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ON THE CONVERGENCE OF GODUNOV SCHEME FOR NONLINEAR HYPERBOLIC SYSTEMS

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Abstract

The authors consider systems of the form

$$u_t + A(u)u_x = 0, \qquad u \in \mathbb{R}^n,$$

where the matrix $A(u)$ is assumed to be strictly hyperbolic and with the property that the integral curves of the eigenvector fields are straight lines. For this class of systems one can define a natural Riemann solver, and hence a Godunov scheme, which generalize the standard Riemann solver and Godunov scheme for conservative systems. This paper shows convergence and L^1 stability for this scheme when applied to data with small total variation. The main step in the proof is to estimate the increase in the total variation produced by the scheme due to quadratic coupling terms. Using Duhamel's principle, the problem is reduced to the estimate of the product of two Green kernels, representing probability densities of discrete random walks. The total amount of coupling is then determined by the expected number of crossings between two random walks with strictly different average speeds. This provides a discrete analogue of the arguments developed in [3,9] in connection with continuous random processes.

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§1. Introduction

We consider the one dimensional Cauchy problem for an $n \times n$ system of the form

$$u_t + A(u)u_x = 0, \quad u(x,0) = \bar{u}(x). \tag{1.1}$$

Here $A(u)$ is a smooth matrix valued map from a domain $\mathcal{U} \subset \mathbb{R}^n$ into $\mathbb{R}^{n\times n}$, and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. The system is assumed to be strictly hyperbolic, i.e. the matrix $A(u)$ has n real and strictly different eigenvalues at each point $u \in \mathcal{U}$.

We note that even for smooth data a classical solution is only defined locally in time. In general the solution will develop discontinuities in finite time and it is not clear how to prolong the solution beyond this point. However, in the conservative case, i.e. $A(u)$ is the Jacobian of some map $f : \mathcal{U} \to \mathbb{R}^n$, Glimm[13] proved global existence of weak entropy solutions of (1.1) when the data has small total variation and each characteristic field is genuinely nonlinear or linearly degenerate. Under the same assumptions, recent work has

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established the uniqueness and L^1 stability of solutions obtained as limits of the Glimm scheme or by wavefront tracking[5-8].

A major open question is whether other natural methods of approximation yield the same BV solutions. In particular one has considered the methods of vanishing viscosity, relaxation and finite difference schemes.

For scalar equations it is known that all of these methods work in the sense that the approximate solutions converge strongly in L^1 to the unique entropy solution of the conservation law. In addition some results are also available for 2×2 systems[12] and for $n \times n$ Temple class systems[16] using compensated compactness methods. For general conservative $n \times n$ systems, convergence results for vanishing viscosity approximations are known only in the case where the limit solution is piecewise smooth[14,18].

A basic method for proving the convergence of a sequence of approximate solutions is the use of a compactness argument in the space BV. If one can show an a priori estimate on the total variation of the approximate solutions, then Helly's theorem guarantees convergence (of a subsequence) toward a BV function. Using recent uniqueness results (see [10]) one can then show that this limit is the appropriate entropy weak solution.

Recently such a priori BV estimates have been established for systems of the form (1.1) under the assumption that the integral curves of the eigenvector fields of $A(u)$ are straight lines in state space. For these straight-line systems one can define a Riemann solver which extends the usual Riemann solver when the system is in conservative form. Bianchini and Bressan[4] showed that the approximate solutions obtained by vanishing viscosity converge and yield an L^1 Lipschitz continuous semigroup which is consistent with the natural Riemann solver for piecewise constant data. Similar results have been obtained by Bressan and Shen[9] and Bianchini[3] in connection with relaxation.

The aim of the present paper is to show that, under the same straight-line assumption, the approximate solutions constructed by the Godunov scheme converge to the same solution as obtained by vanishing viscosity[4].

The reason for treating this class of systems separately is that in this case the oscillation generated in one component of the solution is only due to the interaction of waves from different families. The straight-line assumption is a transversality condition which makes it possible to control the creation of new oscillations in the solution. The influence of other terms in the general case seems to require other methods.

We observe that the same class of systems has been singled out by Arora and Roe as particularly well behaved in connection with numerical computation of slowly moving shocks. Indeed, in [2] they conjectured that if the straight-line assumption is not satisfied, then any numerical scheme using a Godunov or a Roe flux will create spurious oscillations.

Although the results we prove apply to general (nonconservative) straight-line systems, we will throughout the paper indicate the corresponding results for the conservative case.

The rest of the paper is organized as follows. In the next section we define the Riemann solver and the corresponding Godunov scheme and state the main results. We then define strengths and speeds of waves and show that the increase in total variation is due only to transversal coupling (see Lemma 3.1 below). In Section 4 we establish a priori BV bounds on the approximate solutions generated by the scheme. The proof is based on Duhamel's principle and a basic estimate on the product of Green kernels (Lemma 4.1) proved by probabilistic methods in Section 5. Finally, we consider the L^1 stability of the approximate solutions and prove the main theorems.

§2. The Godunov Scheme and Main Results

We assume that the matrix $A(u)$ has n real and distinct eigenvalues $\lambda^k(u)$, $k=1,\cdots,n$, $u\in\mathcal{U}$. By performing the linear change of independent coordinates

$$t' = 2\bar{\lambda}t, \quad x' = x+\bar{\lambda}t, \quad \text{where } \bar{\lambda} > \max_{u\in\mathcal{U},1\le k\le n} |\lambda^k(u)|, \tag{2.1}$$

we can assume that

$$0<\lambda^1(u)<\cdots<\lambda^n(u)<1. \tag{2.2}$$

By possibly restricting to a smaller domain we can also assume that there are constants $\bar{\lambda}^0=0<\bar{\lambda}^1<\cdots<\bar{\lambda}^{n-1}<\bar{\lambda}^n=1$ such that

$$\lambda^k(u)\in(\bar{\lambda}^{k-1},\bar{\lambda}^k), \quad u\in\mathcal{U}, \quad k=1,\cdots,n.$$

The corresponding right and left eigenvectors are denoted by $r^k(u)$, $l^k(u)$, respectively, and normalized such that

$$|r^k(u)|\equiv 1, \quad r^i(u)\cdot l^j(u)\equiv \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\ne j. \end{cases}$$

The basic assumption throughout the paper is that the integral curves of the eigenvector fields are straight lines. This can be expressed by the relation

$$(Dr^k(u))r^k(u)=0, \qquad u\in\mathcal{U}, \quad k=1,\cdots,n, \tag{2.3}$$

where Dr^k denotes the Jacobian matrix of r^k. In the conservative case when $A(u)$ is a Jacobian matrix this condition implies that the shock curves and rarefaction curves coincide and are straight lines in $\mathcal{U}$ (see [17]).

2.1. Riemann Solver and Godunov Scheme

We now describe a natural way of solving the Riemann problem for (1.1), that is, when the initial data consists of two nearby constant states separated by a jump discontinuity,

$$\bar{u}(x)=\begin{cases} u_- & \text{for } x<0, \\ u_+ & \text{for } x>0. \end{cases} \tag{2.4}$$

For $u\in\mathcal{U}$, $k=1,\cdots,n$, let $R^k(\cdot)(u)$ denote the integral curve of r^k through u, parametrized by arc length,

$$R^k(\sigma)(u)=u+\sigma r^k(u).$$

Given $u_-,u_+\in\mathcal{U}$, by strict hyperbolicity and the Implicit Function Theorem (if necessary we restrict $\mathcal{U}$ further) there exist unique intermediate states $\omega_0=u_-,\omega_1,\cdots,\omega_n=u_+$ and wave strengths σ^k such that

$$\omega_k=R^k(\sigma^k)(\omega_{k-1}), \qquad k=1,\cdots,n.$$

For each k, define the scalar function

$$F^k(\sigma)=\int_0^\sigma \lambda^k\left(R^k(s)(\omega_{k-1})\right)\,ds,$$

and let $z^k(x,t)$ be the unique (self similar) entropy solution to the scalar Riemann problem

$$z_t + F^k(z)_x = 0, \qquad z(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ \sigma^k & \text{for } x > 0 \end{cases} \tag{2.5}$$

We can now define a self similar solution of the original Riemann problem by using the solution of these scalar Riemann problems to move along the integral curves. More precisely, we first define the map $\mathcal{R}(u_-, u_+) : [0,1] \to \mathcal{U}$ by

$$\mathcal{R}(u_-, u_+)(\xi) = R^k(z^k(\xi,1))(\omega_{k-1}) \qquad \text{for } \xi \in [\bar\lambda^{k-1}, \bar\lambda^k], \quad k = 1, \cdots, n. \tag{2.6}$$

This map is well defined since

$$z^k(x,t) = \begin{cases} 0 & \text{for } x/t < \bar\lambda^{k-1}, \\ \sigma^k & \text{for } x/t > \bar\lambda^k. \end{cases}$$

We define the solution of the Riemann problem (1.1),(2.4) to be the function $u(x,t)$ given by

$$u(x,t) = \begin{cases} u_- & \text{for } x/t < 0, \\ \mathcal{R}(u_-, u_+)(x/t) & \text{for } x/t \in [0,1], \\ u_+ & \text{for } x/t > 1. \end{cases} \tag{2.7}$$

It is easily verified that this solution coincides with the usual Riemann solution when the system can be written in conservative form. For general, nonconservative systems of the form (1.1), there is no natural definition of weak solutions. Still, in the case where (2.3) is satisfied, the definition above is the appropriate one in the sense that it coincides with the limit of vanishing viscosity approximations (see [4]).

Having defined a Riemann solver we can now define the corresponding Godunov scheme. By the assumption (2.2) the Courant-Friedrichs-Lewy condition (see [15]) is satisfied with equal time and space step $\Delta x = \Delta t$. The constant value of the approximate solution in the i-th cell at time $j\Delta t$ is denoted by $u_{i,j}$. The Godunov scheme is now defined inductively as follows: At time $t = 0$ we let $u_{i,0}$ denote the cell average over the i-th cell of the initial data $\bar u$,

$$u_{i,0} = \frac{1}{\Delta x} \int_{i\Delta x}^{(i+1)\Delta x} \bar u(\xi)\, d\xi.$$

Given the values $u_{i-1,j}$ and $u_{i,j}$ in the $(i-1)$-th and i-th cells, respectively, at time $j\Delta t$, $j \geq 0$, we define the value $u_{i,j+1}$ as the x-average at time Δt over the interval $[0, \Delta x]$ of the Riemann solution with left state $u_{i-1,j}$ and right state $u_{i,j}$. Using the notation above the scheme can be written as

$$u_{i,j+1} = \frac{1}{\Delta x} \int_0^{\Delta x} \mathcal{R}(u_{i-1,j}, u_{i,j})(x/\Delta t)\, dx = \int_0^1 \mathcal{R}(u_{i-1,j}, u_{i,j})(\xi)\, d\xi. \tag{2.8}$$

The scheme is well defined as long as the values $u_{i,j}$ remain in the domain of definition of the matrix A. In the conservative case when $A(u) = Df(u)$ this is the standard Godunov scheme which may be rewritten in finite difference form

$$u_{i,j+1} = u_{i,j} - [f(u_{i,j}) - f(u_{i-1,j})], \quad i \in \mathbb{Z}, \quad j \in \mathbb{Z}_+. \tag{2.9}$$

For each Δx the scheme gives an approximate solution $u_{\Delta x}$ of (1.1) defined for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ in the following way. At each time step $j\Delta t$ we define $u_{\Delta x}$ to be equal to the cell average $u_{i,j}$ on each cell $[i\Delta x, (i+1)\Delta x] \times \{j\Delta t\}$. For intermediate times $t \in (j\Delta t, (j+1)\Delta t)$, we let $u_{\Delta x}$ be the function obtained by patching together the solutions of the Riemann problems given by the cell averages at time $j\Delta t$.

2.2. Main Results

Our main results are the uniform BV bounds, the L^1 stability, and the convergence of the approximate solutions to (1.1) produced by the Godunov scheme (2.8). We assume that the normalization (2.2) is satisfied and that $\Delta x = \Delta t$. For a fixed time step $j\Delta t$ we let $u(j)$ denote the discrete function $i \mapsto u_{i,j}$. The total variation of $u(j)$ is then given by

$$\mathrm{T.V.}[u(j)] = \sum_{i\in\mathbb{Z}} |u_{i,j} - u_{i-1,j}|.$$

Theorem 2.1. *Suppose that $A(u)$ is strictly hyperbolic for each $u \in \mathcal{U}$ and that the normalized eigenvectors satisfy* (2.3). *Then there exist constants $\delta_0, \delta_1 > 0$ such that the following holds. For each initial data $\bar{u}$ with*

$$\mathrm{T.V.}[\bar{u}] < \delta_0, \tag{2.10}$$

the corresponding solution $u_{i,j}$ of the Godunov scheme (2.8) *is well defined for all time steps $j \in \mathbb{Z}_+$ and satisfies*

$$\mathrm{T.V.}[u(j)] < \delta_1 \quad \text{for all } j \in \mathbb{Z}_+. \tag{2.11}$$

Furthermore, there exists a constant L such that for all pairs of initial data $\bar{u}, \bar{v}$ satisfying (2.10)*, the corresponding solutions satisfy*

$$\|u(j) - v(j')\|_{\mathrm{L}^1} \leq L \cdot (|j - j'| \cdot \Delta t + \|\bar{u} - \bar{v}\|_{\mathrm{L}^1}). \tag{2.12}$$

This stability result implies the convergence of the Godunov scheme to the same solution as given by the method of vanishing viscosity[4]. Let δ_0 and L be as in Theorem 2.1 and let $\mathcal{D}$ denote the set of $\mathrm{L}^1_{\mathrm{loc}}$-functions with total variation bounded by δ_0. We then have the following theorem.

Theorem 2.2. *As the discretization parameter $\Delta x = \Delta t$ tends to zero, the approximate solutions given by the Godunov scheme* (2.8) *converge to the same limit as given by the method of vanishing viscosity. This limit can be characterized as the trajectory of a semigroup $S : \mathcal{D} \times \mathbb{R}^+ \to \mathcal{D}$ with the properties*

(i) *For every $\bar{u}, \bar{v} \in \mathcal{D}$ and every $t, s \geq 0$,*

$$\|S_t\bar{u} - S_s\bar{v}\|_{\mathrm{L}^1} \leq L \cdot (|t - s| + \|\bar{u} - \bar{v}\|_{\mathrm{L}^1}). \tag{2.13}$$

(ii) *For every piecewise constant initial data $\bar{u} \in \mathcal{D}$, there exists a positive time $\tau > 0$ such that the semigroup trajectory $S_t\bar{u}$ on $[0, \tau]$ coincides with the function obtained by patching together the solutions of the Riemann problems given by $\bar{u}$ and solved according to* (2.7).

In particular, for conservative systems the Godunov scheme yields the trajectories of the Standard Riemann Semigroup.

§3. Wave Strengths and Wave Speeds

In order to study the increase in the total variation of the approximate solutions constructed by the scheme, we consider the equations for the strengths of waves. We obtain these from (2.8) by projecting along the characteristic curves. A basic observation is that, up to higher order terms, these equations can be interpreted as equations for discrete random walks on the underlying grid. For each characteristic family we have a random walk for which the characteristic speed gives the probability for a particle to have a unit forward jump. The idea is to use this probabilistic interpretation to simplify the analysis of the equations for the wave strengths.

We measure the strengths of waves in terms of arc length along the integral curves R^k. Given a left state u_- and a right state u_+ connected by a single k-wave of strength σ^k, the speed of the wave is

$$\lambda^k(u_-,u_+)=\int_0^1 \lambda^k(u_-+\xi\sigma^k r^k(u_-))\,d\xi. \tag{3.1}$$

The increase in total variation introduced by the scheme can now be studied as follows. Given three consecutive states u_-, u_0, and u_+, let $\bar u_-$ and $\bar u_+$ be the two resulting states given by the scheme (2.8). That is, let

$$\bar u_-=\int_0^1 \mathcal{R}(u_-,u_0)(\xi)\,d\xi, \qquad \bar u_+=\int_0^1 \mathcal{R}(u_0,u_+)(\xi)\,d\xi. \tag{3.2}$$

The strengths of the k-waves, $k=1,\cdots,n$, in the three Riemann problems (u_-,u_0), (u_0,u_+), and $(\bar u_-,\bar u_+)$ are denoted by σ_-^k, σ_+^k, and $\bar\sigma^k$, respectively. The intermediate states in the two Riemann problems (u_-,u_0) and (u_0,u_+) are denoted by $u_-^1,\cdots,u_-^{n-1}$ and $u_+^1,\cdots,u_+^{n-1}$, respectively. Now define the map

$$\Psi^k(u_0;\sigma_-;\sigma_+)=\bar\sigma^k-(\lambda^k(u_-^{k-1},u_-^k)\sigma_-^k+[1-\lambda^k(u_+^{k-1},u_+^k)]\sigma_+^k), \tag{3.3}$$

where $\sigma_\pm=(\sigma_\pm^1,\cdots,\sigma_\pm^n)$. The map Ψ^k measures the change in the amount of waves in the k-th family produced by the Godunov scheme. Note that in the scalar case Ψ^k vanishes identically. In general Ψ^k is nonzero because of the nonlinear coupling among the wave families. However, for straight-line systems we have the following representation.

Lemma 3.1. *Suppose that the straight-line assumption* (2.3) *holds and let* Ψ^k *be defined by* (3.3). *Then there exist smooth functions* $A_{p,q}^k$, $B_{p,q}^k$, *and* $C_{p,q}^k$ *depending on* u_0,σ_-,σ_+ *such that*

$$\Psi^k(u_0;\sigma_-;\sigma_+)=\sum_{1\le p\ne q\le n}(A_{p,q}^k\sigma_+^p\sigma_+^q+B_{p,q}^k\sigma_+^p\sigma_-^q+C_{p,q}^k\sigma_-^p\sigma_-^q) \tag{3.4}$$

for all $u_0\in\mathcal{U}$ *and for all sufficiently weak strengths* $\sigma_\pm^1,\cdots,\sigma_\pm^n$.

Proof. We first show that Ψ^k has the property that

$$\Psi^k(u_0;0,\cdots,0,\sigma_-^k,0,\cdots,0;0,\cdots,0,\sigma_+^k,0,\cdots,0)\equiv 0.$$

Using the same notation as above, these values of the strengths correspond to

$$u_0=u_-+\sigma_-^k r_0^k, \qquad u_+=u_0+\sigma_+^k r_0^k,$$

where $r_0^k=r^k(u_0)=r^k(u_-)=r^k(u_+)$. From the definition of the Riemann solver it follows that

$$\bar u_-=u_0-\lambda^k(u_-,u_0)\sigma_-^k r_0^k, \qquad \bar u_+=u_+-\lambda^k(u_0,u_+)\sigma_+^k r_0^k. \tag{3.5}$$

Hence

$$\bar u_+-\bar u_-=(\lambda^k(u_-,u_0)\sigma_-^k+[1-\lambda^k(u_0,u_+)]\sigma_+^k)r_0^k,$$

such that the Riemann problem $(\bar{u}_-, \bar{u}_+)$ is solved by a single wave of family k and with strength $(\lambda^k(u_-, u_0)\sigma_-^k + [1-\lambda^k(u_0, u_+)]\sigma_+^k)$. This shows that

$$\bar{\sigma}^k = \lambda^k(u_-, u_0)\sigma_-^k + [1-\lambda^k(u_0, u_+)]\sigma_+^k,$$

that is,

$$\Psi^k(u_0; 0, \cdots, 0, \sigma_-^k, 0, \cdots, 0; 0, \cdots, 0, \sigma_+^k, 0, \cdots, 0) = 0.$$

From the straight-line assumption (2.3) it also follows that

$$\Psi^k(u_0; 0, \cdots, 0, \sigma_-^i, 0, \cdots, 0; 0, \cdots, 0, \sigma_+^i, 0, \cdots, 0) = 0 \quad \text{for } i \neq k.$$

We now apply a standard representation result for functions of several variables: a smooth function $F : \mathbb{R}^{2n} \to \mathbb{R}$ can be represented in the form

$$\begin{aligned} F(x_1, \cdots, x_n; y_1, \cdots, y_n) = F(0;0) &+ \sum_{i=1}^{n} x_p G_p(x_p, x_{p+1}, \cdots, x_n; y_1, \cdots, y_n) \\ &+ \sum_{i=1}^{n} y_p H_p(y_p, y_{p+1}, \cdots, y_n), \end{aligned}$$

where the functions G_p, H_p, $p = 1, \cdots, n$, are smooth (see [11, p.193]). Suppose that the function F has the property that it vanishes along pairs of coordinate axes in the sense that

$$F(0, \cdots, 0, x_p, 0, \cdots, 0; 0, \cdots, 0, y_p, 0, \cdots, 0) \equiv 0.$$

From the representation it immediately follows that

$$\begin{aligned} &G_p(x_p, 0, \cdots, 0; 0, \cdots, 0, y_p, 0, \cdots, 0) \equiv 0, \\ &H_p(y_p, 0, \cdots, 0) \equiv 0. \end{aligned}$$

A first order Taylor expansion of the functions G_p, H_p then yields the representation

$$F(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum_{1 \le p \neq q \le n} (A_{p,q} x_p x_q + B_{p,q} x_p y_q + C_{p,q} y_p y_q),$$

where $A_{p,q}$, $B_{p,q}$, $C_{p,q}$ are smooth functions of $x_1, \cdots, x_n, y_1, \cdots, y_n$. Applying this to the function $\Psi^k(u_0; \cdot; \cdot)$ gives the representation (3.4).

This lemma shows that the increase in total variation is due only to transversal coupling terms, i.e. terms involving two different characteristic families.

Returning to the scheme (2.8), we let $\sigma_{i,j}^k$ denote the strength of the k-wave in the solution of the Riemann problem $(u_{i-1,j}, u_{i,j})$, and we let $\lambda_{i,j}^k$ be the corresponding speed given by (3.1). By Lemma 3.1 the strengths satisfy the following system of equations,

$$\sigma_{i,j+1}^k = \lambda_{i-1,j}^k \sigma_{i-1,j}^k + \big(1 - \lambda_{i,j}^k\big)\sigma_{i,j}^k + Q_{i,j}^k \tag{3.6}$$

for $k = 1, \cdots, n$, where the quadratic coupling terms are given by

$$Q_{i,j}^k = \sum_{1 \le p \neq q \le n} \Big(A_{p,q}^k \sigma_{i,j}^p \sigma_{i,j}^q + B_{p,q}^k \sigma_{i,j}^p \sigma_{i-1,j}^q + C_{p,q}^k \sigma_{i-1,j}^p \sigma_{i-1,j}^q \Big). \tag{3.7}$$

Observe that, if the coupling term $Q_{i,j}^k$ vanishes identically, then (3.6) describes the dynamics of a discrete random walk whose particle at (i, j) jumps with probability $\lambda_{i,j}^k$ to $(i+1, j+1)$, and with probability $(1 - \lambda_{i,j}^k)$ to $(i, j+1)$.

§4. A Priori Bounds on the Total Variation

We seek an a priori bound on the total variation of the approximate solution. We do this by bounding the total amount of waves in each family uniformly in time. That is, we establish estimates on the sums

$$\mathrm{V}^k(j) := \sum_{i=-\infty}^{\infty} |\sigma_{i,j}^k|, \qquad k = 1, \cdots, n, \tag{4.1}$$

that are independent of the time step j. We will derive a functional relation (see (4.14)) which implies that if the initial amount of waves is sufficiently small, then it remains small for all times. This result implies that the scheme is well defined for all times, and that the sequence of approximate solutions converges to a function of bounded variation. The functional relation is deduced by applying Duhamel's principle, a comparison lemma, and a basic estimate on the product of two Green kernels (see Lemma 4.1).

We start by assuming that the scheme is well defined up to time step j and we let C_0 be a constant that dominates the absolute values of all $A_{p,q}^k$, $B_{p,q}^k$, and $C_{p,q}^k$. From (3.6) it follows that

$$\mathrm{V}^k(j+1) \le \mathrm{V}^k(j) + \sum_{i=-\infty}^{+\infty} |Q_{i,j}^k| \le \mathrm{V}^k(0) + \sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} |Q_{i,m}^k|. \tag{4.2}$$

Defining the magnitude $\mathrm{Q}^k(j)$ by

$$\mathrm{Q}^k(j) := \sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} |\mathrm{Q}_{i,m}^k|, \tag{4.3}$$

and using (2.7), we obtain the following bound

$$\begin{aligned}\mathrm{Q}^k(j) &\le C_0 \cdot \sum_{p\ne q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} \big(|\sigma_{i,m}^p||\sigma_{i,m}^q| + |\sigma_{i-1,m}^p||\sigma_{i,m}^q| + |\sigma_{i-1,m}^p||\sigma_{i-1,m}^q|\big)\\ &\le 2C_0 \cdot \sum_{p\ne q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} \big(|\sigma_{i,m}^p||\sigma_{i,m}^q| + |\sigma_{i-1,m}^p||\sigma_{i,m}^q|\big) =: E_1 + E_2,\end{aligned} \tag{4.4}$$

where E_1 denotes the first part of the sum (including the terms $|\sigma_{i,m}^p||\sigma_{i,m}^q|$), and E_2 denotes the second part.

For a given set of k-speeds $\lambda_{i,j}^k \in (\bar\lambda^{k-1}, \bar\lambda^k)$, $k = 1, \cdots, n$, $(i,j) \in \mathbb{Z}\times\mathbb{Z}_+$, consider the linear homogeneous difference equation

$$\rho_{i,j+1} = \lambda_{i-1,j}^k \rho_{i-1,j} + \big(1 - \lambda_{i,j}^k\big)\rho_{i,j}. \tag{4.5}$$

Denote by $\Gamma^k(i,j;i',j')$ the corresponding Green kernel. In other words, for $j \ge j'$, $\Gamma^k(i,j;i',j') = \rho_{i,j}$ is the value of the solution of (4.5) at the node (i,j), with initial data $\rho_{i',j'} = \delta_{(i',j')}$. Here, $\delta_{(i',j')}$ is given by

$$\delta_{(i',j')}(i,j) = \begin{cases} 1 & \text{if } (i,j) = (i',j'),\\ 0 & \text{otherwise.}\end{cases} \tag{4.6}$$

Note that the Green function is non-negative. By Duhamel's principle we can write the solution of the linear, non-homogeneous system (3.6) in the form

$$\sigma^k_{i,j} = \sum_{l=-\infty}^{+\infty} \Gamma^k(i,j;l,0)\sigma^k_{l,0} + \sum_{r=0}^{j-1}\sum_{l=-\infty}^{+\infty} \Gamma^k(i,j;l,r)\mathrm{Q}^k_{l,r} =: \alpha^k_{i,j} + \beta^k_{i,j}. \tag{4.7}$$

We thus have

$$\begin{aligned} E_1 &\leq 2C_0 \cdot \sum_{p\neq q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} (|\alpha^p_{i,m}||\alpha^q_{i,m}| + |\alpha^p_{i,m}||\beta^q_{i,m}| + |\beta^p_{i,m}||\alpha^q_{i,m}| + |\beta^p_{i,m}||\beta^q_{i,m}|) \\ &= 2C_0 \cdot (S_1 + S_2 + S_3 + S_4). \end{aligned} \tag{4.8}$$

To proceed, we need an estimate on the total interaction between solutions of two systems of the form (4.5) with strictly different speeds. We postpone the proof of the following key estimate to the next section.

Lemma 4.1 (Total Interaction). *Assume that $p < q$ and that $0 < \lambda^p_{i,j} < \lambda < \tilde\lambda < \lambda^q_{i',j'} < 1$ for all $(i,j),(i',j') \in \mathbb{Z}\times\mathbb{Z}_+$. Let $\rho^p_{i,j}$ and $\rho^q_{i,j}$ denote the solutions of (4.5) with $k=p$, $k=q$ and with initial data ρ^p_0 and ρ^q_0, respectively. Then we have the estimate*

$$\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} |\rho^p_{i,m}||\rho^q_{i,m}| \leq C(p,q)\cdot\Big(\sum_{i=-\infty}^{+\infty}|\rho^p_0(i)|\Big)\cdot\Big(\sum_{i=-\infty}^{+\infty}|\rho^q_0(i)|\Big), \tag{4.9}$$

where the constant $C(p,q)$ satisfies

$$C(p,q) \leq \frac{1}{\tilde\lambda - \lambda}. \tag{4.10}$$

Thanks to Lemma 4.1, we can now estimate each term on the right hand side of (4.8). Consider first S_1. Recalling (4.7) and using (4.9) we obtain

$$\begin{aligned} S_1 &= \sum_{p\neq q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} |\alpha^p_{i,m}||\alpha^q_{i,m}| \\ &\leq \sum_{p\neq q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty}\Big(\sum_{l=-\infty}^{+\infty}|\Gamma^p(i,m;l,0)||\sigma^p_{l,0}|\Big)\Big(\sum_{h=-\infty}^{+\infty}|\Gamma^q(i,m;h,0)||\sigma^q_{h,0}|\Big) \\ &= \sum_{p\neq q}\sum_{l=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty}|\sigma^p_{l,0}||\sigma^q_{h,0}|\Big(\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty}|\Gamma^p(i,m;l,0)||\Gamma^q(i,m;h,0)|\Big) \\ &\leq \sum_{p\neq q}\sum_{l=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty} C(p,q)|\sigma^p_{l,0}||\sigma^q_{h,0}| \leq C_1\cdot\sum_{p\neq q}\mathrm{V}^p(0)\cdot\mathrm{V}^q(0), \end{aligned}$$

where $C_1 := \max_{p\neq q} C(p,q)$. Concerning the sum S_2, we have

$$\begin{aligned} S_2 &= \sum_{p\neq q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty}|\alpha^p_{i,m}||\beta^q_{i,m}| \\ &\leq \sum_{p\neq q}\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty}\Big(\Big|\sum_{l=-\infty}^{+\infty}\Gamma^p(i,m;l,0)\sigma^p_{l,0}\Big|\Big)\Big(\sum_{r=0}^{j-1}\sum_{h=-\infty}^{+\infty}|\Gamma^q(i,m;h,r)||\mathrm{Q}^q_{h,r}|\Big) \\ &= \sum_{p\neq q}\sum_{r=0}^{j-1}\sum_{h=-\infty}^{+\infty}|\mathrm{Q}^q_{h,r}|\cdot\Big[\sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty}\Big(\Big|\sum_{l=-\infty}^{+\infty}\Gamma^p(i,m;l,0)\sigma^p_{l,0}\Big|\Big)|\Gamma^q(i,m;h,r)|\Big]. \end{aligned}$$

Here we have used the convention that the Green functions are zero for negative times. We now observe that the term $\sum_{l=-\infty}^{+\infty} \Gamma^p(i,m;l,0)\sigma_{l,0}^p$ is the solution of (4.5) (with $k=p$) at time step m and with data $\sum_{l=-\infty}^{+\infty} \Gamma^p(i,r;l,0)\sigma_{l,0}^p$ given at time step r. Similarly, $\Gamma^q(i,m;h,r)$ is the solution of (4.5) (with $k=q$) at time step m and with data $\delta_{(h,r)}$ given at time step r. An application of Lemma 4.1, and the fact that the L^1 norm of a solution of (4.5) is non-increasing, thus give

$$S_2 \le \sum_{p\neq q}\sum_{r=0}^{j-1}\sum_{h=-\infty}^{+\infty} |\mathrm{Q}_{h,r}^q| \cdot C(p,q)\cdot \mathrm{V}^p(r) \le C_1 \cdot \sum_{p\neq q} \mathrm{Q}^q(j-1)\cdot \mathrm{V}^p(0),$$

where Q^q was defined at (4.3). Using the same arguments to estimate S_3 and S_4, we obtain

$$E_1 \le C_1 \cdot \sum_{p\neq q} (\mathrm{V}^p(0)+\mathrm{Q}^p(j-1))(\mathrm{V}^q(0)+\mathrm{Q}^q(j-1)). \tag{4.11}$$

The quantity E_2 in (4.4) can be estimated in an entirely similar way. From (4.4) we thus obtain

$$\mathrm{Q}^k(j) \le 2C_1 \cdot \sum_{p\neq q} (\mathrm{V}^p(0)+\mathrm{Q}^p(j-1))(\mathrm{V}^q(0)+\mathrm{Q}^q(j-1)). \tag{4.12}$$

We define

$$\mathbf{V}(j) = \sum_{k=1}^{n} \mathrm{V}^k(j), \qquad \mathbf{Q}(j) = \sum_{k=1}^{n} \mathrm{Q}^k(j). \tag{4.13}$$

From (4.2) and (4.12)–(4.13) we thus have

$$\mathbf{V}(j) \le \mathbf{V}(0) + \mathbf{Q}(j-1), \quad \mathbf{Q}(j) \le C_2 \cdot (\mathbf{V}(0)+\mathbf{Q}(j-1))^2, \tag{4.14}$$

where $C_2 = 2n\cdot C_1$. It follows that, if the total amount of waves $\mathbf{V}(0)$ in the initial data is sufficiently small, then $\mathbf{Q}(j)$, and hence also the total amount of waves $\mathbf{V}(j)$, remains small for all time steps. Relying on the key estimate, this completes the proof of the first part of Theorem 2.1.

§5. Bounds on the Total Interaction

In this section we provide a proof of Lemma 4.1. By linearity it suffices to prove (4.9) in the case where the initial data have the Dirac form

$$\rho_0^p = \delta_{(l,0)}, \quad \rho_0^q = \delta_{(h,0)}.$$

In this case the lemma states that

$$\mathcal{E}_{l,h} := \sum_{m=0}^{j}\sum_{i=-\infty}^{+\infty} \Gamma^p(i,m;l,0)\Gamma^q(i,m;h,0) \le C(p,q). \tag{5.1}$$

To prove (5.1), we first consider the case where the jump probabilities $\lambda_{i,j}^k$ do not depend on i,j. We will then extend to the case of variable coefficients by a comparison argument. In these proofs we use the probabilistic interpretation of (4.5) as an equation for random walks on the grid $\mathbb{Z}\times\mathbb{Z}_+$.

5.1. Case 1. The Coefficients Satisfy $0<\lambda_{i,j}^p \equiv \lambda < \tilde{\lambda} \equiv \lambda_{i,j}^q < 1$.

The following lemma yields the result in the case of constant coefficients in (4.5).

Lemma 5.1. *Assume that* $0 < \lambda < \tilde{\lambda} < 1$, *and let* $G_{i,j}$, $\tilde{G}_{i,j}$ *be the solutions of the two* (*decoupled*) *difference equations*

$$G_{i,j+1} = \lambda G_{i-1,j} + (1-\lambda)G_{i,j}, \tag{5.2}$$

$$\tilde{G}_{i,j+1} = \tilde{\lambda}\tilde{G}_{i-1,j} + (1-\tilde{\lambda})\tilde{G}_{i,j}, \tag{5.3}$$

with initial data $G_0 = \delta_{(l,0)}$, $\tilde{G}_0 = \delta_{(h,0)}$, *respectively. We then have the following estimate*

$$E_{l,h} := \sum_{j=0}^{+\infty} \sum_{i=-\infty}^{+\infty} G_{i,j}\tilde{G}_{i,j} \le \frac{1}{\tilde{\lambda} - \lambda}. \tag{5.4}$$

Proof. The solutions of (5.2),(5.3) with the prescribed initial data are given explicitly by the binomial distributions,

$$G_{i,j} = \binom{j}{i-l} \lambda^{i-l}(1-\lambda)^{j-i+l}, \qquad \tilde{G}_{i,j} = \binom{j}{i-h} \tilde{\lambda}^{i-h}(1-\tilde{\lambda})^{j-i+h}. \tag{5.5}$$

First consider the case where $l = h = 0$, i.e. the case where both particles start at the origin. In this case the sum can be evaluated by using properties of the Legendre polynomials L_N. These are given by Rodrigues' formula (see [1]),

$$L_N(y) = \frac{1}{2^N N!} \frac{d^N}{dy^N}[(y^2-1)^N],$$

and a straightforward calculation shows that

$$(z-1)^j L_j\left(\frac{z+1}{z-1}\right) = \sum_{i=-\infty}^{+\infty} \binom{j}{i}^2 z^i.$$

Using the fact that the generating function of the Legendre polynomials is

$$\sum_{j=0}^{+\infty} \alpha^j L_j(y) = \frac{1}{\sqrt{1-2\alpha y + \alpha^2}},$$

we calculate the sum $E_{0,0}$ as follows. Let

$$z = \frac{\lambda\tilde{\lambda}}{(1-\lambda)(1-\tilde{\lambda})}, \quad \alpha = (1-\lambda)(1-\tilde{\lambda})(1-z), \quad y = \frac{z+1}{z-1}.$$

Then,

$$E_{0,0} = \sum_{j=0}^{+\infty} \alpha^j L_j(y) = \frac{1}{\tilde{\lambda} - \lambda}. \tag{5.6}$$

Next let the particles start at $(l,0)$, $(h,0) \in \mathbb{Z} \times \mathbb{Z}_+$, respectively. We observe that $E_{l,h}$ represents the expected number of collisions between a λ-path given by (5.2) and a $\tilde{\lambda}$-path given by (5.3). Let $P_{l,h}$ denote the probability that the two paths never collide. Also, let F be the expected number of future collisions of two paths starting from the same point, that is, $F = E_{0,0} - 1$. We then have the relation, $E_{l,h} = (1-P_{l,h})(1+F) = (1-P_{l,h})E_{0,0} \le E_{0,0}$.

5.2. Case 2. The Coefficients Satisfy $0<\lambda^p_{i,j}<\lambda<\tilde{\lambda}<\lambda^q_{i,j}<1$.

To establish (5.1) in the general case where the jump probabilities $\lambda^k_{i,j}$ vary with (i,j) we use a comparison argument, showing that the double sum in (5.1) is majorized by the corresponding sum in (5.4). For this purpose we shall interpret the Green kernels Γ^p and G as transition probabilities for two random walks π^p and π defined on the same underlying probability space and with the property that, if the π^p-path starts to the left of the π-path, then it remains to the left for all later time steps. More precisely, let $X_1, X_2, \cdots$ be a sequence of random variables on some underlying probability space Ω. Assume that they are independent and uniformly distributed with values in $[0,1]$. For each $\omega \in \Omega$ and any starting point (i_0, j_0), we construct two paths $j \mapsto \pi^p(j) = \pi^p(j;\omega,i_0,j_0)$ and $j \mapsto \pi(j) = \pi(j;\omega,i_0,j_0)$ by defining $\pi^p(j_0) = \pi(j_0) = i_0$, and

$$\begin{aligned} \pi^p(j+1) &= \begin{cases} \pi^p(j) & \text{if } X_j(\omega) \in [0, 1-\lambda^p_{i,j}], \\ \pi^p(j)+1 & \text{if } X_j(\omega) \in (1-\lambda^p_{i,j}, 1], \end{cases} \\ \pi(j+1) &= \begin{cases} \pi(j) & \text{if } X_j(\omega) \in [0, 1-\lambda], \\ \pi(j)+1 & \text{if } X_j(\omega) \in (1-\lambda, 1] \end{cases} \end{aligned} \tag{5.7}$$

for $j \geq j_0$. By this construction we obtain

$$\begin{aligned} \Gamma^p(i,j;i_0,j_0) &= \text{Prob.}[\ \omega \in \Omega : \quad \pi^p(j;\omega,i_0,j_0) = i\ ], \\ G(i-i_0, j-j_0) &= \text{Prob.}[\ \omega \in \Omega : \quad \pi(j;\omega,i_0,j_0) = i\ ]. \end{aligned}$$

Since $\lambda^k_{i,j} < \lambda$ for every (i,j), it follows that

$$\pi^p(j;\omega,i'_0,j'_0) \leq \pi(j;\omega,i_0,j_0) \qquad \text{for each } \omega \in \Omega, \text{ provided } i'_0 \leq i_0. \tag{5.8}$$

Similarly, for $\omega \in \Omega$ and any starting point (i_0,j_0) we define a $\tilde{\pi}$-path $j \mapsto \tilde{\pi}(j) = \tilde{\pi}(j;\omega,i_0,j_0)$ obeying the $\tilde{\lambda}$-statistics given by (5.3), and a π^q-path

$$j \mapsto \pi^q(j) = \pi^q(j;\omega,i_0,j_0)$$

obeying the $\lambda^q_{i,j}$-statistics given by (4.5) with $k = q$. These paths are such that

$$\tilde{\pi}(j;\omega,i_0,j_0) \leq \pi^q(j;\omega,i'_0,j'_0) \qquad \text{provided } i_0 \leq i'_0. \tag{5.9}$$

We now observe that $\mathcal{E}_{l,h}$ is the expected number of collisions between a π^p-path and a π^q-path, given that they start at $(l,0),(h,0) \in \mathbb{Z} \times \mathbb{Z}_+$, respectively. For $(i,j) \in \mathbb{Z} \times \mathbb{Z}_+$, let $\mathcal{P}^{(i,j)}_{l,h}$ denote the probability that the paths collide for the first time at the point (i,j). Also let $\mathcal{F}^{(i,j)}$ be the expected number of future collisions between a π^p-path and a π^q-path when they both start from the point (i,j). From (5.8),(5.9) it follows that $\mathcal{F}^{(i,j)} \leq F$, for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}_+$, where F is as in the proof of Lemma 5.1. We thus have

$$\begin{aligned} \mathcal{E}_{l,h} &= \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}_+} \mathcal{P}^{(i,j)}_{l,h} \cdot (1 + \mathcal{F}^{(i,j)}) \\ &\leq (1+F) \cdot \Big(\sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}_+} \mathcal{P}^{(i,j)}_{l,h} \Big) \leq (1+F) = E_{0,0}. \end{aligned}$$

This completes the proof of Lemma 4.1.

§6. L¹ Stability

Following [4,6] we prove L^1 stability by a linearization argument. Consider two initial data $\bar{u}, \bar{v}$ which we join by the path defined by

$$\theta \mapsto \bar{u}^\theta = \theta\bar{u} + (1-\theta)\bar{v}, \quad \theta \in [0,1]. \tag{6.1}$$

Let $u^\theta_{i,j}$ denote the approximate solution computed with the Godunov scheme applied to the initial data $\bar{u}^\theta$ and let $u_{i,j} = u^1_{i,j}, v_{i,j} = u^0_{i,j}$. We then consider the equation for the infinitesimal perturbation $du^\theta_{i,j}/d\theta$. In analogy with the proof of the BV bounds we will project this tangent vector along eigenvectors and study how the components evolve from one time-step to the next.

We first establish a result corresponding to Lemma 2.1. Given two states u_-, u_+ let $\tilde{u}$ be the resulting state given by the Godunov scheme, i.e.

$$\tilde{u} = \int_0^1 \mathcal{R}(u_-, u_+)(\xi)\, d\xi. \tag{6.2}$$

Using the definition of the Riemann solver we can write this more explicitly as

$$\begin{aligned}
\tilde{u} &= \sum_{m=1}^{n} \int_{\bar{\lambda}^{m-1}}^{\bar{\lambda}^m} R(z^m(\xi,1))(\omega_{m-1})\, d\xi \\
&= \sum_{m=1}^{n} \int_{\bar{\lambda}^{m-1}}^{\bar{\lambda}^m} \left[\omega_{m-1} + z^m(\xi,1) r^m(\omega_m)\right] d\xi \\
&= \sum_{m=1}^{n} \left\{ (\bar{\lambda}^m - \bar{\lambda}^{m-1})\omega_{m-1} + \sigma^m[\bar{\lambda}^m - \lambda^m(\omega_{m-1},\omega_m)] r^m(\omega_m) \right\},
\end{aligned} \tag{6.3}$$

where we have applied the same notation as in Section 2.1.

Assume now that the right and left states depend on a parameter θ, $u_- = u_-^\theta$, $u_+ = u_+^\theta$. The resulting state $\tilde{u} = \tilde{u}^\theta$ is given by (6.3) where now also the strengths and intermediate states will depend on θ, $\omega_m = \omega_m^\theta$, $\sigma^m = \sigma^{\theta,m}$. Differentiation with respect to the parameter yields

$$\frac{d\tilde{u}^\theta}{d\theta} = \sum_{m=1}^{n} \left\{ (\bar{\lambda}^m - \bar{\lambda}^{m-1}) \frac{d\omega_{m-1}^\theta}{d\theta} + \frac{d}{d\theta}\left[\sigma^{\theta,m}\left(\bar{\lambda}^m - \lambda^m(u_-^\theta, u_+^\theta)\right) r^m(\omega_m^\theta)\right] \right\}.$$

We decompose the tangent vectors $\frac{du_\pm^\theta}{d\theta}$ and $\frac{d\tilde{u}^\theta}{d\theta}$ along the right eigenvectors at $u_\pm^\theta$ and $\tilde{u}^\theta$, respectively,

$$\frac{du_\pm^\theta}{d\theta} = \sum_{m=1}^{n} \nu_\pm^{\theta,m} r^m(u_\pm^\theta), \qquad \frac{d\tilde{u}^\theta}{d\theta} = \sum_{m=1}^{n} \tilde{\nu}^{\theta,m} r^m(\tilde{u}^\theta).$$

For each fixed value of the parameter $\theta = \theta_*$ the tangent vector $\frac{d\tilde{u}^\theta}{d\theta}\Big|_{\theta=\theta_*}$ is uniquely given by $u_-^{\theta_*}$, $\sigma^{\theta_*} = (\sigma^{\theta_*,1}, \cdots, \sigma^{\theta_*,n})$ and $\nu_\pm^{\theta_*} = (\nu_\pm^{\theta_*,1}, \cdots, \nu_\pm^{\theta_*,n})$. Hence there exists a (smooth) map $\Xi : \mathbb{R}^{4n} \to \mathbb{R}^n$ such that

$$\frac{d\tilde{u}^\theta}{d\theta}\Big|_{\theta=\theta_*} = \Xi(u_-^{\theta_*}, \sigma^{\theta_*}, \nu_-^{\theta_*}, \nu_+^{\theta_*}).$$

With this notation we have $\tilde{\nu}^{\theta,k} = l^k(\tilde{u}^\theta) \cdot \Xi(u_-^\theta, \sigma^\theta, \nu_-^\theta, \nu_+^\theta)$. Now consider the functions $\Phi^k : \mathbb{R}^{4n} \to \mathbb{R}$, $k = 1, \cdots, n$, defined by

$$\Phi^k(u_-, \sigma, \nu_-, \nu_+) = l^k(\tilde{u}) \cdot \Xi(u_-, \sigma, \nu_-, \nu_+) - \left(\lambda^k(u_-)\nu_-^k + [1 - \lambda^k(u_+)]\nu_+^k\right),$$

where u_+ is obtained from u_- by moving along integral curves according to $\sigma = (\sigma^1, \cdots, \sigma^n)$, and $\tilde{u}$ is given by (6.2).

Clearly, if both u_- and u_+ are independent of θ, then so is $\tilde{u}$, whence

$$\Phi^k(u_-, \sigma, 0, 0) = 0 \quad \text{for all } u_-, \sigma.$$

We also have that $\Phi^k(u_-, 0, \nu_-, \nu_+) = 0$ for all u_-, ν_-, ν_+. To obtain this we observe that if, for a particular value θ_* of the parameter, we have $\sigma^{\theta_*} = 0$, then $\tilde{u}^{\theta_*} = u_+^{\theta_*} = u_-^{\theta_*}$, and

$$\frac{d\sigma^{\theta,k}}{d\theta}\Big|_{\theta=\theta_*} = \nu_+^{\theta_*,k} - \nu_-^{\theta_*,k}.$$

Furthermore, the rate of change of the intermediate states is given by

$$\frac{d\omega_m^\theta}{d\theta}\Big|_{\theta=\theta_*} = \sum_{l=1}^m \nu_+^{\theta_*,l} r^l(u_-^{\theta_*}) + \sum_{l=m+1}^n \nu_-^{\theta_*,l} r^l(u_-^{\theta_*}).$$

Substitution into (6.3) yields

$$\tilde{\nu}^{\theta_*,k} = l^k(u_-^{\theta_*}) \cdot \Big(\frac{d\tilde{u}^\theta}{d\theta}\Big|_{\theta=\theta_*}\Big) = [1-\lambda^k(u_-^{\theta_*})]\nu_+^{\theta_*,k} + \lambda_k(u_-^{\theta_*})\nu_-^{\theta_*,k},$$

that is, $\Phi^k(u_-^{\theta_*}, 0, \nu_-^{\theta_*}, \nu_+^{\theta_*}) = 0$. It follows that the function Φ^k can be represented in the form

$$\Phi^k(u_-, \sigma, \nu_-, \nu_+) = \sum_{1\le p,q\le n} \Big[A_{p,q}^k \sigma^p \nu_-^q + B_{p,q}^k \sigma^p \nu_+^q\Big],$$

where $A_{p,q}^k$ and $B_{p,q}^k$ are smooth functions of $u_-, \sigma, \nu_-, \nu_+$.

Returning to the scheme we thus have the following system of equations for the components $\nu_{i,j}^{\theta,k}$ of the tangent vectors $du_{i,j}^\theta/d\theta$,

$$\nu_{i,j+1}^{\theta,k} = \lambda^k(u_{i-1,j}^\theta)\nu_{i-1,j}^{\theta,k} + [1-\lambda^k(u_{i,j}^\theta)]\nu_{i,j}^{\theta,k} + \mathcal{Q}_{i,j}^{\theta,k}, \tag{6.4}$$

where the coupling terms are given by

$$\mathcal{Q}_{i,j}^{\theta,k} = \sum_{1\le p,q\le n} \Big[A_{p,q,i,j}^{\theta,k} \sigma_{i,j}^{\theta,p} \nu_{i-1,j}^{\theta,q} + B_{p,q,i,j}^{\theta,k} \sigma_{i,j}^{\theta,p} \nu_{i,j}^{\theta,q}\Big].$$

Here $A_{p,q,i,j}^{\theta,k}$ and $B_{p,q,i,j}^{\theta,k}$ denote $A_{p,q}^k$ and $B_{p,q}^k$, respectively, evaluated at the point $(u_{i,j}^\theta, \sigma_{i,j}^\theta, \nu_{i-1,j}^\theta, \nu_{i,j}^\theta)$. The L^1 norm of the difference between $u_{i,j} = u_{i,j}^\theta\big|_{\theta=1}$ and $v_{i,j} = v_{i,j}^\theta\big|_{\theta=0}$ at time step $(j+1)$ is now given by

$$\begin{aligned}\|u(j+1) - v(j+1)\|_{\mathrm{L}^1} &= \sum_{i=-\infty}^{+\infty} |u_{i,j+1} - v_{i,j+1}|\Delta x \le \Delta x \cdot \Big(\sum_{i=-\infty}^{+\infty} \int_0^1 \Big|\frac{du_{i,j+1}^\theta}{d\theta}\Big| d\theta\Big) \\ &= \Delta x \cdot \Big(\int_0^1 \sum_{i=-\infty}^{+\infty} \sum_{k=1}^n |\nu_{i,j+1}^{\theta,k}| \, d\theta\Big) =: \Delta x \cdot \mathcal{N}(j+1). \end{aligned}\tag{6.5}$$

Using (6.4) we have the following estimate for $\mathcal{N}(j+1)$,

$$\begin{aligned}\mathcal{N}(j+1) &\le \int_0^1 \sum_{i=-\infty}^{+\infty} \sum_{k=1}^n \big\{\lambda^k(u_{i-1,j}^\theta)|\nu_{i-1,j}^{\theta,k}| + [1-\lambda^k(u_{i,j}^\theta)]|\nu_{i,j}^{\theta,k}| + |\mathcal{Q}_{i,j}^{\theta,k}|\big\} \, d\theta \\ &= \mathcal{N}(j) + \int_0^1 \Big\{\sum_{i=-\infty}^{+\infty} \sum_{k=1}^n |\mathcal{Q}_{i,j}^{\theta,k}|\Big\} \, d\theta. \end{aligned}\tag{6.6}$$

We thus have that

$$\mathcal{N}(j+1) \le \mathcal{N}(0) + \int_0^1 \sum_{i=-\infty}^{+\infty} \sum_{k=1}^n \sum_{s=0}^j |\mathcal{Q}_{i,s}^{\theta,k}| \, d\theta. \tag{6.7}$$

A simple estimate now shows that for a suitable constant C_3,

$$\int_0^1 \sum_{i=-\infty}^{+\infty} \sum_{k=1}^n \sum_{s=0}^j |\mathcal{Q}_{i,s}^{\theta,k}| \, d\theta \le C_3 \cdot \big(\max_{0\le s\le j} \max_{0\le\theta\le 1} \mathbf{V}^\theta(s) \cdot \mathcal{N}(s)\big),$$

where $\mathbf{V}^\theta(s)$, the total strength of waves in $u^\theta(s)$, is defined by (3.13). From the result in Section 4 it follows that, if the total amount of waves in the two initial data $\bar{u}$ and $\bar{v}$ are sufficiently small, then $\mathbf{V}^\theta(s)$ is uniformly bounded and small (with respect to both the time s and the parameter θ). Thus, for data with sufficiently small total variation we have

$$C_4 := C_3 \cdot \big(\max_{0\le s\le j} \max_{0\le\theta\le 1} \mathbf{V}^\theta(s) \big) < 1,$$

such that $\mathcal{N}(j+1) \le \mathcal{N}(0) + C_4 \cdot \max_{0\le s\le j} \mathcal{N}(s)$. Substituting this back into (6.5) we conclude that there exists a Lipschitz constant L such that

$$\|u(j) - v(j)\|_{\mathrm{L}^1} \le L \cdot \|u(0) - v(0)\|_{\mathrm{L}^1}.$$

The L^1 Lipschitz continuity of an approximate solution with respect to time is easily checked. Up to a change in the Lipschitz constant L, this concludes the proof of Theorem 2.1.

§7. Convergence of the Godunov Scheme

To establish the convergence of the scheme we argue as in [4]. We first define a semigroup by using the approximate solutions $u_{\Delta x}$ defined in Section 2.1. Given $\bar{u} \in \mathcal{D}$, choose a sequence of grid lengths Δx_μ with the property that $\lim_{\mu\to\infty} \Delta x_\mu \to 0$, and a sequence of initial data $\bar{u}_\mu$ such that $\bar{u}_\mu \to \bar{u}$ in $\mathrm{L}^1_{\mathrm{loc}}$. Here $\bar{u}_\mu$ is assumed to be piecewise constant with points of discontinuity lying in the set $\Delta x_\mu \cdot \mathbb{Z}$. For each μ we know by the first part of Theorem 2.1 that the scheme gives a global approximate solution u_μ corresponding to the initial data $\bar{u}_\mu$. These approximate solutions have uniformly bounded total variation such that Helly's compactness theorem[10] implies the existence of a sub-sequence (still denoted by u_μ) which converges in $\mathrm{L}^1_{\mathrm{loc}}$ to a function u for all times $t \ge 0$. By a diagonalization argument we may assume that this convergence holds for all initial data in a countable dense subset $\mathcal{D}'$ of $\mathcal{D}$. Now given a general element $\bar{v} \in \mathcal{D}$ we approximate it by functions $\bar{v}_\mu \in \mathcal{D}'$ with corresponding approximate solutions v_μ converging to a function v. By the second part of Theorem 2.1 the limit is unique and independent of the particular sequence used in the approximation. We can thus define

$$S_t\bar{v} := v(\cdot,t) = \mathrm{L}^1_{\mathrm{loc}} - \lim_{\mu\to\infty} v_\mu(\cdot,t).$$

It remains to show that this semigroup is compatible with the Riemann solver given in Section 2.1. By the same argument as in Corollary 9.2 in [10], it suffices to show that this is the case when each jump in $\bar{u}$ is solved by one single wave. By finite speed of propagation, it is enough to consider one of these jumps. Assume therefore that the left and right states u_-, u_+ are connected by a single wave of the k-th family, say, with strength σ^k. According to the definitions in Section 2.1 the solution of the Riemann problem (u_-, u_+) is then given by

$$u(x,t) = u_- + z(x,t) r^k(u_-), \tag{7.1}$$

where $z(x,t)$ solves the scalar Riemann problem (2.5). For a given Δx the approximate Godunov-solution is given as

$$u_{\Delta x}(x,t) = u_- + z_{\Delta x}(x,t) r^k(u_-),$$

where $z_{\Delta x}$ is the approximate solution of (2.5) computed by the scalar Godunov scheme. For the scalar case it is well-known that, as the discretization parameter Δx approaches zero, the approximate solution given by the Godunov scheme converges to the unique entropic solution $z(x,t)$ of (2.5) (see [15]). Since all states lie on the k-th integral curve through u_-, this demonstrates the convergence in this case. This completes the proof of Theorem 2.2.

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