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A UNIFORM GROWTH ESTIMATES OF SOLUTIONS OF ASYMPTOTICALLY ELLIPTIC OPERATORS**

ZHANG Liqun*

Abstract

This paper introduces a generic eigenvalue flow of a parameter family of operators, where the corresponding eigenfunction is continuous in parameters. Then the author applies the result to the study of polynomial growth L-harmonic functions. Under the assumption that the operator has some weakly conic structures at infinity which is not necessarily unique, a Harnack type uniform growth estimate is obtained.

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§1. Introduction

Let us consider an elliptic operator L defined on R^n by

$$L = \frac{\partial}{\partial x_i}\Big[a^{ij}(x)\frac{\partial}{\partial x_j}\Big] \tag{1.1}$$

with $|a^{ij}|_{L^\infty(R^n)} \le C$ and $\lambda I \le (a^{ij}(x)) \le \Lambda I$ for some positive constants λ, Λ. The solution u of $Lu = 0$ is called the L-harmonic function.

In a recent work[6], Lin studied the L-harmonic functions and gave a Liouville type theorem, provided the operator is asymptotically conic. We are concerned here with the polynomial growth L-harmonic functions, that is, the L-harmonic function satisfying $|u| \le C(1+|x|)^d$. This problem, in general, has not been well understood, except the cases that for $n = 2$, L is the Laplace-Beltrami operator and $n \ge 3$ for some conic operators (see [6]) It is not known in general if there is a polynomial growth, nonconstant L-harmonic function on R^n. On the other hand, it is not hard to show there are many nonconstant L-harmonic functions on R^n whenever $a^{ij}(x)$ are local Lipschitz continuous (see [5]). The nonconstant L-harmonic function on R^n grows at least like a power of $|x|$.

The purpose of the article is to generalize the result of [6] to a larger class of elliptic operators by applying the generic eigenvalue flow. The operators we consider here are close

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*Institute of Mathematics, Academia Sinica, Beijing 100080, China.
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to, in some weak sense at the infinity, a family of elliptic operator on R^n. We first study the eigenvalues of the family of limiting operators.

It is well known that eigenvalues are continuously dependent on parameters in some nice space. We are interested in the continuity of eigenfunctions for one-parameter family of elliptic operators. Whenever the eigenfunction is continuously dependent on the parameters for the limiting operator, we can utilize it to construct almost L-harmonic functions for the original operator. And we have good estimate for those almost L-harmonic functions. Then we can obtain an estimate of the L-harmonic functions. In [18], K. Uhlenbeck proved that in the generic sense eigenvalues have one-dimensional eigenspace. Clearly, if all eigenvalues are of simple multiplicity, then the eigenfunctions are continuously dependent on the parameter. But in some cases, eigenspaces are not always one-dimensional. As we show in an example in [19], eigenvalues do often intersect when the parameter varies. And the study of intersection of those eigenvalues is useful in applications, especially for the multiple parameter case.

In Section 2, we first consider the G-convergent operator and then study the eigenfunctions on the cross section of the limiting operators. We shall apply the transversality theory to construct eigenfunction flows, where the eigenfunctions are continuous in the parameter. Then we apply it to the study of polynomial growth L-harmonic functions in Section 3. Here we make use of the so-called three annuli lemma, inspired by an idea of J. Cheeger. The three annuli lemma was used before by L. Simon[13], Cheeger and Tian[2], etc. Finally, we obtain an a priori estimate for the growth of L-harmonic functions.

§2. Weakly Asymptotically Conic Operators and Generic Eigenfunction Flow

For any bounded Lipschitz domain Ω in R^n, the operator L given in (1.1) defines a map from $H_0^1(\Omega)$ to $H_0^{-1}(\Omega)$ by Lu. We know that for any $f \in H_0^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ satisfying $Lu = f$. Let $C(\lambda, \Lambda)$ be the space of all linear operators given by (1.1) with $\lambda I \le (a^{ij}(x) \le \Lambda I$. We say that a sequence of operators L_k in $C(\lambda, \Lambda)$ is G-convergent to L, as $k \to \infty$, if for any $f, g \in H^{-1}(\Omega)$,

$$\lim_{k\to\infty} \langle g, L_k^{-1} f\rangle = \langle g, L^{-1} f\rangle.$$

There are many properties of G-convergence (see [6]). For convenience, we list a few basic properties.

(i) If L_k is G-convergent to L, $L_k u_k = f_k$ and $f_k \to f$ in $H^{-1}(\Omega)$, as $k \to \infty$, then $u_k \to u$ weakly in $H_0^1(\Omega)$ as $k \to \infty$ and $Lu = f$.

(ii) If L_k is G-convergent to L, $L_k u_k = f$, $u_k \in H_0^1(\Omega)$, then there exists $u \in H_0^1(\Omega)$ such that $u_k \to u$ weakly in $H_0^1(\Omega)$ and

$$\langle L_k u_k, u_k\rangle \to \langle Lu, u\rangle \text{ as } \ k \to \infty.$$

(iii) Let $L_k u_k = 0$ in B_R and L_k is G-convergent to L as operator $H_0^1(\Omega_{B_R}) \to H^{-1}(\Omega_{B_R})$. Suppose $|u_k|_{L^2(B_R)} \le 1$ and $u_k \to u$ weakly in L^2. Then $Lu = 0$ in B_R and $u_k \to u$ weakly in $H^1_{\rm loc}(B_R)$.

It is easy to see that $a_k^{ij}(x) \to a^{ij}(x)$ a.e in Ω implies that L_k is G-convergent to L.

We say that a sequence of operators L_k in $C(\lambda, \Lambda)$ defined on R^n is G-convergent to L, if L_k is G-convergent to L for any bounded Lipschitz domain Ω in R^n.

Now we consider a family of operators in R^n which will be our limit operator. Also the parameter t can be allowed to vary in a compact set of some manifolds, and we only consider $t \in S^1$ for simplicity.

Let $\Delta_{g(t)}$ be the Laplace-Beltrami operator on R^n with the metric

$$g(t) = dr^2 + r^2 g_{ij}(t,\theta) d\theta^i d\theta^j,$$

where (r,θ) is the polar-coordinate on R^n, $\theta \in S^{n-1}$ and $g_{ij}(t,\theta)$ is Lipschitz continuous and periodic in t with period of 2π.

We say that an operator L satisfies Assumption (S_1), if for any sequence $\{r_k\}$, $r_k \to \infty$, there exist a subsequence, still denoted by r_k, and a $t_0 \in [0, 2\pi)$ such that

$$L_{r_k} = \frac{\partial}{\partial x_i}\Big(a^{ij}(r_k x)\frac{\partial}{\partial x_j}\Big)$$

is G-convergent to $\Delta_{g(t_0,\theta)}$ as $k \to \infty$.

Remark 2.1.[6] L satisfies Assumption (S_1) implies that L is weakly asymptotically conic.

Example 2.1. Consider R^n with the metric

$$g = dr^2 + r^2(2 + \sin(\log(\log(e + r)))d\theta^2$$

in the polar coordinate. The corresponding Laplace-Beltrami operator on R^n satisfies the assumption (S_1), where the limit operator is given by the metric $g(t) = dr^2 + r^2(2+\sin t)d\theta^2$. In fact, for any sequence $\{r_k\}$, there exists a subsequence, still denoted by $\{r_k\}$, such that the corresponding Laplace-Beltrami operator is G-convergent to $\Delta_{g(t_0)}$ with some $t_0 \in [0, 2\pi)$.

Now we consider the eigenfunctions defined on $(S^{n-1}, g(t))$ where the metric tensor $g(t)$ depends on the parameter t twice continuously differentiable and periodic with period of 2π. Then we know that eigenvalues are continuous differentiable in t and

$$0 = \lambda_0(t) < \lambda_1(t) \le \lambda_2(t) \le \cdots.$$

Let $G = \{g(t) | g(t)$ is a metric of S^{n-1}, $g(t) \in C^2([0, 2\pi], T(M) \bigotimes T(M))\}$.

Let $S_k = \{u \in H^2_k(S^{n-1}), \int_{S^{n-1}} u^2 = 1\}$. And let

$$\mathcal{M}_2 = \{g | g \in C^2(T(S^{n-1}) \bigotimes T(S^{n-1})) \text{ is the metric of } S^{n-1}\}.$$

We consider the map $\phi : S_k \times R \times \mathcal{M}_2 \mapsto H_{k-2}(S^{n-1})$ given by

$$\phi(u, \lambda, g) = \Delta_g u + \lambda u = L_g u. \tag{2.1}$$

By the study of the regular value of ϕ, K. Uhlenbeck obtained the following result in [18].

Lemma 2.1. *For a metric $g \in \mathcal{M}_2$, the set $\{g \in \mathcal{M}_2 | \Delta_g$ has one-dimensional eigenfunction$\}$ is residual in $\mathcal{M}_2$.*

Therefore for $g(t) \in G$ we may assume that $\Delta_{g(0)}$ has one-dimensional eigenfunction. Consider the eigenvalues of $\Delta_{g(t)}$ for t varying. As we proved in an example in [19], they may have intersections.

For a fixed C_0, we consider those eigenvalues of $\Delta_{g(t)}$ with $\lambda_k(t) < C_0$. Let t_0 be the first point where

$$\cdots < \lambda_{k-1}(t_0) < \lambda_k(t_0) = \cdots = \lambda_{k+l}(t_0) < \lambda_{k+l+1}(t_0) < C_0. \tag{2.2}$$

Moreover, we may assume that those eigenvalues which are less than C_0 are one-dimensional in a small left neighborhood of t_0 and $t \neq t_0$. Since we assume $\Delta_{g(0)}$ has one-dimensional eigenfunctions, the normalized $\phi_k(t)$ with $\lambda_k(t) < C_0$ is in $C(S^{n-1} \times [0, \epsilon))$ for some small $\epsilon > 0$. And $\phi_k(t)$ is a continuous flow, if $\lambda_k(t)$ does not intersect other eigenvalues. We have proved in [19] that it can be continuously defined at the intersection point, up to a small perturbation. In fact, we have the following results in [19].

Lemma 2.2. *Let t_0 be as above. Then for a residual set of $g(t) \in G$,*

$$\lim_{t \to t_0} (\phi_k(t), \phi_{k+1}(t), \cdots, \phi_{k+l}(t)) \tag{2.3}$$

exists by a proper choice of the sign of those eigenfunctions.

Moreover the eigenfunction flow has a very important property, which is proved in [19].

Lemma 2.3. *If $g(t)$ is a periodic function of t with period of 2π, then for a residual set of $g(t) \in G$ the eigenfunction flow is a periodic function with period of 2π. Moreover, the corresponding eigenvalue $\lambda_k(t)$ satisfies*

$$\lambda_k(t) \le C_1 \lambda_k(0), \tag{2.4}$$

where the constant C_1 only depends on the Lipschitz norm of $g(t)$ in t.

Remark 2.2. Although we defined the eigenfunction flow for generic $g(t)$ in G, the estimate in (2.4) only depends on the Lipschitz constants of $g(t)$ in t. Therefore in application we can approximate the Lipschitz $g(t)$ by twice differentiable $g_\epsilon(t)$ and the estimate in(2.4) is uniform in ϵ.

§3. Uniform Growth Estimates

In this section, we study the uniform growth estimates of L-harmonic function with polynomial growth by using the generic eigenfunction flow and the three annuli lemma. This technique was used before by several authors (see [2, 12, 13,19]).

In the following discussion we only prove our result in a simple case. That is, we assume that $g(t)$ is continuously differentiable and 2π-periodic in t, where t depends on the subsequence of $\{r_k\}$. Our proof still works when $g(t)$ is only Lipschitz in t. We may assume, after a generic perturbation, $(S^{n-1}, g_\epsilon(t))$ has a continuous eigenfunction flow and $g_\epsilon(t, \theta) \to g(t, \theta)$ as $\epsilon \to 0$ in C^1. Let $\lambda_{j,\epsilon}(t)$ denote its eigenvalue. Let $A_{a,b}$ be the annulus on R^n with inner radius a and outer radius b.

We consider L-harmonic functions defined on R^n with $|u| \le C(1 + r^d)$, which is called at most d order growth L-harmonic function. Note the eigenvalue $\lambda_{j,\epsilon}(t)$ is a 2π periodic function. Let $\phi_{j_0,\epsilon}(t)$, $\phi_{j_1,\epsilon}(t)$, $\cdots$, $\phi_{j_{m_1},\epsilon}(t)$ be all those eigenfunctions whose eigenvalues intersect a number λ_{d_2} which will be given later with some $t \in [0, 2\pi]$. We stress that there are only finite number of such $\lambda_{j_k,\epsilon}(t)$ and the number is independent of ϵ.

We assume that L also satisfies (S_2) : For any $\delta > 0$, there exist $\{r_{k,i}\}$, $\{B_{k,i}\}$ and a division of $[0, 2\pi]$ with $B_{k,i} r_{k,i} = r_{k,r+1}$, $B_{k,I} r_{k,I} = r_{k+1,1}$, such that for any sequence s_k, $s_k \in [r_{k,i}, B_{k,i} r_{k,i}]$, L_{s_k} is G-convergent to $\Delta_{g(\overline{t_i})}$ as $k \to \infty$ where $t_i \le \overline{t_i} \le t_{i+1}$, $0 = t_1 < \cdots < t_i < \cdots < t_{I+1} = 2\pi$, and for any t, $s \in [t_i, t_{i+1}]$ we have $|g(t) - g(s)| < \delta$.

Let $\Phi_{k,i}$: $(r_{k,i}, B_{k,i} r_{k,i}) \times_r (S^{n-1}, g(t_i)) \mapsto A_{r_{k,i}, \Omega_{k,i} r_{k,i}}$ be the homeomorphic map. For i fixed, let $A^k_{1,B_{k,i}}$ be the rescaled $A_{r_{k,i}, B_{k,i} r_{k,i}}$ under the metric $\frac{1}{r_{k,i}^2} g$.

Now we let

$$\overline{\phi}_{j,\epsilon}(t_i) = \phi_{j,\epsilon}(t_i) \circ \Phi^{-1}_{k,i}, \tag{3.1}$$

where $\phi_{j,\epsilon}(t_i)$ is an eigenfunction on $(S^{n-1}, g_\epsilon(t_i))$. We define a new function $v_{j,\epsilon}$ by

$$v_{j,\epsilon} = \Big[\fint_{\partial B_r} u\overline{\phi_{j,\epsilon}}(t_i)\sqrt{|g_\epsilon|}d\theta \Big] \overline{\phi}_{j,\epsilon}(t_i) \tag{3.2}$$

for $r_{k,i} \le r \le B_{k,i} r_{k,i}$, where $\fint_{\partial B_r} = \frac{1}{\mathrm{vol}_{g(t_i)}(\partial B_r)} \int_{\partial B_r}$. It is easy to see that $v_{j,\epsilon} \in L^2_{\mathrm{loc}}(R^n)$. We note that $a_j r^{p_j(t_i)} \phi_j(t_i)$ is a harmonic function on $(0,\infty) \times_r (S^{n-1}, g(t_i))$, where $\lambda_j(t_i) = p_j(t_i)(p_j(t_i) + n - 2)$, $p_j(t_i) > 0$. We shall show that for i fixed and ϵ small, $v_{j,\epsilon}$ is near $a_j r^{p_j(t_i)} \phi_j(t_i)$ in some sense.

Given $B > 1$ and a family of annuli $\{A_{a_k, Ba_k}\}$, let $A^k_{1,B}$ be the rescaled annulus under the metric $\frac{1}{a_k^2} g$. Then we know there exists a subsequence of $\{a_k\}$, still denoted by a_k, and a metric $g(\bar{t})$, such that L_{a_k} is G-convergent to $\Delta_{g(\bar{t})}$. For k large, let $\Phi_k : A^k_{1,B} \mapsto (1, B) \times_r (S^{n-1}, g(\bar{t}))$ be a homeomorphic map. Put $u_k = u \circ \Phi_k^{-1}$, where u is a given L-harmonic function on R^n. From the property (iii) of G-convergence, we have the following lemma.

Lemma 3.1. *There exists a subsequence of* $\Big\{ \frac{u_k}{|u|_{L^\infty\left(A^k_{\frac{1}{2},2B}\right)}} \Big\}$, *still denoted by itself, such that*

$$\frac{u_k}{|u|_{L^\infty(A^k_{\frac{1}{2},2B})}} \to u_0 \qquad in \quad L^2((1,B) \times_r (S^{n-1}, g(\bar{t}))). \tag{3.3}$$

Furthermore, u_0 is a harmonic function.

Now, we let $\lambda^* = \inf\limits_{t \in [0,2\pi)} \lambda_1(t) > 0$, $d_0 < 0$ and $d_0(d_0 + n - 2) < \lambda^*$. Then we have the following three annuli lemma.

Lemma 3.2. *Given $B > 1$, there exists a k_0, such that for $k > k_0$,*

$$\fint_{A_{B^k, B^{k+1}}} u^2 < \frac{1}{2}\Big(B^{2d_0} \fint_{A_{B^{k-1}, B^k}} u^2 + B^{-2d_0} \fint_{A_{B^{k+1}, B^{k+2}}} u^2 \Big), \tag{3.4}$$

where $\fint_{A_{B^k,B^{k+1}}} u^2 = \frac{1}{\mathrm{vol}_{g(\bar{t})}(A_{B^k,B^{k+1}})} \fint_{A_{B^k,B^{k+1}}} u^2 dV_{g(\bar{t})}$. *So if*

$$\fint_{A_{B^k, B^{k+1}}} u^2 > B^{2d_0} \fint_{A_{B^{k-1}, B^k}} u^2,$$

then

$$\fint_{A_{B^{k+1}, B^{k+2}}} u^2 > B^{2d_0} \fint_{A_{B^k, B^{k+1}}} u^2.$$

Proof. We first prove (3.4) on the cone $(0, \infty) \times_r (S^{n-1}, g(\bar{t}))$. A harmonic function in a cone has the following expansions

$$u = \sum a_j r^{p_j} \phi_j + \sum b_j r^{-(n-2)-p_j} \phi_j,$$

where $p_j \ge 0$ and $p_j(p_j + n - 2) = \lambda_j$, λ_j and ϕ_j are respectively the eigenvalue and the eigenfunction in $(S^{n-1}, g(\bar{t}))$. Then we only need to show that

$$\frac{r^{2p_j+n}|_{B^k}^{B^{k+1}}}{r^n|_{B^k}^{B^{k+1}}} < \frac{1}{2}\Big(B^{2d_0} \frac{r^{2p_j+n}|_{B^{k-1}}^{B^k}}{r^n|_{B^{k-1}}^{B^k}} + B^{-2d_0} \frac{r^{2p_j+n}|_{B^{k+1}}^{B^{k+2}}}{r^n|_{B^{k+1}}^{B^{k+2}}} \Big) \tag{3.5}$$

or

$$\frac{r^{2q_j+n}|_{B^{k+1}}^{B^k}}{r^n|_{B^k}^{B^{k+1}}} < \frac{1}{2}\Big(B^{2d_0} \frac{r^{2q_j+n}|_{B^k}^{B^{k-1}}}{r^n|_{B^{k-1}}^{B^k}} + B^{-2d_0} \frac{r^{2q_j+n}|_{B^{k+2}}^{B^{k+1}}}{r^n|_{B^{k+1}}^{B^{k+2}}} \Big) \tag{3.6}$$

for $q_j = -(n-2) - p_j < \frac{n}{2}$, or for some p_j with $2(n-2) + 2p_j = n$,

$$\frac{\ln r|_{B^k}^{B^{k+1}}}{r^n|_{B^k}^{B^{k+1}}} < \frac{1}{2}\Big(B^{2d_0}\frac{\ln r|_{B^{k-1}}^{B^k}}{r^n|_{B^{k-1}}^{B^k}} + B^{-2d_0}\frac{\ln r|_{B^{k+1}}^{B^{k+2}}}{r^n|_{B^{k+1}}^{B^{k+2}}}\Big). \tag{3.7}$$

(3.5)–(3.7) can be reduced to

$$1 < \frac{1}{2}(B^{2d_0-2p_j} + B^{-2d_0+2p_j}) \tag{3.8}$$

or

$$1 < \frac{1}{2}(B^{2d_0-2q_j} + B^{-2d_0+2q_j}). \tag{3.9}$$

From our assumption we know $d_0 \neq q_j$, $d_0 \neq p_j$, and then (3.8) and (3.9) are true.

Now we prove Lemma 3.2 by contradictions. Suppose there exists a subsequence $\{l_k\}$ such that on annuli $A_{B^{l_k-1},B^{l_k}}$, $A_{B^{l_k},B^{l_k+1}}$ and $A_{B^{l_k+1},B^{l_k+2}}$ inequality (3.4) is not true. Then we apply Lemma 3.1 on $A_{B^{l_k-1},B^{l_k+2}}$, and deduce a contradiction by taking limit.

Lemma 3.3. *The limit u_0 in Lemma 3.2 has the following expansion in the cone $(1,B)\times_r (S^{n-1}, g(\bar{t}))$:*

$$u_0 = \sum a_j r^{p_j(\bar{t})}\phi_j(\bar{t}), \tag{3.10}$$

where $p_j(\bar{t}) \geq 0$ and $p_j(\bar{t})(p_j(\bar{t}) + n - 2) = \lambda_j(\bar{t})$.

Proof. From Lemma 3.2, we deduce that there exists a k_1 such that for $k > k_1$,

$$⨍_{A_{B^k,B^{k+1}}} u^2 > B^{2d_0} ⨍_{A_{B^{k-1},B^k}} u^2. \tag{3.11}$$

Otherwise we have for all $k > k_0$,

$$⨍_{A_{B^k,B^{k+1}}} u^2 < B^{2d_0} ⨍_{A_{B^{k-1},B^k}} u^2. \tag{3.12}$$

We note that the convergence in (3.3) is true in $L^2((\delta,\delta^{-1}) \times_r (S^{n-1}, g(\bar{t}))$, where δ is any small positive constant. Then (3.12) is also true for u_0 on any two annuli in $(\delta,\delta^{-1}) \times_r (S^{n-1}, g(\bar{t}))$. It follows that

$$u_0 = \sum b_j r^{-(n-2)-p_j}\phi_j. \tag{3.13}$$

Therefore there exists a ball B_{B^k} in R^n where u achieves its maximum in the interior of B_{B^k}. This contradicts the maximum principle. Then we apply (3.11) to its limit and Lemma 3.3 follows easily.

Now we go back to the function v_j. We have the following growth estimate.

Lemma 3.4. *Given $B > 1$, $\delta > 0$ and $m_1 > 0$, letting*

$$p_{m^*} = \inf_{t\in[0,2\pi]} p_{j(t),\epsilon}, \quad p_M = \sup_{t\in[0,2\pi]} p_{j(t),\epsilon},$$

we see that there exist R, ϵ_0 and a division I_0 such that for $\epsilon < \epsilon_0$, $a > R$, if

$$⨍_{A_{a,Ba}} v_j^2 \geq \frac{1}{3m_1} ⨍_{A_{a,Ba}} u^2, \tag{3.14}$$

then

$$⨍_{A_{Ba,B^2a}} v_j^2 \geq B^{2p_{m^*}-\delta} ⨍_{A_{a,Ba}} v_j^2, \tag{3.15}$$

$$⨍_{A_{Ba,B^2a}} v_j^2 \leq B^{2p_M+\delta} ⨍_{A_{a,Ba}} v_j^2. \tag{3.16}$$

Proof. We prove the lemma again by contradictions. Suppose there exist $\delta > 0$, $B > 1$ and $\{a_k\}$, $a_k \to \infty$ and ϵ sufficiently small, such that for any fixed division I, we have

$$\fint_{A_{Ba_k,B^2a_k}} v_j^2 \le B^{2p_{m^*}-\delta} \fint_{A_{a_k,Ba_k}} v_j^2, \tag{3.17}$$

$$\fint_{A_{Ba_k,B^2a_k}} v_j^2 \ge \frac{1}{3m_1} \fint_{A_{a_k,Ba_k}} u. \tag{3.18}$$

Furthermore, we may assume that there exists a $\bar{t}$, $t_{i-1} < \bar{t} \le t_i$, such that L_{a_k} is G-convergent to $\Delta_{g(\bar{t})}$. We choose ϵ small so that

$$|\mathrm{vol}_{g(t)}(S^{n-1}) - \mathrm{vol}_{g_\epsilon(t)}(S^{n-1})| < \epsilon_1, \quad |p_{j,\epsilon}(\bar{t}) - p_j(\bar{t}) < \epsilon_1.$$

By Lemma 3.1, we may assume $\frac{u}{|u|_{B^\infty(A^k_{1,B})}} \to u_0$ and u_0 satisfies (3.10). If the division I is large enough so that

$$|\phi_{j,\epsilon}(t_l) - \phi_{j,\epsilon}(\bar{t})|_{L^\infty(S^{n-1})} \le \frac{\epsilon_1}{\mathrm{vol}(S^{n-1})}, \quad l = i-1, i,$$

then

$$\Big|\fint_{(S^{n-1},g(\bar{t}))} \phi_{j,\epsilon}(t_l)\phi_{j,\epsilon}(\bar{t}) - 1\Big| < \epsilon_1, \quad l = i-1, i,$$

$$\Big|\fint_{(S^{n-1},g(\bar{t}))} \phi_{j,\epsilon}(t_l)\phi_{\alpha,\epsilon}(\bar{t})\Big| < \epsilon_1, \quad l = i-1, i, \quad \alpha \ne j.$$

Then from (3.17) we deduce

$$\begin{aligned} &\frac{1}{\mathrm{vol}_{g_\epsilon(\bar{t})}(A_{B,B^2})} \int_B^{B^2} a_j^2 r^{2p_{j,\epsilon}(\bar{t})+n-1} dr - 2\epsilon_1 \fint_{A_{B,B^2}} u_0^2 \\ &\le B^{2p_{m^*}-\delta}\Big[\frac{1}{\mathrm{vol}_{g_\epsilon(\bar{t})}(A_{1,B})} \int_1^B a_j^2 r^{2p_{j,\epsilon}(\bar{t})+n-1} dr + 2\epsilon_1 \fint_{A_{1,B}} u_0^2\Big]. \end{aligned} \tag{3.19}$$

And from (3.18) we deduce

$$\frac{1}{\mathrm{vol}_{g_\epsilon(\bar{t})}(A_{1,B})} \int_1^B a_j^2 r^{2p_{j,\epsilon}(\bar{t})+n-1} dr \ge \Big(\frac{1}{3m_1} - \epsilon_1\Big) \fint_{A_{1,B}} u_0^2. \tag{3.20}$$

Since $p_{j,\epsilon}(\bar{t}) \ge p_{m^*}$ for ϵ_1 sufficiently small, (3.19) and (3.20) give a contradiction. Similarly we can prove (3.16).

Now let $\lambda_d = d(d+n-2)$ and $j_{m_0,\epsilon}$ be the first integer satisfying

$$\min_{j > j_{m_0,\epsilon}, 0 \le t \le 2\pi} \lambda_{j,\epsilon}(t) > \lambda_d + \frac{\delta}{2}, \tag{3.21}$$

where $\lambda_{j,\epsilon}(t)$ are eigenvalues on $(S^{n-1}, g_\epsilon(t))$.

Let $\lambda_{d_1} = \max\limits_{j \le j_{m_0,\epsilon}, 0 \le t \le 2\pi} \lambda_{j_{m_0,\epsilon}}(t)$, $\lambda_{d_1} = d_1(d_1+n-2)$ with $d_1 > 0$. Choose $\lambda_{d_2} = \lambda_{d_1} + \delta$ and d_2 satisfying

$$\lambda_{d_2} = d_2(d_2+n-2), \quad d_2 > 0. \tag{3.22}$$

As we mentioned before, let $\lambda_{j_0,\epsilon}(t), \lambda_{j_1,\epsilon}(t), \cdots \lambda_{j_{m_1},\epsilon}(t)$ be all the eigenvalues which intersect λ_{d_2} for some $t \in [0, 2\pi]$. Obviously $j_{0,\epsilon} > j_{m_0,\epsilon}$ and then

$$\min_{0 \le l \le m_1, 0 \le t \le 2\pi} \lambda_{j_l,\epsilon}(t) > \lambda_d.$$

Put $v = \sum\limits_{j_0 \le j \le j_m} v_{j,\epsilon}$. It is easy to check that for any $\epsilon_1 > 0$ there exists an R_0 such that for $a > R$ and ϵ sufficiently small

$$\Big| \fint_{A_{a,Ba}} u^2 - \fint_{A_{a,Ba}} (u-v)^2 - \fint_{A_{a,Ba}} v^2 \Big| \le \epsilon_1 \fint_{A_{a,Ba}} u^2. \tag{3.23}$$

Since u is at most order d growth, from Lemma 3.4 and (3.23) we deduce that there exists an R_1 such that for $a > R_1$,

$$\fint_{A_{a,Ba}} (u-v)^2 > \fint_{A_{a,Ba}} u^2. \tag{3.24}$$

For $u-v$ we have the following growth estimate which is also called the three annulus lemma.

Lemma 3.5. *Let $B > 1$ and d_2 be given as before. Then there exist a k_0 and a division I_0 such that for $k > k_0$ and ϵ small,*

$$\fint_{A_{B^k,B^{k+1}}} (u-v)^2 < \frac{1}{2}\Big[B^{2d_2} \fint_{A_{B^{k-1},B^k}} (u-v)^2 + B^{-2d_2} \fint_{A_{B^{k+1},B^{k+2}}} (u-v)^2\Big]. \tag{3.25}$$

Therefore if

$$\fint_{A_{B^k,B^{k+1}}} (u-v)^2 > B^{2d_2} \fint_{A_{B^{k-1},B^k}} (u-v)^2,$$

then

$$\fint_{A_{B^{k+1},B^{k+2}}} (u-v)^2 > B^{2d_2} \fint_{A_{B^k,B^{k+1}}} (u-v)^2.$$

Proof. We prove this lemma again by contradictions. Suppose that there exists a sequence $\{k_l\}$, $k_l \to \infty$, such that (3.25) is not true for any given I_0 and small ϵ. As in the proof of Lemma 3.4, we may assume $L_{B^{k_l}}$ is G-convergent to $\Delta_{g(\bar t)}$ with $t_{i-1} < \bar t \le t_i$, and $\frac{u}{|u|_{L^\infty\left(A^{k_l}_{1,B^3}\right)}} \to u_0$, u_0 satisfying (3.10). We choose ϵ small enough so that

$$|\mathrm{vol}_{g(t)}(S^{n-1}) - \mathrm{vol}_{g_\epsilon(t)}(S^{n-1})| < \epsilon_1$$

and for $j = j_0, \cdots, j_{m_1}$,

$$|p_j(t) - p_{j,\epsilon}(t)| < \epsilon_1, \quad |\phi_{j,\epsilon}(\bar t) - \phi_j(\bar t)|_{L^2(S^{n-1})} < \epsilon_1.$$

If I is large enough so that

$$|\phi_{j,\epsilon}(t_l) - \phi_{j,\epsilon}(\bar t)|_{L^\infty(S^{n-1})} \le \frac{\epsilon_1}{m_1 \mathrm{vol}(S^{n-1})}, \quad l = i-1, i, \quad j = j_0, j_1, \cdots j_{m_1},$$

then we deduce

$$\begin{aligned}
&\sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{B,B^2}} r^{2p_j(\bar t)+n-1} \phi_j^2(\bar t) - C\epsilon_1 \fint_{A_{B,B^2}} u_0^2 \\
&\ge \frac{1}{2}\Big[B^{2d_2}\Big(\sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{1,B}} r^{2p_j(\bar t)+n-1} \phi_j^2(\bar t) + C\epsilon_1 \fint_{A_{1,B}} u_0^2\Big) \\
&\quad + B^{-2d_2}\Big(\sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{B^2,B^3}} r^{2p_j(\bar t)+n-1} \phi_j^2(\bar t) + C\epsilon_1 \fint_{A_{B^2,B^3}} u_0^2\Big)\Big],
\end{aligned} \tag{3.26}$$

where $A_{1,B} = (1, B) \times_r (S^{n-1}, g(\bar t))$. Since $d_2 \ne p_j(\bar t)$ for $j \ne j_0, \cdots j_{m_1}$, for ϵ_1 small (3.26) is impossible.

From Lemma 3.4 and Lemma 3.5, we can have the growth estimate.

Lemma 3.6. *If u is an L-harmonic function with growth order at most d, then for $B > 1$ fixed, there exists a k_0 such that for $k > k_0$,*

$$\fint_{A_{B^k,B^{k+1}}} u^2 \le B^{2d_2} \fint_{A_{B^{k-1},B^k}} u^2. \tag{3.27}$$

Proof. By Lemma 3.4, we only need to show that (3.27) is true for $k > k_0$ and some large I for $u - v$. From Lemma 3.5, if (3.27) is not true for $u - v$, then for $k > k_0$ and ϵ small

$$\fint_{A_{B^k,B^{k+1}}} (u-v)^2 > B^{2d_2} \fint_{A_{B^{k-1},B^k}} (u-v)^2. \tag{3.28}$$

By induction, we have

$$\fint_{A_{B^k,B^{k+1}}} (u-v)^2 \ge B^{2d_2(k-k_0-2)} \fint_{A_{B^{k_0+1},B^{k_0+2}}} (u-v)^2. \tag{3.29}$$

We note $d_2 > d$, (3.23) and (3.24), then (3.29) is impossible.

We actually proved the following growth estimates for L-harmonic functions.

Theorem 3.1. *Under the assumption of (S_2), for given $B > 1$, then there exist $C = C(L)$, $k_0 = k_0(L)$ such that for any at most order d growth L-harmonic function u, we have*

$$\fint_{A_{B^k,B^{k+1}}} u^2 \le B^{2Cd(k-k_0-2)} \fint_{A_{B^{k_0+1},B^{k_0+2}}} u^2$$

for $k > k_0$.

Now we have the following corollary.

Theorem 3.2. *Under the assumption of (S_2), then the dimension of at most order d growth L-harmonic functions is not more than the dimension of at most order Cd growth harmonic functions on its tangent cone at infinity. Here $C = C(L)$.*

Proof. Let u, w be two linear independent L-harmonic functions on R^{n-1} with growth order at most d.

Put $w_k = \gamma_k w$, $u_k = \alpha_k u - \beta_k w$ such that

$$\fint_{A_{B^{k-1},B^{k+2}}} {w_k}^2 = 1,$$

$$\fint_{A_{B^{k-1},B^{k+2}}} {u_k}^2 = 1,$$

$$\fint_{A_{B^{k-1},B^{k+2}}} w_k u_k = 0.$$

We may assume, as before, there exists a subsequence $\{k_l\}$ so that

$$L_{B^{k_l}} \to \Delta_{g(\bar{t})},$$

where the convergence is L-convergent. And u_{k_l} and w_{k_l} have limit $\overline{u}$ and $\overline{w}$ respectively. By Lemma 3.3 and Lemma 3.6, we can deduce as before

$$\overline{u} = \sum_{0 \le p_j \le d_2} a_j r^{p_j} \phi_j, \tag{3.30}$$

$$\overline{w} = \sum_{0 \le p_j \le d_2} c_j r^{p_j} \phi_j. \tag{3.31}$$

Moreover

$$\fint_{A^{\infty}_{L^{-1},L^2}} \overline{uw} = 0. \tag{3.32}$$

Then our theorem follows easily from (3.30)–(3.32).

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