, C_n$ such that $C_i \notin \mathrm{ran}\,\tau_{A_1 A_i}$ and $\|C_i\| < \epsilon/2^i$ $(i = 2, 3, \cdots, n)$.

Set

$$K = \begin{bmatrix} K_1 & C_2 & \dots & C_n \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{bmatrix}.$$

Then K is compact and $\|K\| < \epsilon$. It is easily seen that $(T_\phi^*)^{(n)} + K \in (SI)$.

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