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ON THE ZEROS OF QUOTIENTS OF BESSEL FUNCTIONS

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Abstract

The paper studies the zeros of $2mI_m/I_{m-1}-(m+1)I_1/I_0$, where I_m are the Bessel functions.

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§1. The Main Results

In this paper we establish the following result concerning the Bessel functions $I_m(x)$:

Theorem 1.1. (i) *For any $m \geq 2$ there exists a unique positive solution $x = x_m$ of the equation*

$$\frac{I_m(x)}{I_{m-1}(x)} - \frac{m+1}{2m}G(x) = 0, \tag{1.1}$$

where

$$G(x) = \frac{I_1(x)}{I_0(x)}. \tag{1.2}$$

(ii) *If $2 \leq l < m$, then*

$$x_l < x_m. \tag{1.3}$$

This theorem is used in the study of free boundary problems[1].

The following result will be used in the proof of (1.3):

Theorem 1.2. *The function $G(x)$ is concave for $0 < x < \infty$; more precisely, $G''(x) < 0$ for $0 < x < \infty$.*

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§2. Proof of Theorem 1.1 (i)

We recall (see, for instance, [3]) that $I_m(x)$ satisfies the differential equation

$$I''_m(x) + \frac{1}{r} I'_m(x) - \Big(1 + \frac{m^2}{x^2}\Big) I_m(x) = 0 \tag{2.1}$$

and is given by

$$I_m(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{m+2k}}{k!\Gamma(m+k+1)}. \tag{2.2}$$

Furthermore

$$I'_m(x) + \frac{m}{x} I_m(x) = I_{m-1}(x), \quad m \geq 1, \tag{2.3}$$

$$I'_m(x) - \frac{m}{x} I_m(x) = I_{m+1}(x), \tag{2.4}$$

$$I_m(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} e^x \Big[1 + O\Big(\frac{1}{x}\Big)\Big] \quad \text{if } x \to \infty, \tag{2.5}$$

$$I_m(x)I_n(x) = \sum_{k=0}^{\infty} \frac{\Gamma(m+n+2k+1)(x/2)^{m+n+2k}}{k!\Gamma(m+k+1)\Gamma(n+k+1)\Gamma(m+n+k+1)}. \tag{2.6}$$

Consider the function

$$f_m(x) = \frac{I_0(x)I_m(x)}{I_1(x)I_{m-1}(x)}, \quad m \geq 2. \tag{2.7}$$

Theorem 2.1. *If $m \geq 2$, then*

$$\frac{d}{dx} f_m(x) > 0 \quad \textit{for all} \quad x > 0. \tag{2.8}$$

To prove the theorem we first study the functions

$$S_m(x) = \log I_m(x). \tag{2.9}$$

Lemma 2.1. *For any $m \geq 1$,*

$$S'_m(x) - S'_{m-1}(x) > 0 \quad \textit{for all } x > 0, \tag{2.10}$$

and, consequently,

$$S'_m(x) - S'_0(x) > 0 \quad \textit{for all } x > 0. \tag{2.11}$$

Proof. From (2.1) we have

$$S''_m + (S'_m)^2 + \frac{1}{x} S'_m = 1 + \frac{m^2}{x^2}. \tag{2.12}$$

Similarly

$$S''_{m-1} + (S'_{m-1})^2 + \frac{1}{x} S'_{m-1} = 1 + \frac{(m-1)^2}{x^2},$$

so that

$$(S_m - S_{m-1})'' + (S'_m + S'_{m-1})(S'_m - S'_{m-1}) + \frac{1}{x}(S'_m - S'_{m-1}) = \frac{m^2 - (m-1)^2}{x^2} > 0. \tag{2.13}$$

As $x \to 0$

$$S_m = m \log x + O(1), \quad S_{m-1} = (m-1) \log x + O(1),$$

so that

$$S_m - S_{m-1} = \log x + O(1), \quad S'_m - S'_{m-1} = \frac{1}{x} + O(1).$$

Thus (2.10) holds if x is small.

If (2.10) does not hold for all $x > 0$, then there is a smallest value $x = x_0 > 0$ for which (2.10) is not satisfied. Clearly

$$\begin{aligned} S'_m(x_0) - S'_{m-1}(x_0) &= 0, \\ S''_m(x_0) - S''_{m-1}(x_0) &\le 0. \end{aligned} \tag{2.14}$$

On the other hand, from (2.13) and (2.14) we infer that

$$S''_m(x_0) - S''_{m-1}(x_0) = \frac{m^2 - (m-1)^2}{x_0^2} > 0,$$

which is a contradiction.

Lemma 2.2. *For any* $m \ge 1$,

$$S'_m(x) - S'_0(x) < \frac{m}{x} \quad \textit{for all } x > 0. \tag{2.15}$$

Proof. In view of (2.9), (2.15) is equivalent to $\frac{I'_m(x)}{I_m(x)} - \frac{I'_0(x)}{I_0(x)} < \frac{m}{x}$ or, by (2.4), to

$$\frac{m}{x} + \frac{I_{m+1}(x)}{I_m(x)} - \frac{I'_0(x)}{I_0(x)} < \frac{m}{x}$$

which is the same as

$$\frac{I_{m+1}}{I_m} < \frac{I_1}{I_0} \quad \text{for } x > 0. \tag{2.16}$$

Using the product formula (2.6) it was proved in [2] that each term in the power series of $I_{m+1}(x)I_{m-1}(x)$ is smaller than each term in the power series of $(I_m(x))^2$, so that $\frac{I_{m+1}}{I_m} < \frac{I_m}{I_{m-1}}$ and (2.16) then follows by iterating this inequality.

Proof of Theorem 2.1. Introduce the function

$$\phi(x) = (S_m(x) - S_{m-1}(x)) - (S_1(x) - S_0(x)). \tag{2.17}$$

Then the assertion (2.8) is equivalent to the inequality

$$\phi'(x) > 0 \quad \text{if } x > 0. \tag{2.18}$$

Writing (2.13) for $m = 1$ and subtracting from (2.13), we obtain

$$\phi'' + \{(S'_m + S'_{m-1})(S'_m - S'_{m-1}) - (S'_1 + S'_0)(S'_1 - S'_0)\} + \frac{1}{x}\phi' = \frac{2(m-1)}{x^2} > 0 \text{ if } m \ge 2. \tag{2.19}$$

For x small,

$$\begin{aligned} \frac{I_m}{I_{m-1}}\frac{I_0}{I_1} &= \frac{(\frac{x}{2})^m[\frac{1}{\Gamma(m+1)} + (\frac{x}{2})^2\frac{1}{\Gamma(m+2)}]}{(\frac{x}{2})^{m-1}[\frac{1}{\Gamma(m)} + (\frac{x}{2})^2\frac{1}{\Gamma(m+1)}]}\, \frac{1 + (\frac{x}{2})^2}{\frac{x}{2}[1 + \frac{1}{2}(\frac{x}{2})^2]}[1 + O(x^2)] \\ &= \frac{1}{m}\, \frac{1 + \frac{z}{m+1}}{1 + \frac{z}{m}}\, \frac{1 + z}{1 + \frac{z}{2}}(1 + O(z)) \quad (z = (x/2)^2) \\ &= \frac{1}{m}\Big[1 + \Big(\frac{1}{2} - \frac{1}{m(m+1)}\Big)z\Big] + O(z^2) \end{aligned}$$

and, since $m \ge 2$, (2.18) holds for x small.

If (2.18) does not hold for all $x > 0$, then there is a smallest $x = x_0$ such that

$$\varphi'(x_0) = 0 \tag{2.20}$$

and, clearly, $\varphi''(x_0) \le 0$, so that, by (2.19),

$$\frac{2(m-1)}{x^2} \le \{(S'_m + S'_{m-1})(S'_m - S'_{m-1}) - (S'_1 + S'_0)(S'_1 - S'_0)\} \text{ at } x = x_0. \tag{2.21}$$

Writing (2.20) in the form

$$S'_m - S'_{m-1} = S'_1 - S'_0 \quad \text{at} \quad x = x_0, \tag{2.22}$$

we see that the right-hand side of (2.21) is equal to

$$\begin{aligned}
&[(S'_m + S'_{m-1}) - (S'_1 + S'_0)](S'_1 - S'_0)\\
&= [(S'_{m-1} + (S'_1 - S'_0) + S'_{m-1}) - (S'_1 + S'_0)](S'_1 - S'_0)\\
&= (2S'_{m-1} - 2S'_0)(S'_1 - S'_0)
\end{aligned}$$

which, by Lemmas 2.2 and 2.3, is smaller than $2\frac{m-1}{x}\frac{1}{x}$; this is a contradiction to (2.21).

From (2.2) and (2.5) we see that

$$f_m(x) \to 1 \quad \text{if } x \to \infty,$$
$$f_m(x) \to \frac{1}{m} \quad \text{if } x \to 0,$$

so that, if $m \geq 2$, $f_m(x) - \frac{m+1}{2m}$ is negative for small x and positive for large x. Hence we have

Corollary 2.1. *For any $m \geq 2$ there exists a unique positive solution $x = x_m$ of the equation*

$$\frac{I_0(x_m)}{I_1(x_m)}\frac{I_m(x_m)}{I_{m-1}(x_m)} = \frac{m+1}{2m}. \tag{2.23}$$

This is precisely the assertion (i) of Theorem 1.1.

§3. Proof of Theorem 1.1 (ii)

By (2.4)

$$\frac{I'_{m-1}(x)}{I_{m-1}(x)} - \frac{m-1}{x} = \frac{I_m(x)}{I_{m-1}(x)}, \tag{3.1}$$

so that the equation

$$\frac{I_0(x)}{I_1(x)}\frac{I_m(x)}{I_{m-1}(x)} = \frac{m+1}{2m} \tag{3.2}$$

can be written also in the form

$$\frac{I'_{m-1}(x)}{I_{m-1}(x)} = \frac{m-1}{x} + \frac{m+1}{2m}\frac{I_1(x)}{I_0(x)}. \tag{3.3}$$

By Corollary 2.1 this equation has unique positive solution x_m and, setting

$$z_m(x) = \frac{I'_{m-1}(x)}{I_{m-1}(x)}, \tag{3.4}$$

we have

$$\begin{aligned}
z_m(x) &< \frac{m-1}{x} + \frac{m+1}{2m}G(x) \quad \text{if } x < x_m,\\
z_m(x) &> \frac{m-1}{x} + \frac{m+1}{2m}G(x) \quad \text{if } x > x_m
\end{aligned} \tag{3.5}$$

for any $m \geq 1$.

We shall prove

Theorem 3.1. *For any $m \geq 1$,*

$$z_{m+1}(x) < \frac{m}{x} + \frac{m+2}{2(m+1)}G(x) \quad \text{at } x = x_m, \tag{3.6}$$

so that $x_{m+1} > x_m$.

Combining this with (3.5) we conclude that $x_{m+1} > x_m$, and this completes the proof of Theorem 1.1 (ii).

In order to prove Theorem 3.1 we shall need several facts about the function G.

Lemma 3.1. *The function* $G(x)$ *satisfies:*

$$0 < G(x) < 1 \quad \text{if } x > 0. \tag{3.7}$$

Proof. By (2.10) (with $m = 1$),

$$G'(x) > 0 \text{ for all } x > 0. \tag{3.8}$$

It is also easily seen that

$$G(x) = \frac{x}{2} + O(x^2) \text{ as } x \to 0. \tag{3.9}$$

Since further, by (2.5),

$$G(x) \to 1 \quad \text{if } x \to \infty, \tag{3.10}$$

the lemma follows.

Using the relation $I_1' = -I_0/x + I_0$ we find that G satisfies the differential equation

$$G' + G^2 + \frac{1}{x}G = 1. \tag{3.11}$$

This equation will be needed later on.

We shall also need Theorem 1.2, i.e.,

$$G''(x) < 0 \text{ for all } x > 0. \tag{3.12}$$

The proof, which is somewhat lengthy, will be given in Section 5.

Proof of Theorem 3.1. From (3.1) and (3.4) we have

$$z_m(x) - \frac{m-1}{m} = \frac{I_m(x)}{I_{m-1}(x)}. \tag{3.13}$$

It will actually be more convenient to work with the function

$$V_m(x) = z_m(x) - \frac{m-1}{x} \quad \Big(= \frac{I_m(x)}{I_{m-1}(x)} \Big). \tag{3.14}$$

By (2.3), (2.4) and (2.2), $V_m(x)$ satisfies

$$V_m'(x) + V_m^2(x) + \frac{2m-1}{x}V_m(x) = 1, \tag{3.15}$$

$$V_m(x) = \frac{x}{2m} + O(x^2), \quad x \to 0. \tag{3.16}$$

Similarly

$$V_{m+1}'(x) + V_{m+1}^2(x) + \frac{2m+1}{x}V_{m+1}(x) = 1, \tag{3.17}$$

$$V_{m+1}(x) = \frac{x}{2(m+1)} + O(x^2), \quad x \to 0. \tag{3.18}$$

Theorem 3.1 can be restated as follows: If

$$V_m(x_m) = \frac{1}{2}\,\frac{m+1}{m}G(x_m), \tag{3.19}$$

then

$$V_{m+1}(x_m) < \frac{1}{2}\,\frac{m+2}{m+1}G(x_m), \tag{3.20}$$

or

$$W(x_m) < \frac{1}{2}\ \frac{m+1}{m} G(x_m), \tag{3.21}$$

where

$$W(x) = \frac{(m+1)^2}{m(m+2)} V_{m+1}(x). \tag{3.22}$$

The reason for introducing the function W is to make the right-hand side of (3.21) the same as for (3.19). We intend to compare $V_m(x)$ with $W(x)$ for $0 < x \le x_m$.

By (3.17), W satisfies the differential equation

$$W' + \frac{m(m+2)}{(m+1)^2} W^2 + \frac{2m+1}{x} W = \frac{(m+1)^2}{m(m+2)}, \quad x > 0, \tag{3.23}$$

and, by (3.18),

$$W(x) = \frac{m+1}{2m(m+2)} x + O(x^2), \quad x \to 0.$$

In view of (3.16),

$$W(x) < V_m(x) \text{ for } \ x \text{ near } 0. \tag{3.24}$$

If we can show that

$$\frac{(m+1)^2}{m(m+2)} - \frac{m(m+2)}{(m+1)^2} W^2 - \frac{2m+1}{x} W < 1 - W^2 - \frac{2m-1}{x} W \quad \text{for } 0 < x \le x_m, \tag{3.25}$$

then, by comparison, we deduce that $W(x) < V_m(x)$ for all $0 < x \le x_m$, and (3.21) follows. Indeed, otherwise there is a smallest $\bar{x}$ such that $0 < \bar{x} \le x_m$ and $(W - V_m) = 0$, $(W - V_m)' \ge 0$ at $\bar{x}$; however, in view of (3.25) at $x = \bar{x}$ and (3.15),(3.23), we also have $W'(\bar{x}) < V_m'(\bar{x})$, which is a contradiction.

We have thus reduced the proof of Theorem 3.1 to establishing the inequality (3.25), or

$$\frac{1}{m(m+2)} + \frac{1}{(m+1)^2} W^2 < \frac{2}{x} W \quad \text{for } 0 < x \le x_m. \tag{3.26}$$

The above analysis shows that as long as (3.26) holds for $0 < x < \tilde{x}$, $W(x) < V_m(x)$ for $0 < x < \tilde{x}$. Hence, in proving (3.26) it suffices to consider functions $W(x)$ satisfying

$$W(x) < V_m(x). \tag{3.27}$$

Since

$$V_m(x) < \frac{1}{2}\ \frac{m+1}{m} G(x) \quad \text{if } x < x_m,$$

we may replace (3.27) by the simpler inequality

$$W(x) < \frac{1}{2}\ \frac{m+1}{m} G(x), \quad 0 < x < x_m. \tag{3.28}$$

Lemma 3.2. *There holds*

$$\frac{m+1}{m(m+2)} G(x) \le W(x) \quad \textit{for all } x > 0. \tag{3.29}$$

Proof. We shall construct a subsolution $\widetilde{W}(x) = \lambda G(x),\ \lambda > 0$, to the function W. By (3.23), this means that λ has to be such that

$$\lambda G' + \lambda^2 \frac{m(m+2)}{(m+1)^2} G^2 + \frac{(2m+1)\lambda}{x} G < \frac{(m+1)^2}{m(m+2)},$$

or, in view of (3.11),

$$G^2\Big[\lambda^2\frac{m(m+2)}{(m+1)^2}-\lambda\Big]+\frac{2m\lambda}{x}G<\frac{(m+1)^2}{m(m+2)}-\lambda.$$

But for $\lambda\le(m+1)/[m(m+2)]$, this inequality is a consequence of

$$\frac{2m\lambda}{x}G<\frac{(m+1)^2}{m(m+2)}-\lambda,$$

or

$$\frac{2}{x}G<1,\ \text{i.e.,}\ I_1(x)<\frac{x}{2}I_0(x),$$

which is indeed true for all $x>0$ (by comparing the two series term-wise).

As $x\to 0$

$$\lambda G\sim\frac{\lambda x}{2},\quad W\sim\frac{m+1}{2m(m+2)}x,$$

so that the inequality $\lambda G(x)<W(x)$ holds for x near zero if $\lambda<\frac{m+1}{m(m+2)}$. But then, by comparison, $\lambda G(x)<W(x)$ for all $x>0$. This yields the assertion (3.29).

We summarize: In order to complete the proof of Theorem 3.1 or Theorem 1.1 (ii) it suffices to prove that (3.26) holds for $W(x)$ satisfying (3.28) and (3.29). We shall state this in a different way:

If we introduce the function

$$\Phi(z,x)=\frac{1}{m(m+2)}+\frac{z^2}{(m+1)^2}G^2(x)-\frac{2}{x}zG(x),\tag{3.30}$$

then, in view of (3.28) and (3.29), what we have to prove is the following

Lemma 3.3. *There holds*

$$\Phi(z,x)<0\ \ \textit{for}\ 0<x<x_m,\quad z\in\Big(\frac{m+1}{m(m+2)},\frac{m+1}{2m}\Big).\tag{3.31}$$

Proof. Set $z_1=\frac{m+1}{m(m+2)}$, $z_2=\frac{m+1}{2m}$. Since the function $z\to\Phi(z,x)$ is a parabola, it is sufficient to prove (3.31) just at the extreme points z_1 and z_2. But

$$\Phi(z_1,x)<0\ \text{ reduces to }\ 1+\frac{G^2}{m(m+2)}<\frac{2(m+1)}{x}G,$$

$$\Phi(z_2,x)<0\ \text{ reduces to }\ \frac{1}{m+2}+\frac{G^2}{4m}<\frac{m+1}{x}G,$$

and since (by Lemma 3.2) $0<G<1$, it suffices to prove that

$$1+\frac{1}{m(m+2)}<\frac{2(m+1)}{x}G,\quad \frac{1}{m+2}+\frac{1}{4m}<\frac{m+1}{x}G.$$

Noting that the second inequality is a consequence of the first one, it remains to prove that the function

$$F(x)=2(m+1)G(x)-\theta_m x\tag{3.32}$$

is positive for $0<x\le x_m$, where

$$\theta_m=\frac{(m+1)^2}{m(m+2)}.\tag{3.33}$$

Observe that for x near 0,

$$F(x)\sim(m+1)x-\frac{(m+1)^2}{m(m+2)}x=ax,\quad a>0,$$

so that $F(0) = 0$, $F'(0) > 0$. By (3.12), $F(x)$ is a concave function. Hence, in order to prove that $F(x) > 0$ for all $0 < x \le x_m$ it suffices to show that $F(x_m) > 0$, i.e., that

$$2(m+1)G(x_m) - \theta_m x_m > 0. \tag{3.34}$$

Since the proof of this inequality is quite lengthy, it is given in the next section.

§4. Proof of (3.34)

Introduce the positive solution $\bar{V}_m$ of

$$\bar{V}_m^2(\xi) + \frac{2m-1}{\xi}\bar{V}_m(\xi) = 1, \tag{4.1}$$

i.e.,

$$\bar{V}_m(\xi) = -\frac{2m-1}{2\xi} + \Big[\Big(\frac{2m-1}{2\xi}\Big)^2 + 1\Big]^{1/2}. \tag{4.2}$$

Note that, by (4.1),

$$\bar{V}_m(\xi) < \frac{\xi}{2m-1}. \tag{4.3}$$

Differentiating (4.1) we obtain

$$2\bar{V}_m\bar{V}_m' + \frac{2m-1}{\xi}\bar{V}_m' = \frac{2m-1}{\xi^2}\bar{V}_m \tag{4.4}$$

and hence, upon using (4.3),

$$\bar{V}_m' = \frac{\frac{2m-1}{\xi^2}\bar{V}_m}{2\bar{V}_m + \frac{2m-1}{\xi}} < \frac{\frac{1}{\xi}}{\frac{2m-1}{\xi}} = \frac{1}{2m-1}. \tag{4.5}$$

Lemma 4.1. *There holds*

$$V_m(x) \ge \Big(\frac{2m-1}{2m}\Big)^{1/2}\bar{V}_m\Big(\Big(\frac{2m-1}{2m}\Big)^{1/2}x\Big) \tag{4.6}$$

for all $x > 0$.

Proof. Consider the function

$$z(x) = \lambda\bar{V}_m(\lambda x), \quad \lambda > 0 \tag{4.7}$$

and set $\xi = \lambda x$. Then

$$\begin{aligned} z' + z^2 + \frac{2m-1}{x}z - 1 &= \lambda^2\Big[\bar{V}_m'(\xi) + \bar{V}_m^2(\xi) + \frac{2m-1}{\xi}\bar{V}_m(\xi) - 1\Big] + \lambda^2 - 1 \\ &= \lambda^2\bar{V}_m'(\xi) + \lambda^2 - 1 \quad \text{(by (4.1))} \\ &< \frac{\lambda^2}{2m-1} + \lambda^2 - 1 \quad \text{(by (4.5))} \\ &= \lambda^2\frac{2m}{2m-1} - 1 \ < \ 0 \end{aligned}$$

if $\lambda^2 < \frac{2m-1}{2m}$, so that z is a subsolution of (3.15). Since also, for x near 0,

$$\begin{aligned} z(x) &\sim \lambda\frac{2m-1}{2\xi}\frac{1}{2}\Big(\frac{2\xi}{2m-1}\Big)^2 \quad \text{(by (4.2))} \\ &= \lambda^2\frac{x}{2m-1} \ < \ \frac{x}{2m} \ \sim \ V_m(x), \end{aligned}$$

we conclude, by comparison, that $z(x) < V_m(x)$ for all $x > 0$, and (4.6) follows.

Lemma 4.2. *The function $\bar V_m(\xi)$ satisfies $\bar V_m'(\xi) > 0$, $\bar V_m''(\xi) < 0$.*

Proof. The first inequality follows from (4.4). To prove the second inequality we set $V = \bar V_m$ and differentiate (4.4). We get

$$2VV'' + 2(V')^2 + \frac{2m-1}{\xi}V'' - \frac{2(2m-1)}{\xi^2}V' + \frac{2(m-1)}{\xi^3}V = 0.$$

Since $V > 0$, we see that

$$\begin{aligned}\mathrm{sgn}V'' &= \mathrm{sgn}\Big\{-2(V')^2 + \frac{2(2m-1)}{\xi^2}V' - \frac{2(2m-1)}{\xi^3}V\Big\}\\ &\le \mathrm{sgn}\Big\{\frac{2(2m-1)}{\xi^2}\Big(V' - \frac{V}{\xi}\Big)\Big\} = \mathrm{sgn}\Big\{\frac{2}{\xi}(-2VV')\Big\} \text{ by (4.4)},\end{aligned}$$

which is negative since $V' > 0$.

The lower bound for V_m (derived in Lemma 4.1) will henceforth be used to deduce that x_m in (3.5) is sufficiently large, which is an important step in the proof of (3.34).

Set

$$\begin{aligned}Q_m(x) &= \frac{2m}{m+1}\Big(\frac{2m-1}{2m}\Big)^{1/2}\bar V_m\Big(\Big(\frac{2m-1}{2m}\Big)^{1/2}x\Big),\\ P_m(x) &= \frac{2m}{m+1}V_m(x) \quad (V_m \text{ was defined in (3.14)}),\\ L_m(x) &= \frac{1}{2(m+1)}\theta_m x = \frac{m+1}{2m(m+2)}x.\end{aligned}$$

By (3.19)

$$P_m(x_m) = G(x_m), \tag{4.8}$$

and (3.34) is equivalent to

$$L_m(x_m) < G(x_m). \tag{4.9}$$

For x near 0, $G(x) \sim \frac{x}{2}$, so that

$$G(0) = L_m(0) = 0, \quad G'(0) > L_m'(0),$$

whereas, for x large, $G(x) < 1 < L_m(x)$. Since G is concave, it follows that there exists a unique point $\tilde x_m$ such that

$$\begin{aligned}L_m(x) < G(x) &\quad \text{if } x < \tilde x_m,\\ L_m(x) > G(x) &\quad \text{if } x > \tilde x_m.\end{aligned} \tag{4.10}$$

To prove (4.9), let $\tilde x$ be the point where

$$L_m(\tilde x) = 1, \quad \text{i.e.,} \quad \tilde x = \frac{2m(m+2)}{m+1}. \tag{4.11}$$

Since $G(x) < 1$, (4.10) implies that

$$\tilde x_m < \tilde x. \tag{4.12}$$

Suppose

$$Q_m(\tilde x) > 1 = L_m(\tilde x). \tag{4.13}$$

Then from the concavity of Q_m (Lemma 4.2) and the fact that $Q_m(0) = L_m(0) = 0$, it follows that

$$Q_m(x) > L_m(x) \quad \text{if } x < \tilde x. \tag{4.14}$$

By monotonicity of Q_m (which follows from Lemma 4.1)

$$Q_m(x) > Q_m(\tilde{x}) > 1 \quad \text{if } x > \tilde{x},$$

whereas by (4.14), (4.10),

$$Q_m(x) > L_m(x) > G(x) \quad \text{if } \tilde{x}_m < x < \tilde{x}.$$

Thus, altogether, $Q_m(x) > G(x)$ if $\tilde{x}_m < x < \infty$ and, since $P_m(x) > Q_m(x)$ for all $x > 0$ (Lemma 4.1), we conclude that

$$P_m(x) > G(x) \quad \text{if } \tilde{x}_m < x < \infty.$$

Hence, by (4.8), $x_m < \tilde{x}_m$ and, recalling (4.10), the assertion (4.9) follows, and this completes the proof of (3.34).

It remains to prove that (4.13) holds.

Set

$$A_m = \frac{m+1}{2m}\Big(\frac{2m}{2m-1}\Big)^{1/2}, \quad \sigma_m = \Big(\frac{2m-1}{2m}\Big)^{1/2}\tilde{x}.$$

Then (4.13) reduces to

$$\bar{V}_m(\sigma_m) > A_m. \tag{4.15}$$

Since

$$\bar{V}_m^2(\sigma_m) + \frac{2m-1}{\sigma_m}\bar{V}_m(\sigma_m) = 1,$$

if

$$A_m^2 + \frac{2m-1}{\sigma_m}A_m < 1, \tag{4.16}$$

then (4.15) follows by monotonicity. Substituting $\tilde{x}$ from (4.11) into σ_m, the inequality (4.16) reduces to

$$\frac{(m+1)^2(3m+1)}{2m(2m-1)(m+2)} < 1,$$

which is valid if $m \geq 4$.

We have thus completed the proof of (3.34) (and therefore also of Theorem 1.1 (ii)) for $m \geq 4$. The proof for $m \leq 3$ can be obtained by explicit calculations. Indeed, the solution of

$$\frac{I_2(x)}{I_1(x)} = \frac{3}{4}\frac{I_1(x)}{I_0(x)}$$

is $x = a = 3.773474$ and

$$\frac{I_3(a)}{I_2(a)} - \frac{2}{3}\frac{I_1(a)}{I_0(a)} = -0.0686071 < 0,$$

whereas the solution of

$$\frac{I_3(x)}{I_2(x)} = \frac{2}{3}\frac{I_1(x)}{I_0(x)}$$

is $x = b = 5.119174$ and

$$\frac{I_4(b)}{I_3(b)} - \frac{5}{8}\frac{I_1(b)}{I_0(b)} = -0.058144 < 0.$$

§5. Proof of Theorem 1.2

The function $\bar{G}(x) = \frac{1+\varepsilon}{2}x$ ($\varepsilon > 0$) is a supersolution of G, i.e., $\bar{G}' > 1 - \bar{G}^2 - \frac{\bar{G}}{x}$. Since also $G(x) \sim \frac{x}{2} < \bar{G}(x)$ for x near 0, it follows that

$$G(x) \le \frac{x}{2} \quad \text{for all } x > 0. \tag{5.1}$$

Differentiating (3.11) we get

$$\begin{aligned} G'' &= -2GG' - \frac{G'}{x} + \frac{1}{x^2}G \\ &= -2G\Big(1 - G^2 - \frac{G}{x}\Big) - \frac{1}{x}\Big(1 - G^2 - \frac{G}{x}\Big) + \frac{G}{x^2} \\ &= -2G + 2G^3 + \frac{3G^2}{x} + \frac{2G}{x^2} - \frac{1}{x} \;\equiv\; K(G, x). \end{aligned} \tag{5.2}$$

If $x \le 1$, then by (5.1)

$$G'' < -2G + xG^2 + \frac{3}{2}G + \frac{1}{x} - \frac{1}{x} = G\Big(-\frac{1}{2} + xG\Big) < 0$$

and thus G is concave.

We next want to show that

$$K(G(x), x) < 0 \quad \text{if } 1 < x < 2. \tag{5.3}$$

To do that note that since $G(x) > G(1) = 0.4464$,

$$\frac{3G^2(x)}{\xi} + \frac{2G(x)}{\xi^2} - \frac{1}{\xi}$$

is monotone decreasing in ξ, $0 < \xi < 1$, and

$$\frac{\partial K}{\partial G} = -2 + 6G^2 + \frac{6G}{x} + \frac{2}{x^2} > 0 \quad \text{if } G = G(\xi),\ \xi > 1,\ 1 < x < 2.$$

Hence, if

$$H(x, y) \equiv -2G(x) + 2G^3(x) + \frac{3G^2(x)}{y} + \frac{2G(x)}{y^2} - \frac{1}{y} < 0 \tag{5.4}$$

for a pair (x, y) with $1 \le y < x \le 2$, then $K(G(\xi), \xi) < 0$ for $y \le \xi \le x$. We shall use this remark with points

$$a_1 = 1.0,\ a_2 = 1.1,\ a_3 = 1.2,\ a_4 = 1.3,\ a_5 = 1.5,\ a_6 = 1.7,\ a_7 = 1.9,\ a_8 = 2.0.$$

By direct computation we find that $H(a_{j+1}, a_j) < 0$ for all j. Hence (5.3) holds and, consequently, $G(x)$ is concave for $1 \le x \le 2$.

It remains to prove the concavity of $G(x)$ for $x > 2$. To do that we shall first derive rather sharp upper and lower bounds on G:

Lemma 5.1. *The function G satisfies*

$$G(x) < 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3} \quad \textit{if } x \ge 2. \tag{5.5}$$

Proof. Writing

$$G = 1 - \frac{1}{2x} + \varphi,$$

(3.11) becomes

$$\varphi' = -2\varphi - \varphi^2 - \frac{1}{4x^2}.$$

By direct calculation one shows that the function

$$\psi(x) = -\frac{1}{8x^2} - \frac{1}{8x^3}$$

satisfies

$$\psi' > -2\psi - \psi^2 - \frac{1}{4x^2},$$

and thus

$$\bar{G} \equiv 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3}$$

is a supersolution to (3.11). Since

$$\bar{G}(2) = 1 - \frac{1}{4} - \frac{1}{32} - \frac{1}{64} \approx 0.703 > 0.697 \approx G(2),$$

the assertion (5.5) follows.

We have

$$6G^2(x) \geq 6G^2(2) \approx 2.91 > 2 \quad \text{if } x \geq 2$$

and, therefore,

$$\frac{\partial K}{\partial G} = -2 + 6G^2 + \frac{6G}{x} + \frac{2}{x^2} > 0 \quad \text{if } x \geq 2.$$

Consequently (recall (5.2)) in order to prove that $G''(x) < 0$, or that $K(G, x) < 0$, it suffices to prove that $K(\bar{G}, x) < 0$ where $\bar{G}(x)$ is the supersolution given by the right-hand side of (5.5). By direct computation,

$$K(\bar{G}, x) = \frac{\bar{G}}{x}\Big\{1 + \frac{1}{2x} - \frac{5}{8x^2} - \frac{3}{32x^3} + \frac{1}{16x^4} + \frac{1}{32x^5}\Big\} - \frac{1}{x},$$

which can be expanded to

$$K(\bar{G}, x) = \frac{1}{x^3}\Big(-1 + \frac{1}{32}\ \frac{1}{x} + \frac{1}{8}\ \frac{1}{x^2} + \frac{23}{256}\ \frac{1}{x^3} - \frac{3}{256}\ \frac{1}{x^4} - \frac{3}{256}\ \frac{1}{x^5} - \frac{1}{256}\ \frac{1}{x^6}\Big).$$

For $x \geq 1$, the sum of the terms with positive sign is less than $1/32 + 1/8 + 23/256$, which is less than 1. Therefore, $K(\bar{G}, x) < 0$ if $x \geq 1$. This completes the proof of the concavity of G.

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