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Γ-CONVERGENCE OF INTEGRAL FUNCTIONALS DEPENDING ON VECTOR-VALUED FUNCTIONS OVER PARABOLIC DOMAINS**

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Abstract

This paper studies Γ-convergence for a sequence of parabolic functionals,

$$F^\varepsilon(u) = \int_0^T \int_\Omega f(x/\varepsilon, t, \nabla u)dxdt \text{ as } \varepsilon \to 0,$$

where the integrand f is nonconvex, and periodic on the first variable. The author obtains the representation formula of the Γ-limit. The results in this paper support a conclusion which relates Γ-convergence of parabolic functionals to the associated gradient flows and confirms one of De Giorgi's conjectures partially.

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§1. Introduction

We begin with the characterization of Γ-convergence in [1, 2].

Definition 1.1. *Let (X, τ) be a first countable topological space and $\{F^h\}_{h=1}^\infty$ be a sequence of functionals from X to $\bar{R} = R \cup \{-\infty, \infty\}$, $u \in X, \lambda \in \bar{R}$. We call*

$$\lambda = \Gamma(\tau) \lim_{h\to\infty} F^h(u)$$

if and only if for every sequence $\{u^h\}$ converging to u in (X, τ)

$$\lambda \le \liminf_{h\to\infty} F^h(u^h), \tag{1.1}$$

and there exists a sequence $\{u^h\}$ converging to u in (X, τ) such that

$$\lambda \ge \limsup_{h\to\infty} F^h(u^h). \tag{1.2}$$

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We call $\lambda = \Gamma(\tau)\lim_{\varepsilon\to a} F^{\varepsilon}(u)$ *if and only if for every* $\varepsilon_h \to a$ $(h\to\infty)$

$$\lambda = \Gamma(\tau)\lim_{h\to\infty} F^{\varepsilon_h}(u).$$

Throughout this paper, we assume that Ω is a bounded open set in R^n. Let $p>1$, $T>0$, and m be a positive integer. Denote

$$\Omega_T = \Omega\times(0,T),\quad V_p(\Omega_T,m) = L^P([0,T],W^{1,p}(\Omega,R^m)),$$

$$V_p^0(\Omega_T,m) = L^p([0,T],W_0^{1,p}(\Omega,R^m)),$$

$$Du(x,t) = \nabla u(x,t) = \Big(\frac{\partial u^i(x,t)}{\partial x_j}\Big)\quad (1\le i\le m, 1\le j\le n)$$

for a vector valued function u.

Consider the fuctionals

$$F_1^{\varepsilon}(v,\Omega) = \int_\Omega f_1\Big(\frac{x}{\varepsilon},Dv\Big)dx,\quad v\in W^{1,p}(\Omega,R^m)\ \ (\varepsilon\to 0^+) \tag{1.3}$$

and the corresponding parabolic functionals in the following form:

$$F^{\varepsilon}(u,\Omega_T) = \int_{\Omega_T} f\Big(\frac{x}{\varepsilon},t,,Du\Big)dxdt,\quad u\in V_p(\Omega_T,m)\ \ (\varepsilon\to 0^+), \tag{1.4}$$

where $f\colon R^{n+1}\times R^{mn}\to R$ is a Caratheodory function satsfying

$$C_1|\lambda|^p \le f(x,t,\lambda)\le C_2(1+|\lambda|^p) \tag{1.5}$$

for some positive constants $C_2 > C_1$.

In 1979, E. De Giorgi[3] conjectured that when a sequence of functionals, for instance, the one in (1.4) or in a more general form, converges in the sence of Γ-convergence to a limiting functional, the corresponding gradient flows will converge as well (maybe after an appropriate change of timescale). Also see [4, p.216] and [5, p.507].

In [6], the author proved the De Giorgi's conjecture for a rather wide kind of functionals. Thus, a natural question is under what conditions the functional sequence like (1.4) can be Γ-convergence.

The first result related to this question appeared in [7]. Because the integrands in [7] have the same scale for the variables x and t, the methods there can not be applied to functionals (1.4) whose integrands are anisotropic in x and t.

In this paper, we will cleverly combine the arguements in [8, 9, 10], all of which study the Γ-convergence of elliptic functionals like (1.3) with the weak-topology of $W^{1,p}(\Omega,R^m)$, to prove that the Γ-convergence holds for the functional (1.4) under Assumption (1.5) and a periodic hypothesis (see (1.8) below). For this purpose, we construct functionals as follows.

Let $Y=(0,1)^n=\{0<y_i<1, i=1,2,\cdots n\}$, $kY=(0,k)^n$, and $k_T = kY\times(0,T)$. For $\lambda\in R^{mn}$ and a.e. $t\in R$, define

$$\bar f(t,\lambda) = \inf_{k\in N}\inf\Big\{|kY|^{-1}\int_{kY} f(y,t,\lambda+D\phi(y,t))dy\colon \phi\in V_p^0(k_T,m)\Big\}, \tag{1.6}$$

here and below $|E| \overset{\text{def.}}{=} L^n(E)$ and L^k is used to denote the k-dimensional Lebesque measure.

Obviously (1.5) implies that $\bar f(t,Du)$ is nonnegative and measurable, so we can define the homogenized functional

$$F(u,\Omega_T) = \int_{\Omega_T}\bar f(t,Du)dxdt,\qquad u\in V_p(\Omega_T,m). \tag{1.7}$$

The main result of this paper is the following theorem.

Theorem 1.1. *If hypotheses* (1.4) *and* (1.5) *are satisfied, and suppose*

$$f(y,t,\lambda) \text{ is } \bar{Y}\text{-periodic on the first variable } y, \tag{1.8}$$

then for every $T>0$ *and every bounded open set* $\Omega \subset R^n$ *with* $L^n(\partial\Omega)=0$,

$$\Gamma(\tau)\lim_{\varepsilon\to 0} F^\varepsilon(u,\Omega_T) = F(u,\Omega_T), \forall u \in V_p(\Omega_T,m),$$

where τ *is taken as the sw-topology of* $V_p(\Omega_T,m)$. (*See the Definition* 1.2 *in* [6] *for the sw-topology.*)

The proof of this theorem will be given in Section 4.

§2. Preliminary Lemmas

We collect some properties of the Γ-limits in [1, 2] which are well-known but important for the coming arguements.

If the lim sup in (1.2) is replaced by lim inf , Definition 1.1 is turned to the definition of low Γ-limit. In this case, we denote it by $\lambda = \Gamma^-(\tau)\lim_{h\to\infty} F^h(u)$. Similarly, we have upper Γ-limit and denote it by $\lambda = \Gamma^+(\tau)\lim\limits_{h\to\infty} F^h(u)$.

Obviously, $\Gamma(\tau)\lim\limits_{h\to\infty} F^h(u)$ exists if and only if $\Gamma^+(\tau)\lim\limits_{h\to\infty} F^h(u) = \Gamma^-(\tau)\lim\limits_{h\to\infty} F^h(u)$.

Lemma 2.1. $F^-(u) = \Gamma^-(\tau)\lim\limits_{h\to\infty} F^h(u)$ *exists for every* $u \in X$, *and* $F^-(u)$ *is lower semicontinuous in* (X,τ). *If* $F(u) = \Gamma(\tau)\lim\limits_{h\to\infty} F^h(u)$ *exists for every* $u \in X$, *then F(u) is also lower semicontinuous in* (X,τ) .

Lemma 2.2. *For each sequence* $\{F^h\}$ *of functionals in* (X,τ), *there exists a subsequence* F^{h_k} *and* F^∞ *from* X *to* $\bar{R}$, *such that*

$$F^\infty(u) = \Gamma(\tau)\lim_{k\to\infty} F^{h_k}(u), \quad \forall u \in X.$$

Lemma 2.3. *Suppose that* $\lambda = \Gamma(\tau)\lim\limits_{\varepsilon\to 0} F^\varepsilon(u)$ *and* $\varepsilon_h \to 0$ $(h\to\infty)$. *Then*

$$\Gamma^-(\tau)\lim_{h\to\infty} F^{\varepsilon_h}(u) = \Gamma(\tau)\lim_{h\to\infty} F^{\varepsilon_h}(u) = \lambda.$$

Lemma 2.4. *Suppose that* $f\colon R\times R \to \bar{R}$. *Then there exists a function* $\delta\colon \varepsilon \to \delta(\varepsilon)$ *such that* $\varepsilon \to 0$ *implies* $\delta(\varepsilon)\to 0$ *and*

$$\limsup_{\varepsilon\to 0} f(\delta(\varepsilon),\varepsilon) \le \limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} f(\delta,\varepsilon). \tag{2.1}$$

Moreover, the opposite inequality for low limits and the equality for limits hold true respectively.

From now on, we restrict ourselves to the sequence of functionals (1.4), or more general functionals

$$F^\varepsilon(u,\Omega\times(a,b)) = \int_a^b \int_\Omega f\Big(\frac{x}{\varepsilon},t,Du\Big)dxdt \quad (\varepsilon\to 0^+). \tag{2.2}$$

We will fix $T>0$ and allow Ω and (a,b) to be arbitrary. Let $S = R^n\times(0,T)$, β_T be the σ -ring generated by the set $\{\Omega\times(a,b)\colon 0\le a<b\le T, \Omega\subset R^n$ are bounded open sets$\}$. Then (S,β_T,L^{n+1}) is a measure space. Let

$$V_{p,\mathrm{loc}} = L^p([0,T],W^{1,p}_{\mathrm{loc}}(R^n,R^m)). \tag{2.3}$$

Lemma 2.5. *Assume that* (1.4), (1.5) *and* (1.8) *are satisfied. Then for every sequence* $\varepsilon \to 0^+$, *there exist a subsequence* $\varepsilon_h \to 0^+ \quad (h \to 0)$ *and a family of* σ*-finite and* σ*-additive measures* $H(u, \Omega \times (a, b))$ *on* β_T, *such that for every* $u \in V_{p,\mathrm{loc}}$, *every finite interval* (a, b) *and every bounded open set* $\Omega \subset R^n$ *with* $L^n(\partial\Omega) = 0$,

$$\Gamma(\tau) \lim_{h\to\infty} F^{\varepsilon_h}(u, \Omega \times (a, b)) = H(u, \Omega \times (a, b)), \tag{2.4}$$

$$0 \le H(u, \Omega \times (a, b)) \le C \int_a^b \int_\Omega (1 + |Du|^p) dxdt, \tag{2.5}$$

where τ *is the sw-topology of* $V_p(\Omega \times (a, b))$.

Proof. We follow the proof of in [9, Theorem 3.1]. D is used to denote the algebra generated by all open cubes in R^{n+1} with rational vertices and E the class of all bounded open sets in R^{n+1}. Applying Lemma 2.2 and a diagonalization argument, we can find a sequence $\varepsilon_h \quad (h \to \infty)$ such that $\Gamma(\tau) \lim\limits_{h\to\infty} F^{\varepsilon_h}(u, Q)$ exists for all $Q \in D$, i.e.,

$$H^-(u, Q) = H^+(u, Q), \quad \forall Q \in D,$$

where $H^-(u, Q) = \Gamma^-(\tau) \lim\limits_{h\to\infty} F^{\varepsilon_h}(u, Q)$ and $H^+(u, Q) = \Gamma^+(\tau) \lim\limits_{h\to\infty} F^{\varepsilon_h}(u, Q)$.

In the same way as in [9, pp.738–739], by Lemma B in [6], we can prove that H^- is (finitely) super-additive and H^+ is sub-additive over D. For $e \in E$, define

$$H(u, e) = \sup_{Q \subset\subset e} H^-(u, Q) = \sup_{Q \subset\subset e} H^+(u, Q), \quad Q \in D.$$

Then $H(u, e)$ is an increasing, inner regular and finitely additive set function. Therefore, the routine methods implies that (2.4) holds and $H(u, \Omega \times (a, b))$ can be extended to a σ-finite and σ-additive measure on β_T (see [11, Proposition 5.5 and Theorem 5.6]. From (2.4) and (1.5), the estimate (2.5) follows immediately.

§3. Γ-Lmits of Layered Affine Functions

Throughout this section, suppose that (1.4), (1.5) and (1.8) are satisfied. τ is used to denote the **sw**-topology of $V_p(\Omega_T, m)$. For simiplicity, $V_p(\Omega_T)$ denotes the space $V_p(\Omega_T, m)$. We intend to determine the Γ-limits of $F^\varepsilon(u, \Omega_T)$ for $u = \lambda(t)\cdot x + a(t)$ with $\lambda \in L^p([0, T], M(m\times n))$ and $a \in L^p([0, T], R^m)$, where we define the norm on $M(m \times n)$, the space of all real $m \times n$ matrices, as the same as on R^{mn}.

Lemma 3.1. *For each* $u_{\lambda,a} = \lambda(t) \cdot x + a(t)$ *with* $\lambda \in L^p([0, T], M(m, n))$ *and* $a \in L^p([0, T], R^m)$, *there exists a sequence of functions* $\{u^\varepsilon\} \subset V_p(\Omega_T)$ *satifying*

$$\{u^\varepsilon - u_{\lambda,a}\} \subset V_p^0(\Omega_T) \quad \textit{and} \quad u^\varepsilon \xrightarrow{\tau} u_{\lambda,a} \quad \textit{in} \quad V_p(\Omega_T) \quad \textit{as} \quad \varepsilon \to 0^+$$

such that

$$\lim_{\varepsilon\to 0^+} F^\varepsilon(u^\varepsilon, \Omega_T) = \int_{\Omega_T} \bar f(t, \lambda) dxdt = F(u_{\lambda,a}, \Omega_T),$$

where $\bar f(t, \lambda)$ *is given by* (1.6) *and* F *by* (1.7).

Proof. Fix $\delta \in (0, 1)$, one can choose $k \in N$ and $\phi^\delta \in V_p^0(k_T, m)$ (see (1.6)) such that

$$\bar f(t, \lambda(t)) \le |kY|^{-1} \int_{kY} f(y, t, \lambda + D\phi^\delta) dy \le \bar f(t, \lambda(t)) + \delta. \tag{3.1}$$

We use E_η^* to denote the extension of $\eta\bar{Y}$ on the ηY-period, and let

$$\Omega_\eta^* = \{e \in E_\eta^*, e \subset \Omega\}, \quad E_\eta = \bigcup_{e\in E_\eta^*} e, \quad \Omega_\eta = \bigcup_{e\in\Omega_\eta^*} e.$$

Then $E_\eta = R^n$. As Ω is bounded, Ω_η^* is a finite set for each $\eta > 0$, and

$$\lim_{\eta\to 0^+} L^n(\Omega\backslash\Omega_\eta) = 0. \tag{3.2}$$

For every $t \in [0,T]$, extend $\phi^\delta(y,t)$ so that it is a kY-periodic function on the variable y. Then define

$$v^{\varepsilon,\delta}(x,t) = \begin{cases} u_{\lambda,a}(x,t) + \varepsilon\phi^\delta(\frac{x}{\varepsilon},t), & \Omega_{\varepsilon k}, \\ u_{\lambda,a}(x,t), & \Omega\backslash\Omega\varepsilon k. \end{cases} \tag{3.3}$$

It is easy to know that $v^{\varepsilon,\delta} \in V_p(\Omega_T)$, $\;v^{\varepsilon,\delta} - u_{\lambda,a} \in V_p^0(\Omega_T)$. For each $D \in \Omega_{\varepsilon k}$, by the periodicity of $g(y,t) = f(y,t,\lambda(t) + D\phi^\delta(y,t))$, we have

$$\begin{aligned}\int_{D\times(0,T)} f\Big(\frac{x}{\varepsilon},t,Dv^{\varepsilon,\delta}\Big)dxdt &= \int_0^T \Big[\varepsilon^n \int_{D/\varepsilon} f(y,t,\lambda(t) + D\phi^\delta(y,t)dy\Big]dt \\ &= L^n(D)\int_0^T dt|kY|^{-1}\int_{kY} f(y,t,\lambda(t)+D\phi^\delta)dy.\end{aligned} \tag{3.4}$$

Summing up the both sides for all $D \in \Omega_{\varepsilon k}$ and applying (3.1), we obtain

$$\begin{aligned}L^n(\Omega_{\varepsilon k})\int_0^T \bar{f}(t,\lambda(t))dt &\le \int_0^T dt\int_{\Omega_{\varepsilon k}} f\Big(\frac{x}{\varepsilon},t,Dv^{\varepsilon,\delta}\Big)dx \\ &\le L^n(\Omega_{\varepsilon k})\int_0^T (\bar{f}(t,\lambda(t)) + \delta)dt.\end{aligned}$$

Thus, it follows from (1.5) and (3.3) that

$$\begin{aligned}&L^n(\Omega_{\varepsilon k})\int_0^T \bar{f}(t,\lambda(t))dt \le \int_{\Omega_T} f\Big(\frac{x}{\varepsilon},t,Dv^{\varepsilon,\delta}\Big)dxdt \\ &\le L^n(\Omega_{\varepsilon k})\int_0^T (\bar{f}(t,\lambda(t))+\delta)dt + \int_0^T dt\int_{\Omega\backslash\Omega_{\varepsilon k}} f\Big(\frac{x}{\varepsilon},t,\lambda(t)\Big)dx \\ &\le L^n(\Omega_{\varepsilon k})\int_0^T (\bar{f}(t,\lambda(t))+\delta)dt + CL^n(\Omega\backslash\Omega_{\varepsilon k})\int_0^T (1+|\lambda|^p)dt.\end{aligned} \tag{3.5}$$

By this estimate and (3.2), we see that

$$\lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\int_{\Omega_T} f\Big(\frac{x}{\varepsilon},t,Dv^{\varepsilon,\delta}\Big)dxdt = \int_{\Omega_T}\bar{f}(t,\lambda(t))dtdx. \tag{3.6}$$

Moreover, we have

$$\|v^{\varepsilon,\delta} - u_{\lambda,a}\|^p_{L^p(\Omega_T)} = \varepsilon^p\sum_{D\in\Omega_{\varepsilon k}}|D|\int_0^T dt|kY|^{-1}\int_{kY}|\phi^\delta|^p dy. \tag{3.7}$$

Applying (3.6), (3.7), and Lemma 2.4, one can find a sequence $\delta(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$ such that $\{u^\varepsilon = v^{\varepsilon,\delta(\varepsilon)} \colon \varepsilon > 0\}$ satisfy that

$$\{u^\varepsilon - u_{\lambda,a}\} \subset V_p^0(\Omega_T), \quad \lim_{\varepsilon\to 0^+}\|u^\varepsilon - u_{\lambda,a}\|^p_{L^p(\Omega_T)} = 0,$$
$$\lim_{\varepsilon\to 0^+} F^\varepsilon(u^\varepsilon,\Omega_T) = F(u_{\lambda,a},\Omega_T).$$

On the other hand, the coercive condition in (1.5) and (3.5) imply that $\{Du^\varepsilon\}$ is bounded in $L^p(\Omega_T, R^{mn})$. Thus, by Lemma B in [6], we obtain $u^\varepsilon \xrightarrow{\mathcal{T}} u_{\lambda,a}$. This proves the desired result.

Lemma 3.2. *Let $u_{\lambda,a}(x,t) = \lambda(t)\cdot x + a(t)$ be the same as in Lemma 3.1. Then for each sequence $u^\varepsilon \xrightarrow{\mathcal{T}} u_{\lambda,a}$ in $V_p(\Omega_T)$ $\quad (\varepsilon \to 0^+)$,*

$$\liminf_{\varepsilon\to 0^+} F^\varepsilon(u^\varepsilon, \Omega_T) \geq F(u_{\lambda,a}, \Omega_T) = \int_{\Omega_T} \bar{f}(t, \lambda(t))dtdx.$$

Proof. (1) Firstly, assume $u^\varepsilon \xrightarrow{\mathcal{T}} u_{\lambda,a}$ and $u^\varepsilon - u_{\lambda,a} \in V_p^0(\Omega_T)$. As Ω is bounded, we find an open cube D whose sides are parallel to axes and whose center concides with the origin, such that $\bar{\Omega} \subset D$. The side length of D is denoted by 2d, and let

$$k_\varepsilon = \Big[\frac{2d}{\varepsilon}\Big] + 3, \qquad a_\varepsilon = \Big[-\frac{d}{\varepsilon}\Big],$$
$$x_\varepsilon = (a_\varepsilon, \cdots, a_\varepsilon) \in R^n, D_\varepsilon = \varepsilon(x_\varepsilon + k_\varepsilon Y),$$

where $[\kappa]$ denote the maximum integer not greater than κ. It is not difficult to get

$$D \subset D_\varepsilon, \quad \lim_{\varepsilon\to 0^+} L^n(D_\varepsilon) = L^n(D). \tag{3.8}$$

Let

$$Q = D\backslash\bar{\Omega}, \quad Q_T = Q \times (0, T). \tag{3.9}$$

Applying Lemma 3.1 to the open set Q, we can choose a sequence

$$v^\varepsilon \to u_{\lambda,a} \ \text{ sw in } \ V_p(Q_T), \quad v^\varepsilon - u_{\lambda,a} \in V_p^0(Q_T)$$

such that

$$\lim_{\varepsilon\to 0^+} F^\varepsilon(v^\varepsilon, Q_T) = \int_Q \bar{f}(t, \lambda)dxdt. \tag{3.10}$$

For fixed $t \in [0, T]$, define

$$\phi^\varepsilon(x,t) = \begin{cases} u^\varepsilon - u_{\lambda,a}, & x \in \bar{\Omega}, \\ v^\varepsilon - u_{\lambda,a}, & x \in D\backslash\bar{\Omega} = Q, \\ 0, & x \in D_\varepsilon\backslash D. \end{cases} \tag{3.11}$$

By the periodicity of $f(y, t, \lambda)$, using the variable transformation, we obtain

$$\begin{aligned}\int_{D_\varepsilon} f\Big(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon(x,t)\Big)dx &= \varepsilon^n \int_{x_\varepsilon + k_\varepsilon Y} f(y, t, \lambda + D_x\phi^\varepsilon(\varepsilon y, t))dy \\ &= (k_\varepsilon\varepsilon)^n |k_\varepsilon Y|^{-1} \int_{k_\varepsilon Y} f(y, t, \lambda + D\psi^\varepsilon(y,t))dy,\end{aligned} \tag{3.12}$$

where $\psi^\varepsilon(y,t) = \varepsilon^{-1}\phi^\varepsilon\big(\varepsilon(y + x_\varepsilon), t\big)$. Obviously, (3.11) gives us $\psi^\varepsilon \in V_p^0\big((k_\varepsilon Y) \times (0,T)\big)$. Thus, we deduce from (3.12) and (1.6) that for each $t \in [0, T]$,

$$|D_\varepsilon|^{-1} \int_{D_\varepsilon} f\Big(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon(x)\Big)dx = |k_\varepsilon Y|^{-1} \int_{K_\varepsilon Y} f(y, t, \lambda + D\psi^\varepsilon(y,t))dy \geq \bar{f}(t, \lambda).$$

Therefore

$$\int_0^T \int_{D_\varepsilon} f\Big(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon\Big)dxdt \geq L^n(D_\varepsilon) \int_0^T \bar{f}(t, \lambda)dt.$$

On the other hand , by (1.5) and (3.8), we have

$$\liminf_{\varepsilon\to 0^+}\int_0^T dt\int_{D_\varepsilon} f\Big(\frac{x}{\varepsilon},t,\lambda+D\phi^\varepsilon\Big)dxdt=\liminf_{\varepsilon\to 0^+}\int_0^T dt\int_D f\Big(\frac{x}{\varepsilon},t,\lambda+D\phi^\varepsilon\Big)dx.$$

This yields

$$\liminf_{\varepsilon\to 0^+} F^\varepsilon\big(u_{\lambda,a}+\phi^\varepsilon, D\times(0,T)\big)\ge\int_0^T\int_D \bar f(t,\lambda)dxdt.$$

Combing this estimate, (3.9), (3.10) with (3.11), we have

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} F^\varepsilon(u^\varepsilon,\Omega_T)&=\liminf_{\varepsilon\to 0^+}[F^\varepsilon(u_{\lambda,a}+\phi^\varepsilon,D\times(0,T))-F^\varepsilon(v^\varepsilon,Q\times(0,T))]\\&\ge\int_0^T\int_D\bar f(t,\lambda)dxdt-\int_0^T\int_Q\bar f(t,\lambda)dxdt\\&=\int_{\Omega_T}\bar f(t,\lambda)dxdt.\end{aligned}$$

(2) In order to remove the restriction $u^\varepsilon-u_{\lambda,a}\in V_p^0(\Omega_T)$, it is sufficient to apply the De Giorgi's arguements and the result of the case (1). See [11] or [8, p.197] for the details.

Definition 3.1. *Let* $\{\Omega_i\colon i=1,2,\cdots,h\}$ *be a finite partition of* Ω *into open sets* (*except for a set of measure zero*), $\lambda_i\in L^p([0,T],M(m,n))$, $a_i\in L^p([0,T],R^m)$. *We call the function*

$$W(x,t)=\begin{cases}\lambda_i(t)\cdot x+a_i(t), & x\in\Omega_i,\\ 0, & x\in\Omega\backslash\bigcup_{i=1}^h\Omega_i,\end{cases}$$

an L^p *-layered affine function on* Ω_T.

Summing up Lemmas 3.1 and 3.2 (observing that Ω may be arbitrary there), Lemmas 2.5 and 2.3, we obtain the following theorem.

Theorem 3.1. *Suppose that* Ω *is a bounded open set in* R^n *with* $L^n(\partial\Omega)=0$, $H(u,\Omega_T)$ *is given by Lemma* 2.5. *Then*

$$\Gamma(\tau)\lim_{\varepsilon\to 0^+}F^\varepsilon(w,\Omega_T)=\int_{\Omega_T}\bar f(t,Dw)dxdt=H(w,\Omega_T)$$

for any w, *an* L^p*-layered affine function on* Ω_T.

§4. A Proof of Theorem 1.2

In this section, we suppose that all the hypotheses of Theorem 1.2 are satisfied. Applying the same arguement as in [9, Section 5], we can prove that for almost $t\in[0,T]$, $\bar f(t,\lambda)$ is convex if $n=2$; and convex with respect to each column vector if $n>2$. This implies that

Lemma 4.1. *For a.e.* $t\in[0,T]$, $\bar f(t,\lambda)$ *is continuous in* $M(m,n)$.

Lemma 4.2. *Suppose* $v\in V_p(\Omega_T)$ $(1<p<\infty)$. *Then there exists a sequence of* L^p*-layered affine functions*

$$v^k(x,t)=\begin{cases}\lambda_i^k(t)\cdot x+a_i^k(t), & x\in\Omega_i,\\ 0, & x\in\Omega\backslash\bigcup_{i=1}^{h_k}\Omega_i,\end{cases}$$

such that $\|v-v^k\|_{V_p(\Omega_T)}\longrightarrow 0$ *as* $k\to\infty$.

Proof. (1) Suppose $1 < p \le 2$. Fix $v \in V_p(\Omega_T)$. For any $\varepsilon > 0$ we can choose $u \in V_2(\Omega_T)$ such that

$$\|u - v\|_{V_p(\Omega_T)} < \varepsilon. \tag{4.1}$$

Because $H \overset{\text{def.}}{=} W^{1,2}(\Omega)$ is a Hilbert space, one can assume that $\{\psi_l\}_{l=1}^{\infty}$ is its complete orthonormal basis. Let $C_l(t) = \langle u(t,\cdot), \psi_l\rangle_H$. Then $C_l(t) \in L^2[0,T]$, and for a.e. $t \in [0,T]$,

$$I_k(t) = \Big\|u - \sum_{l=1}^{k} C_l \psi_l(x)\Big\|_H \longrightarrow 0 \quad (k \to \infty).$$

Thus the dominated convergence theorem implies that for some integer k,

$$\Big\|u - \sum_{l=1}^{k} C_l \psi_l\Big\|_{V_p(\Omega_T)} \le \varepsilon. \tag{4.2}$$

It is well known that there exist piecewise affine functions $\omega_l(x)$ in Ω such that

$$\max_{1\le l\le k} \|\psi_l - \omega_l\|_{W^{1,p}(\Omega)} \le \varepsilon\Big(1 + \sum_{l=1}^{k} \|C_l\|^{-1}_{L^p(\Omega)}\Big).$$

Let $v^\varepsilon(x,t) = \sum\limits_{l=1}^{k} C_l(t)\omega_l(x)$. Then

$$\Big\|v^\varepsilon - \sum_{l=1}^{k} C_l \psi_l\Big\|_{V_p(\Omega_T)} \le C(p)\varepsilon. \tag{4.3}$$

Combing (4.1), (4.2) with (4.3), we get $\|v - v^\varepsilon\|_{V_p(\Omega_T)} \le C(m,n,p)\varepsilon$. Observing that each v^ε can be written as a layered function on Ω_T, we have completed the proof.

(2) Suppose $2 < p < \infty$. Applying Sobolev embedding theorem we can find an integer k, $\frac{k-1}{n} \ge \frac{1}{2} - \frac{1}{p}$, such that $H_1 \overset{\text{def.}}{=} W^{k,2}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. Given $v \in V_p(\Omega_T)$, for $\varepsilon > 0$, one can find $u \in L^p([0,T], W^{k,2}(\Omega))$ such that

$$\|u - v\|_{V_p(\Omega_T)} < \varepsilon. \tag{4.4}$$

Let $\{\psi_l\}_{l=1}^{\infty}$ be the complete orthonormal basis of the Hilbert space H_1. Then

$$C_l(t) \overset{\text{def.}}{=} \langle u(\cdot,t), \psi_l\rangle_{H_1} \in L^p[0,T].$$

The remaining part is entirely the same as the case (1).

Now we are in a position to prove Theorem 1.1. We will use the idea of [9, pp.750–751]. For $u \in V_p(\Omega_T)$, we can extend u so that $u \in V_{p,\text{loc}}$ (recall (2.3)). From Lemma 4.2, choose a sequence of L^p-layered functions $\omega^k(x,t)$, such that

$$\|u - \omega^k\|_{V_p(\Omega_T)} \longrightarrow 0 \ \ (k \to \infty). \tag{4.5}$$

By taking a subsequence, one can assume that $D\omega^k \to Du$ almost everywhere on Ω_T and

$$\bar{f}(t, D\omega^k) \longrightarrow \bar{f}(t, Du) \ \text{ a.e in } \ \Omega_T$$

by virtue of the continuity of $\bar{f}(t,\cdot)$ (see Lemma 4.1).

We deduce, from the absolute continuity of $\int |Du|^p dxdt$, Egoroff theorem, Theorem 3.1 and inequality (2.5), that

$$\liminf_{k\to\infty} \int_{\Omega_T} \bar{f}(t, D\omega^k)dxdt \le \int_{\Omega_T} \bar{f}(t, Du)dxdt.$$

Therfore, by the semi-continuity of $H(u,\Omega_T)$ (see Lemma 2.1),

$$H(u,\Omega_T)\le \liminf_{k\to\infty} H(\omega^k,\Omega_T)=\liminf_{k\to\infty}\int_{\Omega_T}\bar f(t,D\omega^k)dxdt\le\int_{\Omega_T}\bar f(t,Du)dxdt. \tag{4.6}$$

On the other hand, according to Lemma 2.5 and Lebesgue-Nikodym theorem (see [12, §3 of Chapter 3]), we have

$$H(u,\Omega_T)=\int_{\Omega_T}h(x,t)dxdt \tag{4.7}$$

for some $h\in L^1_{\rm loc}(R^n\times(0,T))$ and all $\Omega_T=\Omega\times(0,T)$. By approximation argument, one can easily prove that for a.e $(x,t)\in\Omega_T$, there exists $r_k\to 0^+$ such that

$$\frac{u(x+r_k(y-x),t)-u(x,t)}{r_k}\xrightarrow{\tau} Du(x,t)\cdot(y-x)\quad \text{in}\quad V_p(B(x,1)\times(0,T)). \tag{4.8}$$

Since

$$\begin{aligned}&\Big|\int_0^T h(x,t)dt-\int_0^T|B(x,r_k)|^{-1}\int_{B(x,r_k)}h(y,t)dydt\Big|\\&\le|B(x,r_k)|^{-1}\int_{B(x,r_k)}\Big|\int_0^T[h(x,t)-h(y,t)]dt\Big|dy,\\&\int_0^T h(y,t)dt\in L^1_{\rm loc}(R^n),\end{aligned}$$

we have

$$\int_0^T h(x,t)dt=\lim_{k\to\infty}\int_0^T|B(x,r_k)|^{-1}\int_{B(x,r_k)}h(y,t)dydt\quad\text{for a.e. }\ x\in\Omega. \tag{4.9}$$

Fix k and x, set $r=r_k,\ \ B_r=B(x,r),\ \ B_{r,T}=B_r\times(0,T)$. By (4.7) and Lemma 2.5, we can find a sequence $u^h\xrightarrow{\tau}u$ in $V_p(B_{r,T})(h\to 0)$ such that

$$\begin{aligned}&\int_0^T|B_r|^{-1}\int_{B_r}h(y,t)dydt=|B_r|^{-1}H(u,B_{r,T})\\&=\lim_{h\to\infty}\int_0^T|B_r|^{-1}\int_{B_r}f\Big(\frac{y+\varepsilon_hk_h}{\varepsilon_h},t,Du^h\Big)dydt,\quad\Big(k_h\overset{\text{def.}}{=}\Big[\frac{x(r-1)}{\varepsilon_h}\Big]\Big)\\&\ge\liminf_{h\to\infty}\int_0^T|B_{\frac r2}|^{-1}\int_{B_{\frac r2}}f\Big(\frac{y+x(r-1)}{\varepsilon_h},t,Du^h(y+a_h,t)\Big)dydt\\&\qquad\Big(\text{note that } a_h\overset{\text{def.}}{=}x(r-1)-\varepsilon_hk_h\to0^+\Big)\\&=\liminf_{h\to\infty}\int_0^T|B_{\frac12}|^{-1}\int_{B_{\frac12}}f\Big(\frac{ry}{\varepsilon_h},t,D\big(u^h(x+r(y-x)+a_h,t)-u(x,t)\big)r^{-1}\Big)dydt.\end{aligned} \tag{4.10}$$

Let $u_{r,x}(y,t)=r^{-1}[u(x+r(y-x),t)-u(x,t)]$. Obviously

$$r^{-1}[u^h(x+r(y-x)+a_h,t)-u(x,t)]\xrightarrow{\tau}r^{-1}u_{r,x}\ \text{in}\ \ V_p(B_{\frac12,T})\ \ \text{as } h\to\infty.$$

Let

$$\delta_h=r^{-1}\varepsilon_h,\quad a=|B_{\frac12}|^{-1},\quad F^-(u,Q)=\Gamma^-(\tau)\lim_{h\to\infty}F^{\delta_h}(u,Q).$$

By (4.10), Lemmas 2.1 and 2.3, (4.8) and Theorem 3.1 in that order, we deduce that

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^T |B(x,r_k)|^{-1}\int_{B(x,r_k)} h(y,t)dydt \\
&\geq a\cdot \liminf_{k\to\infty} F^-(u_{r_k,x},B_{\frac{1}{2},T}) \\
&\geq a\cdot F^-(Du(x,t)(y-x),B_{\frac{1}{2},T}) \\
&= a\int_0^T\int_{B(x,\frac{1}{2})} \bar{f}(t,Du(x,t)dtdy \\
&= \int_0^T \bar{f}(t,Du(x,t))dt.
\end{aligned}$$

Combing this estimate with (4.9), we obtain

$$\int_{\Omega_T} h(x,t)dxdt \geq \int_{\Omega_T} \bar{f}(x,Du)dxdt,$$

which together with (4.7) implies the opposite inequality of (4.6). Hence

$$H(u,\Omega_T)=\int_{\Omega_T}\bar{f}(t,Du)dxdt,\quad \forall u\in V_p(\Omega_T).$$

Observing that

$$F(u,\Omega_T)=\int_{\Omega_T}\bar{f}(t,Du)dxdt$$

is independent of $\{\varepsilon_h\}$, we have completed the proof of Theorem 1.1 by Lemma 2.5.

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References

[1] Attouch, H., Variational convergence for functions and operators [M], Appl. Math. Series, Pitman, 1984.

[2] Dal Maso, G., An introduction to Γ-convergence [M], Boston Basel, Birkhäuser, 1993.

[3] De Giorgi, E., New problems in Γ-convergence and G-convergence in free boundary problems [C], Proc. of seminar held in Pavia, September-October 1979, Ist. Naz. Alt. Mat. Francesco Severi, Vol. II, 183–194, Rome, 1980.

[4] Bronsard, L.& Kohn, R. V., Motion by mean curvature as the singular limit of Ginzburg-Landau Dynamics [J], *J. Differ. Eqn.,* **90**(1991), 211–237.

[5] Owen, N. C., Rubinstein, J. & Sternberg, P., Minimizers and gradient flows for singularly perturbed bi-stable potentials with a Dirichlet condition [J], *Proc. R. Soc. Lond. Ser.A,* **429**(1990), 505–532.

[6] Jian, H. Y., A relation between Γ-convergence of functionals and their associated gradient flows [J], *Science in China,* **A42**(1999), 133–139.

[7] Jian, H. Y., Homogenization problems of parabolic minima [J], *Acta Math. Appl. Sinica,* **12**(1996), 318–327.

[8] Müller, S., Homogenization of noncovex integral functionals and elastic materials [J], *Arch. Rat. Mech. Anal.,* **99**(1987), 187–212.

[9] Weinan, E., A class of homogenization problems in the calculus of variations [J], *Commu. Pure Appl. Math.,* **44**(1991), 733–759.

[10] Jian, H. Y., Γ-convergence of noncoercive functionals in vector-valued space $W^{1,p}(\Omega)$ [J] (in Chinese), *Science in China Ser. A,* **24:**3(1994), 233–240.

[11] De Giorgi, E. & Letta, G., Une notion general de convergence faible pour des fontions crotssantes d'ensemble [J], *Ann Scuola Norm. Sup. Pisa CI Sci.,* **4**(1977), 61–99.

[12] Yosida, K., Functional analysis [M], 5th ed, Springer-Verlag, 1975.