Chin. Ann. of Math.
21B: 3(2000),297-322.

DIRECT CONSTRUCT OF PERIODIC SOLUTIONS TO PERTURBED KLEIN-GORDON EQUATIONS BY NEWTON ITERATION**

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Abstract

By introducing the block estimate technique and directly using the Newton iteration method, the author constructs Cantor families of time periodic solutions to a class of nonlinear wave equations with periodic boundary conditions. The Lyapunov-Schmidt decomposition used by J. Bourgain, W. Craig and C. E. Wayne is avoided. Thus this work simplifies their framework for KAM theory for PDEs.

Keywords Klein-Gordon equation, Periodic solution, Newton iteration
1991 MR Subject Classification 35B10, 35L70
Chinese Library Classification O175.27, O175.29 **Document Code** A
Article ID 0252-9599(2000)03-0297-26

§1. Introduction

Near an equilibrium state to a system of nonlinear ODEs, one can regard the nonlinear system as a perturbation of its linearization. It is well known that if none of the characteristic exponents of the linearized system has zero real part, then the phase portrait near the equilibrium of the nonlinear system is topologically equivalent to that of its linearization (cf. [18]). However, if characteristic exponents with zero real part appear, then the linearized system alone does not contain complete topological information of the phase portrait of the nonlinear system: in general, one cannot deduce the stability or the existence of periodic or quasiperiodic solutions to the nonlinear system through linearization. Nevertheless, if the nonlinear system has an energy conservation background, then it may still preserve classes of periodic or quasiperiodic solutions to its linearization (cf. e.g., [22,1,23,24,34]). A typical example is that the system is Hamiltonian and the equilibrium is elliptic. In this case, all the characteristic exponents of the linearized system have zero real part, thus all the solutions to the linearized system are periodic or quasiperiodic. If suitable nonresonant conditions are satisfied, then the Lyapunov center theorem asserts that the nonlinear system can preserve

Manuscript received January 30, 2000.
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**Project supported by the Special Funds for Major State Basic Research Projects of China, the Laboratory of Mathematics for Nonlinear Sciences, Fudan university, and the Institut Sino-Français de Mathématiques Appliquées (ISFMA).

smooth families of periodic solutions to its linearization (cf. [22]), and the KAM theory asserts that the nonlinear system can preserve Cantor families of quasiperiodic solutions to its linearization (cf. [1,14,25]). The reason for Cantor set appearing in the latter result is the small divisor phenomenon.

In recent years, the Lyapunov center theorem and the KAM theory for equilibriums have been extended to some infinitely dimensional Hamiltonian systems described by nonlinear PDEs such as periodic boundary value problems or homogeneous Dirichlet boundary value problems for wave equations of the form

$$u_{tt} - \triangle u + f(x,u) = 0$$

or Schrödinger equations of the form

$$iu_t + \triangle u = h(u,\bar{u})$$

and the periodic boundary value problem for the KdV equation

$$u_t + u_{xxx} + uu_x = 0.$$

The main conclusion is that typical nonlinear Hamiltonian perturbations of a linear Hamiltonian PDE can preserve Cantor families of periodic or quasiperiodic solutions. The results are now concentrated in the case where the space dimension is one.

In general, there are two approaches to investigate this problem. One approach is a direct extention of the classical KAM method proposed independently by S. B. Kuksin[18] and C. E. Wayne[32], namely, the coordinates transformation (in phase space) method. Through this approach, extensive studies have been carried out for quasiperiodic solutions to the KdV equation, the nonlinear Schördinger equations or the nonlinear wave equations (for KdV equation, cf. [17,18,19]; for Schördinger equation, cf. [20]; for wave equation, cf. [2,3,27, 32]). Since a Melnikov nonresonant condition is required in this approach, it is difficult to treat problems with multiple (or approximately multiple) eigenvalues such as problems in multi-dimensional spaces for various equations with various boundary conditions and the periodic boundary value problems for the wave equations or the Schrödinger equations in one space dimension. Another approach is the Fourier transformation method proposed by W. Craig and C. E. Wayne in [11] and extensively developped by J. Bourgain to treat problems in multi-dimensional spaces. By Fourier transformation of periodic or quasiperiodic solutions to be constructed, the problem is translated into a lattice problem whose task is essentially to solve a nonlinear functional equation. This appoach does not require the Melnikov nonresonant condition as in the previous approach, thus it is more suitable for dealing with problems with multiple (or approximately multiple) eigenvalues. Via this approach, W. Craig and C. E. Wayne constructed Cantor families of time periodic solutions to the wave equations[10,11,12] or the Schördinger equations[13] with various boundary conditions in one space dimension, and J. Bourgain constructed Cantor families of time periodic solutions to the wave equations in arbitrarily multi-dimensional spaces with periodic boundary conditions[4] and quasiperiodic solutions to the Schördinger equations in two space dimensions with the same boundary conditions[6].

In [11], W. Craig and C. E. Wayne investigated small amplitude time periodic solutions to the nonlinear wave equations of the form

$$u_{tt} - u_{xx} + g(x,u) = 0 \tag{1.1}$$

with periodic boundary condition

$$u(t, x + 2\pi) = u(t, x) \tag{1.2}$$

or homogeneous Dirichlet boundary condition, where the nonlinear term $g(x, u) = g_1(x)u + g_2(x)u^2 + \cdots$ vanishes at $u = 0$. To describe their result, we consider the periodic boundary condition (the result and the method for homogeneous Dirichlet boundary condition are similar). Suppose that $g(x, u)$ is periodic in the variable x with period 2π. Let $\{\psi_j(x)\}_{j=0}^{\infty}$ be the complete set of eigenfunctions of the Sturm-Liouville operator $-\partial_{xx} + g_1(x)$ with the periodic boundary condition (1.2), and $\omega_0^2 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots$ be the corresponding eigenvalues. For any given $j_0 \in Z^+$, if the eigenvalue $\omega_{j_0}^2 > 0$, the linearized equation

$$u_{tt} - u_{xx} + g_1(x)u = 0 \tag{1.3}$$

of (1.1) has a family of time periodic solutions $r\psi_{j_0}(x)\cos\omega_{j_0}t$ $(r \in R)$ satisfying the periodic boundary condition (1.2). Craig and Wayne proved the following theorem.

Theorem 1.1. *For any given integer $j_0 \geq 0$, there is a generic class of analytic nonlinear terms $g(x, u)$, and for each term in this class there exists a constant $r_* > 0$ sufficiently small and a Cantor set $\mathcal{C} \subseteq [0, r_*)$, such that for any given $\epsilon \in \mathcal{C}$, there exists $\Omega(\epsilon) = \omega_{j_0} + O(\epsilon^2)$ $(\epsilon \to 0)$ and a time periodic solution $u(\Omega(\epsilon)t, x; \epsilon)$ with (angular) frequency $\Omega(\epsilon)$ to the nonlinear wave equation* (1.1) *satisfying the periodic boundary condition* (1.2)*, and this solution is close to a solution $\epsilon\psi_{j_0}(x)\cos\omega_{j_0}t$ to the linearized equation* (1.3) *satisfying the same boundary condition*

$$|u(t, x; \epsilon) - \epsilon\psi_{j_0}(x)\cos t| \leq c\epsilon^2, \quad \forall (t, x) \in \mathbf{R}^2,$$

where $c > 0$ is a constant independent of ϵ. Furthermore, if a fully nonresonant condition is satisfied, then the Cantor set $\mathcal{C}$ has full density at zero.

This result has been extended by Bourgain to the arbitrarily finitely dimensional space case. Consider the nonlinear wave equation in d $(d \geq 1)$ space dimensions

$$u_{tt} - \triangle u + \rho u + u^3 = 0, \tag{1.4}$$

where ρ is a real number. Bourgain[4] proved the following theorem.

Theorem 1.2. *If ρ satisfies a Diophantine condition, then for any given $m \in Z^d \setminus \{0\}$ with $|m|^2 + \rho > 0$, there exists $r_* > 0$ and a Cantor subset $\mathcal{C}$ of $[0, r_*)$, such that for any given $\epsilon \in C$, there is a time periodic solution $u(t, x; \delta)$ with (angular) frequency $\lambda(\epsilon)$ to Equation* (1.4) *with periodic boundary conditions*

$$u(t, x_1, \cdots, x_i + 2\pi, \cdots, x_d) = u(t, x_1, \cdots, x_i, \cdots, x_d), \quad i = 1, \cdots, d, \tag{1.5}$$

this solution is close to the time periodic solution $\epsilon\cos(\langle m, x\rangle + \lambda_m t)$ to the linearized equation $u_{tt} - \triangle u + \rho u = 0$ with the same boundary condition, where $\lambda_m = \sqrt{|m|^2 + \rho}$:

$$u(t, x; \epsilon) = \epsilon\cos(\langle m, x\rangle + \lambda(\epsilon)t) + O(\epsilon^3),$$
$$\lambda(\epsilon) = \lambda_m + O(\epsilon^2) \quad \text{as} \quad \epsilon \longrightarrow 0.$$

Concerning time periodic solutions, there is a difference between the results on ODEs and those on PDEs. As mensioned above, in the case of ODEs, the Lyapunov center theorem asserts that a system of nonlinear equations may preserve smooth families of periodic solutions to the linearized system. However, for a nonlinear PDE, which is an infinitly dimensional dynamical system and has infinitely many degrees of freedom, the small divisor phenomenon which is a character of the KAM theory appears in the study of periodic solutions. As a result, in general only Cantor families of periodic solutions can be preserved. However, Li Tatsien, Qin Tiehu and the author of the present paper found that in some special cases, for example, for the problem above studied by Bourgain, nonlinear perturbations to a linear equation may preserve smooth families of periodic solutions as well. Through constructing travelling wave solutions, we proved the following theorem[30].

Theorem 1.3. *Suppose that* $g(u) \in C^2$, $g(0) = 0$, $g'(0) \stackrel{\triangle}{=} \rho \neq 0$. *Then for any given* $m \in Z^d \setminus \{0\}$ *satisfying* $|m|^2 + \rho > 0$, *there exists* $r_* > 0$, *such that for any given* $\epsilon \in [0, r_*)$, *there exists a periodic travelling wave solution*

$$u(t, x; \epsilon) = p_\epsilon(\langle m, x\rangle + \lambda(\epsilon)t)$$

with time (*angular*) *frequency* $\lambda(\epsilon)$ *to the nonlinear wave equation*

$$u_{tt} - \triangle u + g(u) = 0$$

with the periodic boundary conditions (1.5), *this solution is close to a time periodic solution* $\epsilon\cos(\langle m, x\rangle + \lambda_m t)$ *to the linearized equation* $u_{tt} - \triangle u + \rho u = 0$ *with the same boundary conditions, where* $\lambda_m = \sqrt{|m|^2 + \rho}$:

$$\|u(t, x; \epsilon) - \epsilon\cos(\langle m, x\rangle + \lambda(\epsilon)t)\|_{C^1} \leq c\epsilon^2, \quad |\lambda(\epsilon) - \lambda_m| \leq c\epsilon,$$

where $c > 0$ *is a constant independent of* ϵ. *In particular, if* $g(u) = \rho u + \kappa u^3$ ($\kappa = \pm 1$), *then the above two estimates can be improved as*

$$\|u(t, x; \epsilon) - \epsilon\cos(\langle m, x\rangle + \lambda(\epsilon)t)\|_{C^1} \leq c\epsilon^3, \quad \Big|\lambda^2(\epsilon) - \Big(\lambda_m^2 + \frac{3}{4}\kappa\epsilon^2\Big)\Big| \leq c\epsilon^4.$$

This theorem improved the above result of Bourgain. As well, the travelling wave solution method is much simpler than the method of Newton iteration plus Lyapunov-Schmidt decomposition used in [4].

As mentioned above, the Fourier transformation approach translates problems of constructing periodic (or quasiperiodic) solutions of nonlinear PDEs to a corresponding lattice problem which requires essentially to solve a nonlinear functional equation. To solve this functional equation, one usually applies the Lyapunov-Schmidt decomposition to split it into two equations. One equation, called Q equation, is strongly resonant and finitely dimensional; while the other one, called P equation, has resonance weaker than the Q equation, but meets a small divisor difficulty (even in the case of periodic solutions) since it is infinitely dimensional. First, fixing the frequency Ω and the amplitude ϵ of the solution to be constructed as parameters, applying the Newton iteration method whose fast convergent property allows one to overcome the small divisor difficulty, one solves the P equation which has solution $u(\epsilon, \Omega)$ for a Cantor set of parameters (ϵ, Ω). Then, substituting this families of solutions in the Q equation, it turns out to be an equation of the frequency Ω and the amplitude ϵ. Applying the implicit function theorem, one solves the functional relation between the frequency and the amplitude, and obtains a family of solutions to the original problem parameterized by the frequency (or the amplitude).

Applying the Lyapunov-Schmidt decomposition to solve the strong resonance of the Q equation causes some new difficulties. In the course of solving the P equation by Newton iteration, in order to control the smallness of the small divisors, some values of the frequency and the amplitude parameters must be elliminated, so that the P equation is only solved for the frequency and amplitude parameters (Ω, ϵ) in a Cantor subset of a neighbourhood of $(\omega_{j_0}, 0)$. To solve the Q equation by the implicit function theorem so as to determine the functional relation between the frequency and the amplitude, one has to extend the Cantor family of solutions $u(\epsilon, \Omega)$ to the P equation as a smooth function of (Ω, ϵ) in a neighbourhood of $(\omega_{j_0}, 0)$, and estimate its derivatives with respect to Ω and ϵ. This procedure is technically complicated; in the meantime it increases difficulty in the estimate of the measure of the Cantor parameter set corresponding to the family of obtained solutions.

In this paper, instead of the Lyapunov-Schmidt decomposition, we introduce the block estimate technique, and directly using the Newton iteration method, we construct Cantor families of periodic solutions to a class of nonlinear wave equations. Thus we simplify the framework in [11] (see Section 3 for details). Our method is generally applicable for similar problems; in this paper we only consider the following case: discuss time periodic solutions to the perturbed Klein-Gordon equations of the form

$$u_{tt} - u_{xx} + au - u^3 + f(x, u) = 0 \tag{1.6}$$

with the periodic boundary condition (1.2), where $f(x, u) = \sum\limits_{m=4}^{\infty} f_m(x) u^m$ is an analytical function of (x, u), and is periodic in x with period 2π. For each $j = 0, 1, \cdots$, let $\omega_j = \sqrt{j^2 + a}$, which is a squareroot of an eigenvalues of the Sturm-Liouville operator $-u_{xx} + au$ with the periodic boundary condition (1.2). We will prove the following theorem.

Theorem 1.4. *Suppose that $a > 0$ satisfies the Diophantine condition $|ka - j| > dk^{-\gamma}$, $\forall j, k \in N$ for some constants $d > 0$ and $\gamma > 0$. Then for any given positive integer j_0, there exists $r_* > 0$ and a Cantor subset $\mathcal{C}$ of the interval $(\omega_{j_0} - r_*, \omega_{j_0}]$ which has full density at ω_{j_0}, such that for any given $\Omega \in \mathcal{C}$, the perturbed Klein-Gordon equation* (1.6) *admits a time periodic solution $u(\Omega t, x; \Omega)$ with* (*angular*) *frequency Ω satisfying the periodic boundary condition* (1.2)*, and this solution is close to a periodic solution $\epsilon(\Omega) \cos \Omega t \cos j_0 x$ to the linearized equation $u_{tt} - u_{xx} + au = 0$ with the same boundary condition, where $\epsilon(\Omega) = \frac{4}{3}\sqrt{2\omega_{j_0}(\omega_{j_0} - \Omega)}$,*

$$|u(t, x; \Omega) - \epsilon(\Omega) \cos \Omega t \cos j_0 x| \le c\epsilon(\Omega)^2, \quad \forall (t, x) \in \mathbf{R}^2.$$

As mensioned above, it is difficult to study problems with multiple or approximately multiple eigenvalues via the classical KAM approach, because a Melnikov nonresonant condition is required. After the completion of this work, the author learned that the Melnikov nonresonant condition has been weakened quite recently by L. Chierchia and J. You, this allows them to construct Cantor families of quasiperiodic solutions via the classical KAM approach to periodic boundary problems for a class of one dimensional nonlinear wave equations of the form[8] $u_{tt} - u_{xx} + V(x)u = f(u)$, $\quad f(u) = O(u^2)$, which correspond to approximately double eigenvalues. The problems with infinitely many multiple eigenvalues, as the equation (1.6) discussed in this paper, is still excluded by their method.

§2. Preliminaries

In this section, we introduce a family of Banach spaces of periodic functions $\{\mathcal{H}_\sigma\}_{\sigma>0}$, and discuss linear operators on this family of Banach spaces. Some function norms and operator norms similar to those introduced in this section have appeared in other articles[13,26].

2.1. Periodic Function Space $\mathcal{H}$

Suppose that $u(s, x)$ is a complex valued function defined on R^2, which is periodic in both variables s and x with the common period 2π. If u is suitably smooth, then we can expand it in Fourier series

$$u(s,x)=\sum_{j,k\in Z}u_{jk}e^{i(jx+ks)}. \tag{2.1}$$

Denote $e_z=e^{i(jx+ks)}$ and $|z|=|j|+|k|$ for all double indices $z=(j,k)\in Z^2$. Then (2.1) can be rewritten as $u=\sum\limits_{z\in Z^2}u(z)e_z$.

We now define a family of function spaces $\{H_\sigma\}_{\sigma>0}$ of such periodic functions. For any given $\sigma>0$, the norm $\|\cdot\|_\sigma$ of u is defined by $\|u\|_\sigma=\sum\limits_{z\in Z^2}|u(z)|e^{\sigma|z|}$. The function space H_σ is definded as the set of such periodic functions u with $\|u\|_\sigma<\infty$. It is easy to check that H_σ is a Banach space associated to the norm $\|\cdot\|_\sigma$. Moreover, equipped with the ordinary product of functions as the multiplying operation, H_σ becomes a Banach algebra. In fact,

$$\begin{aligned}\|uv\|_\sigma&=\sum_{z\in Z^2}|(uv)(z)|e^{\sigma|z|}=\sum_{z\in Z^2}\Big|\sum_{p\in Z^2}u(p)v(z-p)\Big|e^{\sigma|z|}\\&\le\sum_{p\in Z^2}|u(p)|e^{\sigma|p|}\sum_{z\in Z^2}|v(z-p)|e^{\sigma|z-p|}\le\|u\|_\sigma\|v\|_\sigma.\end{aligned}$$

Define the function space $\mathcal{H}=\bigcup\limits_{\sigma>0}H_\sigma$.

For any given subset X of Z^2, we can define a projection operator on $\mathcal{H}$, still denoted by X, by

$$u=\sum_{z\in Z^2}u(z)e_z\mapsto Xu=\sum_{z\in X}u(z)e_z.$$

The function spaces $H_{\sigma,X}$ and $\mathcal{H}_X$ are defined as the images of H_σ and $\mathcal{H}$ under the projection operator X, respectively.

Suppose that $f(x,u)=\sum\limits_{m=4}^{\infty}f_m(x)u^m$ is a periodic function of x with period 2π, which is analytic on the domain $\{(x,u)\in C^2\,|\,|\Re x|\le\bar\sigma,|u|\le\delta\}$, where $\bar\sigma>0$ and $\delta>0$ are constants. Thanks to the Cauchy integral formula, we have $\|f_m(x)\|_{\bar\sigma}\le c\delta^{-m}$. Therefore, for any given $0<\sigma<\bar\sigma$ and $0<\delta'<\delta$, for any given $u\in H_\sigma$ with $\|u\|_\sigma\le\delta'$, we have

$$\|f(x,u)\|_\sigma\le c_1\|u\|_\sigma^4,\quad\|f'(x,u)\|_\sigma\le c_2\|u\|_\sigma^3,\quad\|f''(x,u)\|_\sigma\le c_3\|u\|_\sigma^2,$$

where c_1, c_2 and c_3 are positive constants depending on δ', and

$$\begin{aligned}f'(x,u)&=\frac{\partial f}{\partial u}(x,u)=\sum_{m=3}^{\infty}(m+1)f_{m+1}(x)u^m,\\f''(x,u)&=\frac{\partial^2 f}{\partial u^2}(x,u)=\sum_{m=2}^{\infty}(m+1)(m+2)f_{m+2}(x)u^m.\end{aligned}$$

2.2. Linear Operators on $\mathcal{H}$

Suppose that $T:\mathcal{H}\longrightarrow\mathcal{H}$ is a linear operator. If for any given $\sigma>0$, there exists $\tau\in(0,\sigma]$ such that $\|Tu\|_\tau\le c_{\sigma,\tau}\|u\|_\sigma$, $\forall u\in H_\sigma$, then we call T continuous.

It is clear that the composition of two linear continuous operators is also a linear continuous operator. Therefore, since

$$\|\partial_t u\|_{\sigma-\tau}\le c\tau^{-1}\|u\|_\sigma,\quad\|\partial_x u\|_{\sigma-\tau}\le c\tau^{-1}\|u\|_\sigma,\qquad\forall\sigma>\tau>0,\ \ \forall u\in\mathcal{H}_\sigma,$$

the differential operators $\partial_t^\alpha\partial_x^\beta$ $((\alpha,\beta)\in Z^2,\alpha,\beta\ge0)$ are continuous.

Denote

$$T(z_1, z_2) = (2\pi)^{-2} \int_{[0,2\pi]^2} (Te_{z_2}) \cdot \bar{e}_{z_1}, \quad \forall z_1, z_2 \in Z^2.$$

Then

$$Te_z = \sum_{p \in Z^2} T(p, z) e_p, \quad \forall z \in Z^2.$$

One can check that if $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a linear continuous operator, then

$$(Tu)(z) = \sum_{p \in Z^2} T(z, p) u(p), \quad \forall z \in Z^2.$$

If $T_1, T_2 : \mathcal{H} \longrightarrow \mathcal{H}$ are linear continuous operators, then

$$T_2 T_1(z_1, z_2) = \sum_{p \in Z^2} T_2(z_1, p) T_1(p, z_2), \quad \forall z_1, z_2 \in Z^2.$$

For a linear continuous operator $T : \mathcal{H} \longrightarrow \mathcal{H}$, we define its σ-norm by

$$\|T\|_\sigma = \sup_{z_2 \in Z^2} \sum_{z_1 \in Z^2} |T(z_1, z_2)| e^{\sigma |z_1 - z_2|}.$$

Define $\mathcal{L}_\sigma$ as the set of such linear continuous operators T with $\|T\|_\sigma < \infty$. If $T \in \mathcal{L}_\sigma$, then

$$\|Tu\|_\tau \le \|T\|_\sigma \|u\|_\tau, \quad \forall \tau \in (0, \sigma], \quad \forall u \in H_\tau.$$

In fact,

$$\begin{aligned}
\|Tu\|_\tau &= \sum_{z \in Z^2} |(Tu)(z)| e^{\tau |z|} = \sum_{z \in Z^2} \Big| \sum_{p \in Z^2} T(z, p) u(p) \Big| e^{\tau |z|} \\
&\le \sum_{p \in Z^2} |u(p)| e^{\tau |p|} \sum_{z \in Z^2} |T(z, p)| e^{\sigma |z - p|} \le \|T\|_\sigma \|u\|_\tau.
\end{aligned}$$

It is clear that $\mathcal{L}_\sigma$ equipped with the norm $\|\cdot\|_\sigma$ is a Banach space. Moreover, equipped with the composition of operators as the multiplying operation, it becomes a Banach algebra. In fact,

$$\begin{aligned}
\|T_2 T_1\|_\sigma &= \sup_{z_2 \in Z^2} \sum_{z_1 \in Z^2} |T_2 T_1(z_1, z_2)| e^{\sigma |z_1 - z_2|} \\
&= \sup_{z_2 \in Z^2} \sum_{z_1 \in Z^2} \Big| \sum_{p \in Z^2} T_2(z_1, p) T_1(p, z_2) \Big| e^{\sigma |z_1 - z_2|} \\
&\le \sup_{z_2 \in Z^2} \sum_{p \in Z^2} |T_1(p, z_2)| e^{\sigma |p - z_2|} \sum_{z_1 \in Z^2} |T_2(z_1, p)| e^{\sigma |z_1 - p|} \\
&\le \|T_1\|_\sigma \|T_2\|_\sigma, \quad \forall T_1, T_2 \in \mathcal{L}_\sigma.
\end{aligned}$$

Every function $u \in \mathcal{H}$ generates a linear operator $\langle u \rangle : \mathcal{H} \longrightarrow \mathcal{H}$ in the following way

$$\langle u \rangle v = uv, \quad \forall v \in \mathcal{H}.$$

Since

$$\begin{aligned}
\langle u \rangle (z_1, z_2) &= (2\pi)^{-2} \int_{[0,2\pi]^2} (\langle u \rangle e_{z_2}) \bar{e}_{z_1} = (2\pi)^{-2} \int_{[0,2\pi]^2} u e_{z_2} \bar{e}_{z_1} \\
&= (2\pi)^{-2} \int_{[0,2\pi]^2} u \bar{e}_{z_1 - z_2} = u(z_1 - z_2), \quad \forall z_1, z_2 \in Z^2
\end{aligned}$$

for any given $\sigma > 0$, the function u belongs to H_σ if and only if the operator $\langle u\rangle$ belongs to $\mathcal{L}_\sigma$; moreover, $\|u\|_\sigma = \|\langle u\rangle\|_\sigma$.

Suppose that $X \subseteq Z^2$. For any given linear operator $T : \mathcal{H}_X \longrightarrow \mathcal{H}_X$, for any given $Y \subseteq X$, we denote $T|_Y$ or simply T_Y: $\mathcal{H}_Y \longrightarrow \mathcal{H}_Y$ the restriction of T on $\mathcal{H}_Y$, i.e.,

$$(T|_Y)u = YTu, \quad \forall u \in \mathcal{H}_Y.$$

Suppose that $X_1, X_2 \subseteq Z^2$ and that $T : \mathcal{H}_{X_1} \longrightarrow \mathcal{H}_{X_2}$ is a linear operator. If for any given $\sigma > 0$, there exists $\tau \in (0, \sigma]$ such that

$$\|Tu\|_\tau \le c_{\sigma,\tau}\|u\|_\sigma, \quad \forall u \in H_{\sigma,X_1},$$

then we call T continuous, and denote

$$\|T\|_\sigma = \sup_{z_2\in X_1} \sum_{z_1\in X_2} |T(z_1, z_2)|e^{\sigma|z_1-z_2|},$$

where

$$T(z_1, z_2) = (2\pi)^{-2}\int_{[0,2\pi]^2}(Te_{z_2}) \cdot \bar{e}_{z_1}, \quad \forall z_1 \in X_2, \quad \forall z_2 \in X_1.$$

One can check that if $T : \mathcal{H}_{X_1} \longrightarrow \mathcal{H}_{X_2}$ is a linear continuous operator, then

$$(Tu)(z) = \sum_{p\in X_1} T(z, p)u(p), \quad \forall z \in X_2.$$

If $T_1 : \mathcal{H}_{X_1} \longrightarrow \mathcal{H}_{X_2}$ and $T_2 : \mathcal{H}_{X_2} \longrightarrow \mathcal{H}_{X_3}$ are linear continuous operators, then

$$T_2T_1(z_1, z_2) = \sum_{p\in Z^2} T_2(z_1, p)T_1(p, z_2), \quad \forall z_1 \in X_3, \quad \forall z_2 \in X_1.$$

Moreovere, if the right-hand side of the following inequality makes sense, then

$$\|T_2T_1\|_\sigma \le \|T_2\|_\sigma\|T_1\|_\sigma.$$

2.3. Block Decomposition for Linear Operators

Suppose that $X, Y \subseteq Z^2$, where $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, and $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$. For any given linear operators $T_1 : \mathcal{H}_{X_1} \longrightarrow \mathcal{H}_{Y_1}$, $T_2 : \mathcal{H}_{X_2} \longrightarrow \mathcal{H}_{Y_1}$, $T_3 : \mathcal{H}_{X_1} \longrightarrow \mathcal{H}_{Y_2}$ and $T_4 : \mathcal{H}_{X_2} \longrightarrow \mathcal{H}_{Y_2}$, the notation $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ denotes the linear operator $T : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$, given by $T = T_1X_1 + T_2X_2 + T_3X_1 + T_4X_2$. It is easy to check that

$$\|T\|_\sigma = \max\{\|T_1\|_\sigma + \|T_3\|_\sigma, \|T_2\|_\sigma + \|T_4\|_\sigma\}, \quad \forall \sigma > 0.$$

Consequently, $\|T_l\|_\sigma \le \|T\|_\sigma$, $\forall \sigma > 0, l = 1, 2, 3, 4$.

§3. A Framework for Constructing Periodic Solutions to Nonlinear PDEs

3.1. Translate the Problem in an Equivalent Functional Equation by Fourier Transformation

To construct time periodic solutions to the periodic boundary value problem of a perturbed Klein-Gordon equation

$$u_{tt} - u_{xx} + au - u^3 + f(x, u) = 0, \tag{3.1}$$

$$u(t, x + 2\pi) = u(t, x), \tag{3.2}$$

as usual, we first make scaling in the time variable. Given a frequency $\Omega \in R$, a linear continuous operator $J_\Omega : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$u = \sum_{z \in Z^2} u(z) e_z \mapsto J_\Omega u = \sum_{z=(j,k) \in Z^2} (-k^2\Omega^2 + j^2 + a) u(z) e_z.$$

And, a nonlinear operator $F_\Omega : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $F_\Omega(u) = J_\Omega(u) - u^3 + f(x,u)$. If $u(s,x) \in \mathcal{H}$ is a solution to the functional equation $F_\Omega(u) = 0$, then $w(t,x) = u(\Omega t, x)$ is a time periodic solution with frequency Ω to the perturbed Klein-Gordon equation (3.1) with the periodic boundary condition (3.2). One can check that if a real number a satisfies a Diophantine condition $|ka - j| > dk^{-\gamma}$, $\forall j,k \in Z, k \neq 0$ for some constants $d > 0$ and $\gamma > 0$, then there exist constants $d' > 0$ and $\gamma' > 0$, such that the real number $\omega_{j_0} = \sqrt{j_0^2 + a}$ satisfies the Diophantine condition $|k\omega_{j_0} - j| > d'k^{-\gamma'}$, $\forall j,k \in N$. Therefore, Theorem 1.4 is a direct consequence of the following theorem.

Theorem 3.1. *Let j_0 be a given positive integer. Suppose that there exist $d > 0, \gamma > 0$, such that $\omega_{j_0} = \sqrt{j_0^2 + a}$ satisfies the Diophantine condition*

$$|k\omega_{j_0} - j| > dk^{-\gamma}, \quad \forall j,k \in N. \tag{3.3}$$

For any given $f(x,u) = \sum_{m=4}^{\infty} f_m(x) u^m$, which is a periodic function in x with period 2π, and is analytic on the domain $\{(x,u) \in C^2 \,|\, |\Re x| \le \bar\sigma, |u| \le \delta\}$, where $\bar\sigma > 0$ and $\delta > 0$ are positive constants, for any given constant $\mu > 1$, there exists $r_* > 0$ and a Cantor subset $\mathcal{C}$ of the interval $(\omega_{j_0} - r_*, \omega_{j_0})$ with full density at ω_{j_0} : $\mathrm{meas}\,(\omega_{j_0} - r, \omega_{j_0}) \setminus \mathcal{C} \le c_1 r^\mu, \quad \forall 0 < r < r_*$, such that for any given frequency parameter $\Omega \in \mathcal{C}$, there exists a solution $u_\Omega \in H_{\bar\sigma/2}$ to the equation $F_\Omega(u) = 0$, which is close to a solution $\epsilon(\Omega)\cos s \cos j_0 x$ of the linear equation $J_\Omega(u) = 0$:

$$\|u_\Omega - \epsilon(\Omega)\cos s \cos j_0 x\|_{\bar\sigma/2} \le c_2 \epsilon(\Omega)^2,$$

where $\epsilon(\Omega) = \frac{4}{3}\sqrt{2\omega_{j_0}(\omega_{j_0} - \Omega)}$, $c_1 > 0$ is a constant independent of r, and $c_2 > 0$ is a constant independent of Ω.

3.2. Direct Construct of an Approximate Solution Sequence by Newton Iteration Method

To solve the functional equation $F_\Omega(u) = 0$, Craig, Wayne and Bourgain used Lyapunov-Schmidt and Newton iteration in their works. In this paper, we shall avoid the use of the Lyapunov-Schmidt decomposition and directly use the Newton iteration method to solve this functional equation. The iteration begins with the first approximate solution $u_0 = \epsilon \cos s \cos j_0 x$, where $\epsilon = \epsilon(\Omega) \overset{\triangle}{=} \frac{4}{3}\sqrt{2\omega_{j_0}(\omega_{j_0} - \Omega)}$. For any given $\Omega \in \mathbf{R}$ and $u \in \mathcal{H}$, let $F'_\Omega(u)$ be the derivative $J_\Omega - 3\langle u^2 + f'(x,u)\rangle$ of F_Ω at u. The iteration procedure is given by $v_{n+1} = -\left(F'_{n+1}\right)^{-1} F(u_n)$, $u_{n+1} = u_n + v_{n+1}$, where $F'_{n+1} = F'_\Omega(u_n)$ is the linearized operator of $F_\Omega(u)$ at u.

3.3. Block Decomposition and Frölich-Spencer Estimate, Main Techniques for Estimating the Inverse Operators of the Linearized Operator

To estimate the correction term v_{n+1} of the nth iteration step, we need to estimate the inverse operator of the linearized operator F'_{n+1}, which is the summation of two linear operators J_Ω and $\langle -3u_n^2 + f'(x,u_n)\rangle$. The former one, the diagonal operator J_Ω, is the principle term, while the latter one, $\langle -3u_n^2 + f'(x,u_n)\rangle$, is a perturbation term since $||\langle -3u_n^2 + f'(x,u_n)\rangle||_\sigma \le c||u||_\sigma^2$, and u_n keeps "small" in the iteration procedure. For those frequecy values Ω close to ω_{j_0}, zero is a cluster point of the infinitely many diagonal elements $J_\Omega(z,z) = k^2\Omega^2 - \omega_j^2$ $(z = (j,k) \in Z^2)$ of J_Ω. Therefore, a small divisor difficulty arises when estimating the inverse operators.

First, we distinguish the lattice set of strong resonance $Q = \{(j,k) \in Z^2 | \, j = \pm j_0, k = \pm 1\}$ and weak resonance $P = Z^2 \setminus Q$. When the frequency Ω is taken near ω_{j_0}, for lattice points z in Q, the diagonal elements $J_\Omega(z,z) = \Omega^2 - \omega_{j_0}^2$ tends to zero when the frequency parameter Ω tends to ω_{j_0}. Therefore, the linearized operator F' has a strong resonance on the lattice set Q. To treat with this problem, we introduce the block decomposition technique.

The following lemma, appeared in [6] and independently used by the author in his Ph.D thesis, is important to our block decomposition technique.

Lemma 3.1. *Suppose that $X \subseteq Z^2$ is a lattice set, $L : \mathcal{H}_X \longrightarrow \mathcal{H}_X$ is a linear operator, $L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$. If L_4 and $(L_1 - L_2 L_4^{-1} L_3)$ are invertible, then*

$$L^{-1} = \begin{pmatrix} A & -AL_2L_4^{-1} \\ -L_4^{-1}L_3A & L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1} \end{pmatrix}$$

is the inverse operator of L, where $A = (L_1 - L_2L_4^{-1}L_3)^{-1}$.

Make block decomposition $F'_{n+1} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$, where $L_1 = QF'_{n+1}Q$, $L_2 = QF'_{n+1}P$, $L_3 = PF'_{n+1}Q$, $L_4 = PF'_{n+1}P$. Having obtained an estimate of the inverse operator of L_4, we can apply Lemma 3.1 to estimate the inverse operator of the linearized operator F'_{n+1}.

The estimation of the inverse operator of L_4 meets the small divisor difficulty. We essentially follow the line of Craig and Wayne[11]. The key point is a Fröhlich-Spencer estimate given in Lemma 3.2 below due to Craig, Wayne[11] and Pöschel[26], which is stated here in an explicit form convenient for our use and also gives an explicit expression of the inverse operator (see (3.5)).

Let $d_0 = \frac{1}{2} \min_{j \in Z} |j - a|$. Define $S(\Omega) = \left\{ z \in Z^2 | \, |J_\Omega(z,z)| \le d_0 \right\} \cap P$ as the singular point set of J_Ω. It is easy to verify that if ω_{j_0} satisfies the Diophantine condition (3.3), then for any given $\mu > 1$, the set $\mathcal{D}$ of those frequencies Ω satisfying the Diophantine condition

$$|k\Omega - j| > \frac{1}{2} d k^{-\frac{1}{\alpha}}, \quad \forall j,k \in N,$$

where $\alpha = \frac{1}{\mu(\gamma+1)+1}$, has full density at $\omega_{j_0} = \sqrt{j_0^2 + a}$: there exists $r^* > 0$ and $c_0 > 0$, such that

$$\operatorname{meas}(\omega_{j_0} - r, \omega_{j_0} + r) \setminus \mathcal{D} \le c_0 r^\mu, \quad \forall 0 < r < r_*. \tag{3.4}$$

If the frequency parameter Ω is taken in the Diophantine set $\mathcal{D}$, then the singular point set $S(\Omega)$ of J_Ω is sparse. In fact, we have the following proposition.

Proposition 3.1. *If $\Omega \in \mathcal{D}$, then there exists $d_1 > 0$, such that for any given $z_1 = (j_1, k_1)$ and $z_2 = (j_2, k_2)$ in $S(\Omega)$ with $|z_1| \le |z_2|$ and $z_1 \ne z_2$, we have $|z_1 - z_2| \ge 2d_1 |z_2|^\alpha$.*

We will prove Proposition 3.1 in Section 5.

In the iteration procedure, to control the "smallness" of the "small divisors ", we will eliminate some "bad" values of the frequency parameter Ω, so as to obtain a sequence of "good" frequency parameter sets $\mathcal{C}_0 = \mathcal{D} \cap (\omega_{j_0} - r_*, \omega_{j_0})$ and

$$\mathcal{C}_n = \{\Omega \in \mathcal{C}_{n-1} | \, |F'_n(z,z)|^{-1} \le 2^n |z|^{-\beta}, \|(F'_n|_{B(z)})^{-1}\|_{\sigma_{n-1}} \le 2^n |z|^{-\beta}, \ \forall z \in S(\Omega)\},$$
$$n = 1, 2 \cdots,$$

where $r_* > 0$ is to be determined later, $B(z)$ denotes the spherical neighbourhood $\{p \in P| \, |p - z| \le d_1|z|^\alpha\}$ of z in P, $\beta = \mu(\gamma + 1) + 1$, and d_1 is given by Proposition 3.1. The limit set $\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$ of the sequence of closed sets $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$ is a Cantor set.

Lemma 3.2. *Suppose that $X \subseteq Z^2$, $L : \mathcal{H}_X \longrightarrow \mathcal{H}_X$ is a linear operator. Suppose that S is a subset of X, and there exist constants $d_1 > 0$ and $0 < \alpha < 1$, such that*

$$|z_1 - z_2| > 2d_1\left((1 + |z_1|)^\alpha + (1 + |z_2|)^\alpha\right), \quad \forall z_1, z_2 \in S, \quad z_1 \ne z_2.$$

For any given $z \in S$, let $B(z)$ be the spherical neighbourhood $\{p \in X| \, |p - z| \le d_1(1 + |z|)^\alpha\}$ of z in X. Denote $R = X \setminus S$, $\tilde{R} = X \setminus \Big(\bigcup_{z\in S} B(z)\Big)$,

$$A = L_R \oplus \Big(\bigoplus_{z\in S} L_z\Big), \quad B = L - A, \quad \tilde{A} = L_{\tilde{R}} \oplus \Big(\bigoplus_{z\in S} L_{B(z)}\Big), \quad \tilde{B} = L - \tilde{A},$$

where "$\oplus$" means direct sum. Suppose furthermore that there exists $\sigma > 0$, such that

- *Both the restrictions of the operator L on R and $\tilde{R}$ are invertible, with $\|L_R\|_\sigma < \infty$, $\|L_{\tilde{R}}\|_\sigma < \infty$;*
- *For any given $z \in S$, both the restrictions of the operator L on z and $B(z)$ are invertible; moreover, there exist constants $d_2 > 0$ and $\beta > 0$, such that*

$$\|L_z^{-1}\|_\sigma = \frac{1}{|L(z,z)|} \le d_2|z|^\beta, \quad \|L_{B(z)}^{-1}\|_\sigma \le d_2|z|^\beta.$$

Then there exits a constant $c_0 > 0$ independent of d_1, d_2, α, β and σ, such that for any given $0 < \tau < \sigma$, we have

$$\|A^{-1}B\tilde{A}^{-1}\tilde{B}\|_{\sigma-\tau} \le c_0 d_2(d_2 + \|L_R^{-1}\|_\sigma + \|L_{\tilde{R}}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}} e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma,$$
$$\|\tilde{B}\tilde{A}^{-1}BA^{-1}\|_{\sigma-\tau} \le c_0 d_2(d_2 + \|L_R^{-1}\|_\sigma + \|L_{\tilde{R}}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}} e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma,$$

where $m = 1 + \min_{z\in S}|z|$. If furthermore,

$$c_0 d_2(d_2 + \|L_R^{-1}\|_\sigma + \|L_{\tilde{R}}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}} e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma < 1,$$

then the operators L, $(Id - \tilde{B}\tilde{A}^{-1}BA^{-1})$ and $(Id - A^{-1}B\tilde{A}^{-1}\tilde{B})$ are invertible with

$$\begin{aligned} L^{-1} &= (A^{-1} - \tilde{A}^{-1}BA^{-1})(Id - \tilde{B}\tilde{A}^{-1}BA^{-1})^{-1} \\ &= (Id - A^{-1}B\tilde{A}^{-1}\tilde{B})^{-1}(A^{-1} - A^{-1}B\tilde{A}^{-1}). \end{aligned} \tag{3.5}$$

We point out that in the case of ODEs, the Lyapunov center theory can be extended to some non-Hamiltonian systems (see [24]). Lemma 3.2 applies not only to selfadjoint operators, but also to non-selfadjoint operator; thus it is possible to extend the result in this paper to non-Hamiltonian PDEs.

3.4. Estimate of the Inverse Operators of the Linearized Operators by the Block Decomposition Technique and the Frölich-Spencer Estimate, Convergence of the Approximate Solution Sequence via Nash-Moser Technique

Let $\sigma_* = \bar{\sigma}/2$. Let $\sigma_0 = \frac{\bar{\sigma}+\sigma_*}{2}$,

$$\tau_n = 2^{-(n+2)}(\bar{\sigma} - \sigma_*), \quad \sigma_n = \sigma_{n-1} - 2\tau_n, \quad \forall n = 1, 2, \cdots.$$

Let $B_1 = \emptyset$ and $B_n = \{z \in P| \, |z| \le 2^n\}$, $n = 2, 3, \cdots$. Denote by F_n' the linearization of the nonlinear operator F_Ω at $u_{n-1}(\Omega)$. Make block decomposition

$$F'_n = \begin{pmatrix} F'_{n,QQ} & F'_{n,QP} \\ F'_{n,PQ} & F'_{n,PP} \end{pmatrix},$$

where the linear operators $F'_{n,QQ} = QF'_nQ$ maps $\mathcal{H}_Q$ in $\mathcal{H}_Q$, $F'_{n,QP} = QF'_nP$ maps $\mathcal{H}_P$ in $\mathcal{H}_Q$, $F'_{n,PQ} = PF'_nQ$ maps $\mathcal{H}_Q$ in $\mathcal{H}_P$, and $F'_{n,PP} = PF'_nP$ maps $\mathcal{H}_P$ in $\mathcal{H}_P$.

In Section 5, applying the block decomposition technique and the Frölich-Spencer estimate, combined with the Nash-Moser technique, we will prove that if $r_* > 0$ is sufficiently small, then for any given $\Omega \in \mathcal{C}_n$, the following induction hypotheses **(F'1.an)**–**(F.an)** $(n = 1, 2, \cdots)$ hold:

(F'1.an) For any given lattice set $Y \subseteq P$ satisfying $Y \supseteq \bigcup_{z\in S\cap B_n} B(z)$ and $Y\cap(S\backslash B_n) = \emptyset$, we have

$$\|(F'_n|_Y)^{-1}\|_{\sigma_n+\tau_n} \le c_1 2^{\beta n^2};$$

(F'2.an) the restriction $F'_n|_P$ of the linearized operator F'_n on $\mathcal{H}_P$ is invertible with

$$\|(F'_{n,P})^{-1}w\|_\sigma \le c_2 2^{\beta n^2}\|w\|_{\sigma+\tau_n}, \quad \forall \sigma \le \sigma_n + \tau_n, \quad \forall w \in QH_{\sigma+\tau_n};$$

(F'3.an) $\|Q(F'_n)^{-1}w\|_{\sigma_n} \le c_3(\epsilon^{-2}\|Qw\|_{\sigma_n} + 2^{\beta n^2}\|Pw\|_{\sigma_n+\tau_n})$,
$\|P(F'_n)^{-1}w\|_{\sigma_n} \le c_4(2^{\beta n^2}\|Qw\|_{\sigma_n+\tau_n} + 2^{2\beta n^2}\|Pw\|_{\sigma_{n-1}})$;

(v.an) $\|v_1\|_{\sigma_1} \le c_5\epsilon^2$, $\|v_n\|_{\sigma_n} \le \epsilon^2 e^{-(\frac{3}{2})^n}$, $\quad n = 2, 3, \cdots$;

(u.an) $\|u_n - u_0\|_{\sigma_n} \le c_6\epsilon^2$;

(F.an) $\|F(u_n)\|_{\sigma_n-\tau_{n+1}} \le \epsilon^{\frac{9}{2}} e^{-2(\frac{3}{2})^n}$,

where $\epsilon(\Omega) = \frac{4}{3}\sqrt{2\omega_{j_0}(\omega_{j_0} - \Omega)}$, and c_i are positive constants independent of the frequency parameter Ω and the iteration step number n.

The induction hypotheses **(F.an)** and **(u.an)** $(n = 1, 2, \cdots)$ imply that for any given frequency Ω taken in the Cantor set $\mathcal{C}$, the functional equation $F_\Omega(u) = 0$ admits a solution $u(\Omega) = \sum_{n=0}^{\infty} u_n(\Omega) \in H_{\sigma_*}$, which satisfies $\|u(\Omega) - \epsilon(\Omega)\cos s \cos j_0 x\|_{\sigma*} \le c_2\epsilon(\Omega)^2$.

3.5. Estimate of the Density of the Cantor Parameter Set via Lipschitz Estimates

To prove Theorem 3.1, it remains to estimate the density of the Cantor parameter set $\mathcal{C}$ at the frequency ω_{j_0} of the linearized problem, namely, to prove that

$$\operatorname{meas}(\omega_{j_0} - r, \omega_{j_0}) \setminus \mathcal{C} \le c_1 r^\mu, \quad \forall 0 < r < r_*.$$

According to (3.4), it suffices to prove the following estimates

(Cn) meas $(\omega_{j_0} - r, \omega_{j_0}) \setminus \mathcal{C}_n \le c2^{-n}r^\mu$, $\forall 0 < r < r_*$, $n = 1, 2, \cdots$.

To prove estimates **(Cn)**, we need only to prove the following proposition.

Proposition 3.2. *For any given $\Omega^+, \Omega^- \in \mathcal{C}_{n-1}$, we have*

$$\|u_{n-1}(\Omega^+)^2 - u_{n-1}(\Omega^-)^2\|_{\sigma_n-\tau_n} \le c|\Omega^+ - \Omega^-|,$$
$$|h_{\Omega^+}(z) - h_{\Omega^-}(z)| \le r_*^2 k^2|\Omega^+ - \Omega^-|, \quad \forall z = (j, k) \in S(\Omega^+) \cap S(\Omega^-).$$

According to the construction of $\mathcal{C}_n$ $(n = 1, 2, \cdots)$, $(\omega_{j_0} - r, \omega_{j_0}) \setminus \mathcal{C}_n = \bigcup_{z\in Z^2} (E^{(1)}_{n,z} \cup E^{(2)}_{n,z})$, where

$$E^{(1)}_{n,z} = \{\Omega \in \mathcal{C}_{n-1} \cap (\omega_{j_0} - r, \omega_{j_0})|\ z \in S(\Omega), \quad |F'_{\Omega,n}(z, z)|^{-1} > 2^n|z|^{-\beta}\},$$
$$E^{(2)}_{n,z} = \{\Omega \in \mathcal{C}_{n-1} \cap (\omega_{j_0} - r, \omega_{j_0})|\ z \in S(\Omega), \quad \|(F'_{\Omega,n}|_{B(z)})^{-1}\|_{\sigma_{n-1}} > 2^n|z|^{-\beta}\}.$$

By Lemma 3.1, we can prove that for any given $\Omega \in \mathcal{C}_{n-1}$ and $z \in S(\Omega)$, it holds that

$$\|(F_n'|_{B(z)})^{-1}\|_{\sigma_{n-1}} \le 2\Big(\frac{1}{R}+1\Big), \tag{3.6}$$

where $R \triangleq F_n'(z,z) - h(z)$, and

$$h(z) = h_\Omega(z) = \sum_{p,q\in B^o(z)} F_n'(z,p)\left(F_n'|_{B^o(z)}\right)^{-1}(p,q)F_n'(q,z).$$

In virtue of Proposition 3.2, we obtain

$$|F'_{n,\Omega^+}(z,z) - F'_{n,\Omega^-}(z,z)| \ge ck^2|\Omega^+ - \Omega^-|,$$
$$|R_{\Omega^+} - R_{\Omega^-}| \ge ck^2|\Omega^+ - \Omega^-|, \quad \forall \Omega^+, \Omega^- \in \mathcal{C}_{n-1}.$$

Therefore, for any given $z = (j,k) \in Z^2$, we have meas $E_{n,z}^{(1)} \le c|k|^{-(\beta+2)}$ and meas $E_{n,z}^{(2)} \le c|k|^{-(\beta+2)}$. Then, note that for any given $k \in Z$, there are no more than $[kr+1]$ number of $j \in Z$ such that there exists $\Omega \in (\omega_{j_0} - r, \omega_{j_0})$, and $z = (j,k) \in S(\Omega)$. Moreover, since ω_{j_0} satisfies the Diophantine condition (3.3), for any given $z = (j,k) \in Z^2$, if $|k| < \delta r^{-(\gamma+1)}$, then $E_k = \emptyset$. Consequently, we have

$$\sum_{z\in Z^2} \text{meas } E_{n,z}^{(1)} \cup E_{n,z}^{(2)} \le \sum_{k\in Z, |k|>\delta r^{-(\gamma+1)}} c(|k|r+1)k^{-(\beta+2)} \le cr^\mu,$$

which proves **(Cn)**.

To prove Proposition 1.2, it is sufficient to prove that for any given $\Omega^+, \Omega^- \in \mathcal{C}_n, n = 1, 2, \cdots$, the following Lipschitz estimates are valid.

(F'.bn)

$$\|(F'_{n,\Omega^+})^{-1}w - (F'_{n,\Omega^-})^{-1}w\|_\sigma \le c\tau_n^{-2-8\beta}(\epsilon^+\epsilon^-)^{-2}|\Omega^+ - \Omega^-|\|w\|_{\sigma+3\tau_n},$$
$$\forall \sigma \le \sigma_n - \tau_n, \quad \forall w \in H_{\sigma+3\tau_n},$$

(v.bn) $\|v_n(\Omega^+) - v_n(\Omega^-)\|_{\sigma_n-\tau_n} \le ce^{-(\frac{3}{2})^n}|\Omega^+ - \Omega^-|$,

(F.bn) $\|F_{\Omega^+}(u_n(\Omega^+)) - F_{\Omega^-}(u_n(\Omega^-))\|_{\sigma_n-\tau_n} \le c\left(\epsilon^+ + \epsilon^-\right)^3 e^{-2(\frac{3}{2})^n}|\Omega^+ - \Omega^-|$,

(u.bn) $\|u_n(\Omega^+) - u_n(\Omega^-)\|_{\sigma_n-\tau_n} \le c|\epsilon^+ - \epsilon^-|$,

where $\epsilon^+ = \epsilon(\Omega^+), \epsilon^- = \epsilon(\Omega^-)$.

We will prove in the last section the estimate (3.6) and the Lipschitz estimates **(F'.bn)** –**(u.bn)**, with which we will prove Proposition 3.2.

§4. Linear Estimates

Our task in this section is to prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. Noting that $X_1X_2 = X_2X_1 = 0$, we have

$$\begin{aligned}
&\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}\begin{pmatrix} A & -AL_2L_4^{-1} \\ -L_4^{-1}L_3A & L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1} \end{pmatrix} \\
&= (L_1A - L_2L_4^{-1}L_3A)X_1 + (L_3A - L_4L_4^{-1}L_3A)X_1 \\
&\quad + [-L_1AL_2L_4^{-1} + L_2(L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1})]X_2 \\
&\quad + [-L_3AL_2L_4^{-1} + L_4(L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1})]X_2.
\end{aligned}$$

Then, noting that

$$(L_1A - L_2L_4^{-1}L_3A)X_1 = (L_1 - L_2L_4^{-1}L_3)AX_1 = X_1,$$
$$(L_3A - L_4L_4^{-1}L_3A)X_1 = 0,$$

$$\begin{aligned}&[-L_1AL_2L_4^{-1}+L_2(L_4^{-1}L_3AL_2L_4^{-1}+L_4^{-1})]X_2\\ &=[(-L_1+L_2L_4^{-1}L_3)AL_2L_4^{-1}+L_2L_4^{-1}]X_2=(-L_2L_4^{-1}+L_2L_4^{-1})X_2=0,\\ &[-L_3AL_2L_4^{-1}+L_4(L_4^{-1}L_3AL_2L_4^{-1}+L_4^{-1})]X_2\\ &=(-L_3AL_2L_4^{-1}+L_4L_4^{-1}L_3AL_2L_4^{-1}+L_4L_4^{-1})X_2=X_2,\end{aligned}$$

we have

$$\begin{pmatrix}L_1 & L_2\\ L_3 & L_4\end{pmatrix}\begin{pmatrix}A & -AL_2L_4^{-1}\\ -L_4^{-1}L_3A & L_4^{-1}L_3AL_2L_4^{-1}+L_4^{-1}\end{pmatrix}=Id_X.$$

Similarly, we have

$$\begin{pmatrix}A & -AL_2L_4^{-1}\\ -L_4^{-1}L_3A & L_4^{-1}L_3AL_2L_4^{-1}+L_4^{-1}\end{pmatrix}\begin{pmatrix}L_1 & L_2\\ L_3 & L_4\end{pmatrix}=Id_X.$$

This completes the proof of Lemma 3.1.

To prove Lemma 3.2, we need the following preliminary lemma, a straightforward extension of a similar lemma well known in the standard Banach space theory, whose proof is then omitted.

Lemma 4.1. *Suppose that $X\subseteq Z^2$, and that $T:\mathcal{H}_X\longrightarrow\mathcal{H}_X$ is a linear operator. Suppose furthermore that there exists $\sigma>0$, such that $\|T\|_\sigma<1$. Then*

(1) *the operator series $\sum\limits_{n=0}^{\infty}T^n$ is convergent in $\mathcal{L}_\sigma$, with*

$$\Big\|\sum_{n=0}^{\infty}T^n\Big\|_\sigma\le\frac{1}{1-\|T\|_\sigma};$$

(2) *the limit operator $\sum\limits_{n=0}^{\infty}T^n$ is the inverse operator of (Id_X-T).*

Proof of Lemma 3.2. First, we have

$$\begin{aligned}&L(A^{-1}-\tilde A^{-1}BA^{-1})=(A+B)A^{-1}-(\tilde A+\tilde B)\tilde A^{-1}BA^{-1}=Id-\tilde B\tilde A^{-1}BA^{-1},\\ &(A^{-1}-A^{-1}B\tilde A^{-1})L=A^{-1}(A+B)-A^{-1}B\tilde A^{-1}(\tilde A+\tilde B)=Id-A^{-1}B\tilde A^{-1}\tilde B.\end{aligned}$$

By Lemma 4.1, if $\|\tilde B\tilde A^{-1}BA^{-1}\|_{\sigma-\tau}<1$ and $\|A^{-1}B\tilde A^{-1}\tilde B\|_{\sigma-\tau}<1$, then both $(Id-\tilde B\tilde A^{-1}BA^{-1})$ and $(Id-A^{-1}B\tilde A^{-1}\tilde B)$ are invertible, which implies that L is invertible, with its inverse operator given by

$$L^{-1}=(A^{-1}-\tilde A^{-1}BA^{-1})(Id-\tilde B\tilde A^{-1}BA^{-1})^{-1}=(Id-A^{-1}B\tilde A^{-1}\tilde B)^{-1}(A^{-1}-A^{-1}B\tilde A^{-1}).$$

It is sufficient to show that there exists a constant $c_0>0$ independent of d_1,d_2,α,β and σ, such that for any given $0<\tau<\sigma$, we have

$$\begin{aligned}&\|A^{-1}B\tilde A^{-1}\tilde B\|_{\sigma-\tau}\le c_0d_2(d_2+\|L_R^{-1}\|_\sigma+\|L_{\tilde R}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde B\|_\sigma,\\ &\|\tilde B\tilde A^{-1}BA^{-1}\|_{\sigma-\tau}\le c_0d_2(d_2+\|L_R^{-1}\|_\sigma+\|L_{\tilde R}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde B\|_\sigma.\end{aligned}$$

In what follows, the length $|z_1-z_2|+\cdots+|z_{n-1}-z_n|$ of an arbitary path $(z_1,z_2,\cdots,z_n)$ in Z^2 will be denoted by $|(z_1,z_2,\cdots,z_n)|$.

According to the definition of the norms of the linear operators, we have

$$\begin{aligned}\|A^{-1}B\tilde{A}^{-1}\tilde{B}\|_{\sigma-\tau} &= \sup_{z_2\in X}\sum_{z_1\in X}|A^{-1}B\tilde{A}^{-1}\tilde{B}(z_1,z_2)|e^{(\sigma-\tau)|z_1-z_2|}\\ &= \sup_{z_2\in X}\sum_{z_1\in X}\Big|\sum_{p\in X}\sum_{q\in X}\sum_{r\in X}A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)\Big|e^{(\sigma-\tau)|z_1-z_2|}\\ &\le \sup_{z_2\in X}\{I(z_2)+II(z_2)\},\end{aligned}$$

where

$$\begin{aligned}I(z_2) &= \sum_{z_1\in R}\sum_{p\in X}\sum_{q\in X}\sum_{r\in X}|A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,p,q,r,z_2)|},\\ II(z_2) &= \sum_{z_1\in S}\sum_{p\in X}\sum_{q\in X}\sum_{r\in X}|A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,p,q,r,z_2)|}.\end{aligned}$$

Because the linear operators $A=L_R\oplus\Big(\bigoplus_{z\in S}(L_z)\Big)$ and $\tilde{A}=L_{\tilde{R}}\oplus\Big(\bigoplus_{z\in S}L_{B(z)}\Big)$ are block diagonal, and, the linear operators $B=L-A$ and $\tilde{B}=L-\tilde{A}$ are off-block diagonal, we have

$$\begin{aligned}I(z_2) &= \sum_{z_1\in R}\sum_{p\in R}\sum_{q\in X}\sum_{r\in X}|L_R^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,p,q,r,z_2)|}\\ &= \sum_{z_1\in R}\sum_{p\in R}\sum_{q\in S}\sum_{r\in X}|L_R^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,p,q,r,z_2)|}\\ &= \sum_{z_1\in R}\sum_{p\in R}\sum_{q\in S}\sum_{r\in B(q)}|L_R^{-1}(z_1,p)B(p,q)L_{B(q)}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,p,q,r,z_2)|},\\ II(z_2) &= \sum_{z_1\in S}\sum_{q\in X}\sum_{r\in X}|(L(z_1,z_1))^{-1}B(z_1,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,q,r,z_2)|}\\ &= II_1(z_2)+II_2(z_2),\end{aligned}$$

where

$$\begin{aligned}II_1(z_2) &= \sum_{z_1\in S}\sum_{q\in\tilde{R}}\sum_{r\in X}|(L(z_1,z_1))^{-1}B(z_1,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,q,r,z_2)|}\\ &= \sum_{z_1\in S}\sum_{q\in\tilde{R}}\sum_{r\in\tilde{R}}|(L(z_1,z_1))^{-1}B(z_1,q)L_{\tilde{R}}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,q,r,z_2)|},\\ II_2(z_2) &= \sum_{z_1\in S}\sum_{z\in S}\sum_{q\in B(z)}\sum_{r\in X}|(L(z_1,z_1))^{-1}B(z_1,q)\tilde{A}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,q,r,z_2)|}\\ &= \sum_{z_1\in S}\sum_{z\in S}\sum_{q\in B(z)}\sum_{r\in B(z)}|(L(z_1,z_1))^{-1}B(z_1,q)L_{B(z)}^{-1}(q,r)\tilde{B}(r,z_2)|e^{(\sigma-\tau)|(z_1,q,r,z_2)|}.\end{aligned}$$

Suppose that $q\in S$, $r\in B(q)$. If $z\in B(q)$, then $\tilde{B}(r,z_2)=0$; otherwise

$$\begin{gathered}|(z_1,p,q,r,z_2)|\ge|q-z_2|\ge d_1(1+|q|)^\alpha\ge d_1m^\alpha,\\ e^{-\frac{\tau}{2}|(z_1,p,q,r,z_2)|}\le e^{-\frac{\tau}{2}|q-z_2|}\le e^{-\frac{d_1\tau}{2}(1+|q|)^\alpha}\le c\,(d_1\tau)^{-\frac{2\beta+3}{\alpha}}\,(1+|q|)^{-(2\beta+3)}.\end{gathered}$$

Noting that $\|L_{B(q)}\|_\sigma \le d_2|q|^\beta$, we have

$$\sum_{r\in B(q)} |\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}|L^{-1}_{B(q)}(q,r)|e^{\sigma|q-r|}e^{-\frac{\tau}{2}|(z_1,p,q,r,z_2)|}$$
$$\le c(d_1\tau)^{-\frac{2\beta+3}{\alpha}}(1+|q|)^{-(2\beta+3)}\sum_{r\in B(q)}|\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}\|L_{B(q)}\|_\sigma$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}(1+|q|)^{-3}\|\tilde{B}\|_\sigma.$$

Then, noting that $\sum\limits_{q\in Z^2}(1+|q|)^{-3}<\infty$, we have

$$I(z_2)\le e^{-\frac{d_1\tau}{2}m^\alpha}\sum_{q\in S}\sum_{r\in B(q)}|\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}|L^{-1}_{B(q)}(q,r)|e^{\sigma|q-r|}e^{-\frac{\tau}{2}|(z_1,p,q,r,z_2)|}$$
$$\times\sum_{p\in R}|B(p,q)|e^{\sigma|p-q|}\sum_{z_1\in R}|L^{-1}_R(z_1,p)|e^{\sigma|z_1-p|}$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma\|L^{-1}_R\|_\sigma\sum_{q\in S}(1+|q|)^{-3}$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma\|L^{-1}_R\|_\sigma.$$

Suppose that $z_1\in S$, $q\in\tilde{R}$. Then we have

$$|(z_1,q,r,z_2)|\ge|z_1-q|\ge d_1(1+|z_1|)^\alpha\ge d_1m^\alpha,$$
$$e^{-\frac{\tau}{2}|(z_1,q,r,z_2)|}\le e^{-\frac{\tau}{2}|z_1-q|}\le e^{-\frac{d_1\tau}{2}(1+|z_1|)^\alpha}\le c(d_1\tau)^{-\frac{2\beta+3}{\alpha}}(1+|z_1|)^{-(2\beta+3)}.$$

Consequently, for any given $q\in X$, we have

$$\sum_{z_1\in S}|L(z_1,z_1)|^{-1}|B(z_1,q)|e^{\sigma|z_1-q|}e^{-\frac{\tau}{2}|(z_1,q,r,z_2)|}$$
$$\le\sum_{z_1\in S}d_2|z_1|^\beta|B(z_1,q)|e^{\sigma|z_1-q|}c(d_1\tau)^{-\frac{2\beta+3}{\alpha}}(1+|z_1|)^{-(2\beta+3)}$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}\sum_{z_1\in S}|B(z_1,q)|e^{\sigma|z_1-q|}$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}\|B\|_\sigma.$$

Therefore

$$II_1(z_2)\le e^{-\frac{d_1\tau}{2}m^\alpha}\sum_{r\in\tilde{R}}|\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}\sum_{q\in\tilde{R}}|L^{-1}_{\tilde{R}}(q,r)|e^{\sigma|q-r|}$$
$$\times\sum_{z_1\in S}|L(z_1,z_1)|^{-1}|B(z_1,q)|e^{\sigma|z_1-q|}e^{-\frac{\tau}{2}|(z_1,q,r,z_2)|}$$
$$\le cd_2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma\|L^{-1}_{\tilde{R}}\|_\sigma.$$

Suppose that $z_1\in S$, and there exists $z\in S$, such that $q,r\in B(z)$. If $z\ne z_1$, then we have

$$|(z_1,q,r,z_2)|\ge|z_1-q|\ge|z_1-z|-|z-q|\ge d_1[(1+|z_1|)^\alpha+(1+|z|)^\alpha]\ge d_1m^\alpha.$$

If $z=z_1$, and $z_2\notin B(z_1)$, then we have

$$|(z_1,q,r,z_2)|\ge|z_1-z_2|\ge d_1(1+|z_1|)^\alpha=d_1(1+|z|)^\alpha\ge d_1m^\alpha.$$

If $z = z_1$, and $z_2 \in B(z_1)$, then $\tilde{B}(r, z_2) = 0$. In conclusion, we always have

$$e^{-\frac{\tau}{2}|(z_1,q,r,z_2)|} \le e^{-\frac{d_1\tau}{4}(1+|z_1|)^\alpha} e^{-\frac{d_1\tau}{4}(1+|z|)^\alpha} \le c(d_1\tau)^{-\frac{2\beta+3}{\alpha}}(1+|z_1|)^{-\beta}(1+|z|)^{-(\beta+3)}.$$

On the other hand, for any given $q \in X$, we have

$$\begin{aligned}&\sum_{z_1\in S} |L(z_1,z_1)|^{-1}|B(z_1,q)|e^{\sigma|z_1-q|}(1+|z_1|)^{-\beta}\\ \le &\sum_{z_1\in S} d_2|z_1|^\beta|B(z_1,q)|e^{\sigma|z_1-q|}(1+|z_1|)^{-\beta} \le cd_2\|B\|_\sigma,\end{aligned}$$

and, for any given $z_2 \in X$, we have

$$\begin{aligned}&\sum_{z\in S}\sum_{r\in B(z)} |\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}\sum_{q\in B(z)}|L^{-1}_{B(z)}(q,r)|e^{\sigma|q-r|}(1+|z|)^{-(\beta+3)}\\ \le &\sum_{z\in S}\|\tilde{B}\|_\sigma\|L^{-1}_{B(z)}\|_\sigma(1+|z|)^{-(\beta+3)} \le d_2\|\tilde{B}\|_\sigma\sum_{z\in S}(1+|z|)^{-3} \le cd_2\|\tilde{B}\|_\sigma.\end{aligned}$$

It follows that

$$\begin{aligned}II_2(z_2) \le\ & c(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\sum_{z\in S}\sum_{r\in B(z)}|\tilde{B}(r,z_2)|e^{\sigma|r-z_2|}\sum_{q\in B(z)}|L^{-1}_{B(z)}(q,r)|e^{\sigma|q-r|}\\ &\times\sum_{z_1\in S}|L(z_1,z_1)|^{-1}|B(z_1,q)|e^{\sigma|z_1-q|}e^{-\frac{\tau}{2}|(z_1,q,r,z_2)|}\\ \le\ & cd_2^2(d_1\tau)^{-\frac{2\beta+3}{\alpha}}\|B\|_\sigma\|\tilde{B}\|_\sigma e^{-\frac{d_1\tau}{2}m^\alpha}.\end{aligned}$$

To summarize, we have

$$\|A^{-1}B\tilde{A}^{-1}\tilde{B}\|_{\sigma-\tau} \le cd_2(d_2 + \|L_R^{-1}\|_\sigma + \|L_{\tilde{R}}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma.$$

Similarly, we can prove that

$$\|\tilde{B}\tilde{A}^{-1}BA^{-1}\|_{\sigma-\tau} \le cd_2(d_2 + \|L_R^{-1}\|_\sigma + \|L_{\tilde{R}}^{-1}\|_\sigma)(d_1\tau)^{-\frac{2\beta+3}{\alpha}}e^{-\frac{d_1\tau}{2}m^\alpha}\|B\|_\sigma\|\tilde{B}\|_\sigma.$$

The proof of Lemma 3.2 is completed.

§5. Convergence of the Newton Iteration

First, let us prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality, we may assume that $|z_2| \le 2|z_1|$. We may assume furthermore that $k_1, k_2 > 0$, $j_1, j_2 > 0$, since (j,k) in $S(\Omega)$ implies that $(\pm j, \pm k)$ in $S(\Omega)$. Then we have

$$\begin{aligned}\frac{1}{d}|k_1-k_2|^{-\frac{1}{\alpha}} &\le |(k_1-k_2)\Omega - (j_1-j_2)| \le |k_1\Omega - j_1| + |k_2\Omega - j_2|\\ &\le \frac{|k_1^2\Omega^2 - j_1^2|}{k_1\Omega + j_1} + \frac{|k_2^2\Omega^2 - j_2^2|}{k_2\Omega + j_2} \le (d_0+a)\Big(\frac{1}{k_1\Omega+j_1} + \frac{1}{k_2\Omega+j_2}\Big)\\ &\le c\Big(\frac{1}{|z_1|} + \frac{1}{|z_2|}\Big) \le \frac{3c}{|z_2|}.\end{aligned}$$

Therefore there exists $d_1 > 0$, such that $|z_1 - z_2| \ge |k_1 - k_2| \ge 2d_1|z_2|^\alpha$. This completes the proof of Proposition 3.1.

Throughout this section, we denote by c the various positive constants independent of the frequency parameter Ω and the iteration step number n.

We need to prove that there exists $r_* > 0$ sufficiently small, such that for the monotone sequence of closed sets $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$ defined by $\mathcal{C}_0 = \mathcal{D} \cap (\omega_{j_0} - r_*, \omega_{j_0})$ and

$$\mathcal{C}_n = \{\Omega \in \mathcal{C}_{n-1} |\ |F_n'(z,z)|^{-1} \le 2^n |z|^{-\beta}, \quad \|(F_n'|_{B(z)})^{-1}\|_{\sigma_{n-1}} \le 2^n |z|^{-\beta}, \quad \forall z \in S(\Omega)\},$$
$$n = 1, 2 \cdots,$$

the following induction hypotheses **(F'1.an)–(F.an)** are valid for any given $\Omega \in \mathcal{C}_n$.

(F'1.an) For any given lattice set $Y \subseteq P$ satisfying $Y \supseteq \bigcup_{z \in S \cap B_n} B(z)$ and $Y \cap (S \backslash B_n) = \emptyset$, we have $\|(F_n'|_Y)^{-1}\|_{\sigma_n + \tau_n} \le c_1 2^{\beta n^2}$;

(F'2.an) the restriction $F_n'|_P$ of the linearized operator F_n' on $\mathcal{H}_P$ is invertible with

$$\|(F_{n,P}')^{-1} w\|_\sigma \le c_2 2^{\beta n^2} \|w\|_{\sigma + \tau_n}, \quad \forall \sigma \le \sigma_n + \tau_n, \quad \forall w \in QH_{\sigma + \tau_n};$$

(F'3.an) $\|Q(F_n')^{-1} w\|_{\sigma_n} \le c_3(\epsilon^{-2} \|Qw\|_{\sigma_n} + 2^{\beta n^2} \|Pw\|_{\sigma_n + \tau_n})$,
$\|P(F_n')^{-1} w\|_{\sigma_n} \le c_4(2^{\beta n^2} \|Qw\|_{\sigma_n + \tau_n} + 2^{2\beta n^2} \|Pw\|_{\sigma_{n-1}})$;

(v.an) $\|v_1\|_{\sigma_1} \le c_5 \epsilon^2, \quad \|v_n\|_{\sigma_n} \le \epsilon^2 e^{-(\frac{3}{2})^n}, \quad n = 2, 3, \cdots$;

(u.an) $\|u_n - u_0\|_{\sigma_n} \le c_6 \epsilon^2$;

(F.an) $\|F(u_n)\|_{\sigma_n - \tau_{n+1}} \le \epsilon^{\frac{9}{2}} e^{-2(\frac{3}{2})^n}$,

where $\epsilon(\Omega) = \frac{4}{3}\sqrt{2\omega_{j_0}(\omega_{j_0} - \Omega)}$, and c_i are positive constants independent of the frequency parameter Ω and the iteration step number n.

5.1. Estimates for the First Approximate Solutions

Let $M = \|\cos s \cos j_0 x\|_{\sigma_0}$. We have

(u.a0) $\|u_0\|_{\sigma_0} = M\epsilon$.

Noting that $Q = \{(j,k) \in Z^2 |\ j = \pm j_0, k = \pm 1\},\ P = Z^2 \setminus Q$, we have $\cos s \cos j_0 x \in \mathcal{H}_Q$ and

$$QJ_\Omega = J_\Omega Q = \Big[\frac{9}{16}\epsilon^2 + O(\epsilon^4)\Big] Q. \tag{5.1}$$

We need to prove

(F.a0) $\|PF(u_0)\|_{\sigma_0} \le c\epsilon^3, \quad \|QF(u_0)\|_{\sigma_0} \le c\epsilon^4$.

By definition, $F(u_0) = J_\Omega(u_0) - u_0^3 + f(x, u_0)$. It follows from **(u.a0)** that

$$\|u_0^3\|_{\sigma_0} \le c\epsilon^3, \quad \|f(x, u_0)\|_{\sigma_0} \le c\epsilon^4.$$

Noting that $Pu_0 = 0$ and the projection operator P is commutative with the operator J_Ω, we have $PJ_\Omega(u_0) = 0$. Consequently, the first estimate in **(F.a0)** is valid. Because

$$u_0^3 = \epsilon^3 \Big(\frac{3}{4}\cos s + \frac{1}{4}\cos 3s\Big)\Big(\frac{3}{4}\cos j_0 x + \frac{1}{4}\cos 3j_0 x\Big),$$

we have $Q(u_0^3) = \frac{9}{16}\epsilon^3 \cos s \cos j_0 x$. On the other hand, it follows from (5.1) that $QJ_\Omega(u_0) = \Big[\frac{9}{16}\epsilon^2 + O(\epsilon^4)\Big]\epsilon \cos s \cos j_0 x$. Therefore the second estimate in **(F.a0)** is valid.

5.2. Convergence of the Iteration Sequence

We are to prove the induction hyposeses **(F'1.an)–(F.an)**.

(i) **(v.a1)–(v.an)** $\Longrightarrow$ **(u.an)** is obvious.

(ii) **(u.a(n-1))** and **(v.an)** $\Longrightarrow$ **(F.an)**.

In fact,

$$F(u_n) = F(u_{n-1}) + F_n' v_n - \Big(3u_{n-1} + v_n + \int_0^1 (1-\eta) f''(x, u_{n-1} + \eta v_n) d\eta\Big) v_n^2.$$

According to the construction of v_n, $F(u_{n-1}) + F_n' v_n = 0$. Therefore

$$\|F(u_n)\|_{\sigma_n} \le \Big\|3u_{n-1} + v_n + \int_0^1 (1-\eta) f''(x, u_{n-1} + \eta v_n) d\eta\Big\|_{\sigma_n} \|v_n\|_{\sigma_n}^2$$
$$\le c\epsilon(\epsilon^2 e^{-(\frac{3}{2})^n})^2 \le \epsilon^{\frac{9}{2}} e^{-2(\frac{3}{2})^n}.$$

(iii) Proof of **(F'1.a1)**.

By definition, $F_1' = J_\Omega + \langle -3u_0^2 + f'(x, u_0)\rangle$. Because

$$\|\langle -3u_0^2 + f'(x,u_0)\rangle\|_{\sigma_0} \le \| -3u_0^2 + f'(x,u_0)\|_{\sigma_0} \le c\|u_0\|_{\sigma_0}^2,$$

it follows from **(u.a0)** that

$$\|\langle -3u_0^2 + f'(x,u_0)\rangle\|_{\sigma_0} \le c\epsilon^2.$$

For each $z \notin S$, the absolute value of the diagonal element $J_\Omega(z,z)$ is lager than d_0. Therefore, for any given $Y \subseteq P$ with $Y \cap S = \emptyset$, it holds that $\|J_\Omega|_Y\|_{\sigma_0} \le 1/d_0$. In virtue of Lemma 4.1, the restricted operator $F_1'|_Y = J_\Omega(Id_Y + J_\Omega^{-1}\langle -3u_0^2 + f'(x,u_0)\rangle)$ is invertible with $\|F_1'|_Y\|_{\sigma_0} \le 2/d_0$, provided that $c\epsilon^2/d_0 \le \frac{1}{2}$.

(iv) **(F'1.a(n-1))** $\Longrightarrow$ **(F'1.an)**, $n = 2, 3, \cdots$.

In what follows, for any given $z \in P$, let $B(z)$ be the spherical neighbourhood $\{p \in X|\, |p - z| \le d_1(1 + |z|)^\alpha\}$ of z in P, where $d_1 > 0$ is a constant independent of Ω given in Proposition 3.1. Given any lattice set $Y \subseteq P$ with $Y \supseteq \bigcup_{z \in S \cap B_n} B(z)$ and $Y \cap (S \setminus B_n) = \emptyset$, let $S_n = S \cap (B_n \setminus B_{n-1})$, $R = Y \setminus S_n$ and $\tilde{R} = Y \setminus \Big(\bigcup_{z \in S_n} B(z)\Big)$. Let L be the restriction $F_n'|_Y$ of F_n' on $\mathcal{H}_Y$, and

$$A = L_R \oplus \Big(\bigoplus_{z \in (S \cap B_n)} L_z\Big), \quad B = L - A, \quad \tilde{A} = L_{\tilde{R}} \oplus \Big(\bigoplus_{z \in (S \cap B_n)} L_{B(z)}\Big), \quad \tilde{B} = L - \tilde{A}.$$

Then we have

$$\|B\|_{\sigma_{n-1}} \le \|\langle -3u_{n-1}^2 + f'(x, u_{n-1})\rangle\|_{\sigma_{n-1}} \le c\|u_{n-1}^2\|_{\sigma_{n-1}} \le c\epsilon^2,$$
$$\|\tilde{B}\|_{\sigma_{n-1}} \le \|\langle -3u_{n-1}^2 + f'(x, u_{n-1})\rangle\|_{\sigma_{n-1}} \le c\|u_{n-1}^2\|_{\sigma_{n-1}} \le c\epsilon^2.$$

According to the construction of $\mathcal{C}_n$, if $\Omega \in \mathcal{C}_n$, then for any given $z \in S_n$, the restrictions of the operator L on z and on $B(z)$ are both invertible with

$$\|L_z^{-1}\|_{\sigma_{n-1}} = 1/|L(z,z)| \le 2^n |z|^\beta, \quad \|L_{B(z)}^{-1}\|_{\sigma_{n-1}} \le 2^n |z|^\beta.$$

On the other hand it follows from **(F'1.a(n-1))** that both $F_{n-1}'|_R$ and $F_{n-1}'|_{\tilde{R}}$ are invertible with $\|F_{n-1}'|_R\|_{\sigma_{n-1}} \le c2^{2\beta(n-1)}$ and $\|F_{n-1}'|_{\tilde{R}}\|_{\sigma_{n-1}} \le c2^{2\beta(n-1)}$. Let $\triangle = F_n' - F_{n-1}'$. Then

$$\triangle = \langle -3u_{n-1}^2 + f'(x, u_{n-1})\rangle - \langle -3u_{n-2}^2 + f'(x, u_{n-2})\rangle$$
$$= \Big\langle\Big(-6u_{n-2} - 3v_{n-1} + \int_0^1 f''(x, u_{n-2} + \eta v_{n-1}) d\eta\Big) v_{n-1}\Big\rangle.$$

Therefore, $\|\triangle\|_{\sigma_{n-1}} \le c\epsilon\|v_{n-1}\|_{\sigma_{n-1}} \le c\epsilon^3 e^{-(\frac{3}{2})^{n-1}}$. It follows then that both $L_R = F_n'|_R = F_{n-1}'|_R + \triangle|_R = F_{n-1}'|_R(Id_R + (F_{n-1}'|_R)^{-1}\triangle|_R)$ and $L_{\tilde{R}} = F_n'|_{\tilde{R}} = F_{n-1}'|_{\tilde{R}} + \triangle|_{\tilde{R}} = F_{n-1}'|_{\tilde{R}}(Id_{\tilde{R}} + (F_{n-1}'|_{\tilde{R}})^{-1}\triangle|_{\tilde{R}})$ are invertible; moreover, if ϵ is sufficiently small, then

$$\|(L_R)^{-1}\|_{\sigma_{n-1}} \le 2\|(F_{n-1}'|_R)^{-1}\|_{\sigma_{n-1}} \le c2^{\beta(n-1)^2+1},$$
$$\|(L_{\tilde{R}})^{-1}\|_{\sigma_{n-1}} \le 2\|(F_{n-1}'|_{\tilde{R}})^{-1}\|_{\sigma_{n-1}} \le c2^{\beta(n-1)^2+1}.$$

Consequently, by Lemma 3.2, if ϵ is small enough, then L and $(Id - A^{-1}B\tilde{A}^{-1}\tilde{B})$ are invertible with

$$L^{-1} = (Id - A^{-1}B\tilde{A}^{-1}\tilde{B})^{-1}(A^{-1} - A^{-1}B\tilde{A}^{-1}), \tag{5.2}$$

$$\|(Id - A^{-1}B\tilde{A}^{-1}\tilde{B})^{-1}\|_{\sigma_n} \leq 2. \tag{5.3}$$

It follows from the definition of the σ-norm that

$$\|A^{-1}\|_{\sigma_{n-1}} \leq \max\Big\{ \big\|L_R^{-1}\big\|_{\sigma_n}, \max_{z\in S_n} \big\|L_z^{-1}\big\|_{\sigma_n} \Big\} \leq c2^{\beta(n-1)^2+1}. \tag{5.4}$$

Similarly, one can prove that

$$\|\tilde{A}^{-1}\|_{\sigma_{n-1}} \leq c2^{\beta(n-1)^2+1}. \tag{5.5}$$

On the other hand,

$$\begin{aligned}\|A^{-1}B\tilde{A}^{-1}\|_{\sigma_{n-1}} &= \sup_{z_2\in Y} \Big|\sum_{z_1\in Y}\sum_{p\in Y}\sum_{q\in Y} A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,z_2)\Big| e^{|z_1-z_2|} \\ &\leq \sup_{z_2\in Y}\{I(z_2)+II(z_2)\},\end{aligned} \tag{5.6}$$

where

$$\begin{aligned} I(z_2) &= \sum_{z_1\in S_n}\sum_{p\in Y}\sum_{q\in Y} |A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,z_2)|e^{|z_1-z_2|} \\ &= \sum_{z_1\in S_n}\sum_{q\in Y} |L(z_1,z_1)^{-1}B(z_1,q)\tilde{A}^{-1}(q,z_2)|e^{|z_1-z_2|} \\ &\leq \max_{z_1\in S_n}|L(z_1,z_1)^{-1}| \sum_{q\in S_n}|\tilde{A}^{-1}(q,z_2)|e^{|q-z_2|} \sum_{z_1\in S_n}|B(z_1,q)|e^{|z_1-q|} \\ &\leq \|B\|_{\sigma_{n-1}}\|\tilde{A}^{-1}\|_{\sigma_{n-1}} \max_{z_1\in S_n}|L(z_1,z_1)^{-1}| \\ &\leq c\epsilon^2 2^{\beta(n-1)^2}2^{(1+\beta)n} \leq c\epsilon^2 2^{\beta n^2}, \end{aligned}$$

$$\begin{aligned} II(z_2) &= \sum_{z_1\in R}\sum_{p\in Y}\sum_{q\in Y} |A^{-1}(z_1,p)B(p,q)\tilde{A}^{-1}(q,z_2)|e^{|z_1-z_2|} \\ &= \sum_{z_1\in R}\sum_{p\in R}\sum_{q\in S_n} |L_R^{-1}(z_1,p)B(p,q)L_{B(q)}^{-1}(q,z_2)|e^{|z_1-z_2|} \\ &\leq \sum_{q\in S_n}|L_{B(q)}^{-1}(q,z_2)|e^{|q-z_2|}\sum_{p\in R}|B(p,q)|e^{|p-q|}\sum_{z_1\in R}|L_R^{-1}(z_1,p)|e^{|z_1-p|} \\ &\leq \|L_R^{-1}\|_{\sigma_{n-1}}\|B\|_{\sigma_{n-1}}\max_{q\in S_n}\|L_{B(q)}^{-1}\|_{\sigma_{n-1}} \\ &\leq c\epsilon^2 2^{\beta(n-1)^2}2^{(1+\beta)n} \leq c\epsilon^2 2^{\beta n^2}. \end{aligned}$$

Consequently **(F'1.an)** follows from (5.2)–(5.6).

(v) **(F'1.an)** $\Longrightarrow$ **(F'2.an)**, $n = 1, 2, \cdots$.

Let $R = P\setminus(S\setminus B_n)$, $\tilde{R} = P\setminus\Big(\bigcup_{z\in S\setminus B_n} B(z)\Big)$. Let L be the restriction $F_n'|_P$ of F_n' on $\mathcal{H}_P$,

$$A = L_R\oplus\Big(\bigoplus_{z\in S\setminus B_n} L_z\Big), \quad B = L - A, \quad \tilde{A} = L_{\tilde{R}}\oplus\Big(\bigoplus_{z\in S\setminus B_n} L_{B(z)}\Big), \quad \tilde{B} = L - \tilde{A}.$$

It follows from **(F'1.an)** that both L_R and $L_{\tilde{R}}$ are invertible with

$$\|(L_R)^{-1}\|_{\sigma_n+\tau_n} \le c2^{\beta n^2}, \quad \|(L_{\tilde{R}})^{-1}\|_{\sigma_n+\tau_n} \le c2^{\beta n^2}.$$

Applying again Lemma 3.2, and repeating the discussions in the proof of **(F'1.a(n-1))** $\Longrightarrow$ **(F'1.an)**, we conclude that if ϵ is sufficiently small, then both L and $(Id - A^{-1}B\tilde{A}^{-1}\tilde{B})$ are invertible with

$$\begin{aligned}&L^{-1} = (Id - A^{-1}B\tilde{A}^{-1}\tilde{B})^{-1}(A^{-1} - A^{-1}B\tilde{A}^{-1}),\\ &\|(Id - A^{-1}B\tilde{A}^{-1}\tilde{B})^{-1}\|_{\sigma_n+\tau_n} \le 2.\end{aligned}$$

Then, because

$$\begin{aligned}\|A^{-1}w\|_\sigma &= \sum_{z_1\in P}\Big|\sum_{z_2\in P} A^{-1}(z_1,z_2)w(z_2)\Big|e^{\sigma|z_1|}\\ &\le \sum_{z_1\in P}\sum_{z_2\in P}|A^{-1}(z_1,z_2)w(z_2)|e^{\sigma(|z_1-z_2|+|z_2|)}\\ &= \sum_{z_1\in R}\sum_{z_2\in R}|L_R^{-1}(z_1,z_2)w(z_2)|e^{\sigma(|z_1-z_2|+|z_2|)} + \sum_{z\in S}|L_z^{-1}w(z)|e^{\sigma|z|}\\ &\le \|L_R^{-1}\|_\sigma\|w\|_\sigma + \sum_{z\in S}2^n|z|^\beta e^{-\frac{\tau_n}{2}|z|}|w(z)|e^{\sigma+\frac{\tau}{2}|z|}\\ &\le \|L_R^{-1}\|_\sigma\|w\|_\sigma + c\tau_n^{-\beta}\|w\|_{\sigma+\frac{\tau_n}{2}}\\ &\le c2^{\beta n^2}\|w\|_{\sigma+\frac{\tau_n}{2}}, \quad \forall\sigma\le\sigma_n+2\tau_n, \quad \forall w\in PH_{\sigma+\frac{\tau_n}{2}},\end{aligned}$$

and similarly

$$\|\tilde{A}^{-1}w\|_\sigma \le c2^{\beta n^2}\|w\|_{\sigma+\frac{\tau_n}{2}}, \quad \forall\sigma\le\sigma_n+2\tau_n, \quad \forall w\in PH_{\sigma+\frac{\tau_n}{2}},$$

we conclude that

$$\|(A^{-1} - A^{-1}B\tilde{A}^{-1})w\|_\sigma \le c2^{\beta n^2}\|w\|_{\sigma+\tau_n}, \quad \forall\sigma\le\sigma_n+\tau_n, \quad \forall w\in PH_{\sigma+\tau_n}.$$

This proves **(F'2.an)**.

(vi) **(F'2.an)** $\Longrightarrow$ **(F'3.an)**.

Make the block decomposition $F_n' = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4\end{pmatrix}$, where $L_1 = QF_n'Q$, $L_2 = QF_n'P$, $L_3 = PF_n'Q$, $L_4 = PF_n'P$. It follows from Lemma 3.1 that

$$(F_n')^{-1} = \begin{pmatrix} A & -AL_2L_4^{-1} \\ -L_4^{-1}L_3A & L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1}\end{pmatrix},$$

where $A = (L_1 - L_2L_4^{-1}L_3)^{-1}$. Since $QJ_\Omega P = PJ_\Omega Q = 0$, we have

$$\begin{aligned}&\|L_2\|_{\sigma_{n-1}} \le 3\|(u_{n-1})^2 + f'(x,u_{n-1})\|_{\sigma_{n-1}} \le c\epsilon^2,\\ &\|L_3\|_{\sigma_{n-1}} \le 3\|(u_{n-1})^2 + f'(x,u_{n-1})\|_{\sigma_{n-1}} \le c\epsilon^2.\end{aligned}$$

Therefore, noting that L_4^{-1} is a bounded linear operator which maps $H_{\sigma_*,P}$ in $H_{\frac{\sigma*}{2},P}$, and that $\mathcal{H}_Q$ is a finite dimensional linear space on which all the norms are equivalent, we have $\|L_2L_4^{-1}L_3\|_{\sigma_n} \le c\epsilon^4$. On the other hand,

$$L_1 = Q[J_\Omega + \langle -3(u_{n-1})^2 + f'(x,u_{n-1})\rangle]Q = QJ_\Omega Q - 3Q\langle u_0^2\rangle Q + R,$$

where $\|R\|_\sigma \le c\epsilon^3$. And, since $Q\langle\cos 2s + \cos 2j_0x + \cos 2s\cos 2j_0x\rangle Q = 0$, we have

$$\begin{aligned} QJ_\Omega Q - 3Q\langle u_0^2\rangle Q &= \Big[\frac{9}{16}\epsilon^2 + O(\epsilon^4)\Big]Q - \frac{3}{4}\epsilon^2 Q\langle 1 + \cos 2s + \cos 2j_0x + \cos 2s\cos 2j_0x\rangle Q \\ &= \Big[-\frac{3}{16}\epsilon^2 + O(\epsilon^4)\Big]Q. \end{aligned}$$

Therefore, it follows from Lemma 4.1 that $\|A\|_{\sigma_n} \le c\epsilon^{-2}$. Consequently, for any given $w \in H_{\sigma_{n-1}}$,

$$\begin{aligned} \|Q(F_n')^{-1}w\|_{\sigma_n} &= \|AQw - AL_2L_4^{-1}Pw\|_{\sigma_n} \le \|AQw\|_{\sigma_n} + \|AL_2L_4^{-1}Pw\|_{\sigma_n} \\ &\le c(\epsilon^{-2}\|Qw\|_{\sigma_n} + 2^{\beta n^2}\|Pw\|_{\sigma_n+\tau_n}), \\ \|P(F_n')^{-1}w\|_{\sigma_n} &= \|-L_4^{-1}L_3AQw + (L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1})Pw\|_{\sigma_n} \\ &\le \|L_4^{-1}L_3AQw\|_{\sigma_n} + \|(L_4^{-1}L_3AL_2L_4^{-1} + L_4^{-1})Pw\|_{\sigma_n} \\ &\le c(2^{\beta n^2}\|Qw\|_{\sigma_n+\tau_n} + 2^{2\beta n^2}\|Pw\|_{\sigma_n+2\tau_n}). \end{aligned}$$

(vii) **(F'3.an)** and **(F.a(n-1))** $\Longrightarrow$ **(v.an)**.

When $n = 1$, we have

$$\begin{aligned} &\|Qv_1\|_{\sigma_1} \le c\epsilon^{-2}\|QF(u_0)\|_{\sigma_0} + c\|PF(u_0)\|_{\sigma_0} \le c\epsilon^2, \\ &\|Pv_1\|_{\sigma_1} \le c\|F(u_0)\|_{\sigma_0} \le c\epsilon^2. \end{aligned}$$

When $n \ge 2$, if ϵ is small enough, then we have

$$\|v_n\|_{\sigma_n} \le c\epsilon^{-2}2^{2\beta n^2}\|F(u_{n-1})\|_{\sigma_{n-1}} \le c\epsilon^{\frac{5}{2}}2^{\beta n^2}e^{-2(\frac{3}{2})^{n-1}} \le \epsilon^2 e^{-(\frac{3}{2})^n}.$$

§6. Density Estimate for the Cantor Parameter Set $\mathcal{C}$

First, we prove (3.6). We need the following lemma.

Lemma 6.1. *Suppose that $X \subseteq Z^2$, $L : \mathcal{H}_X \longrightarrow \mathcal{H}_X$ is a linear continuous operator with the off-diagonal part K satisfying $\|K\|_\sigma \le \frac{1}{2}$, where $\sigma > 0$. Suppose furthermore that there exists $z \in X$, such that*

$$\begin{aligned} &|L(p,p)| \ge 1, \quad \forall p \in X, p \ne z, \\ &R \overset{\triangle}{=} L(z,z) - \sum_{p,q\in X\setminus\{z\}} L(z,p)(L_{B^o(z)})^{-1}(p,q)L(q,z) \ne 0. \end{aligned}$$

Then L is invertible with $\|L^{-1}\|_\sigma \le 2\big(\frac{1}{R} + 1\big)$.

Proof. It follows from Lemma 3.1 that

$$L^{-1} = \begin{pmatrix} \frac{1}{R} & -\frac{1}{R}L_2L_4^{-1} \\ -L_4^{-1}L_3\frac{1}{R} & L_4^{-1}L_3\frac{1}{R}L_2L_4^{-1} + L_4^{-1} \end{pmatrix},$$

where $L_2 = X_1LX_2$, $L_3 = X_2LX_1$, $L_4 = X_2LX_2$, and $X_1 = z$, $X_2 = X \setminus \{z\}$. Clearly, $\|L_2\|_\sigma \le \|K\|_\sigma \le \frac{1}{2}$, $\|L_3\|_\sigma \le \|K\|_\sigma \le \frac{1}{2}$. Therefore, thanks to Lemma 4.1 we have $\|L_4^{-1}\|_\sigma \le 2$. Consequently,

$$\Big\|\frac{1}{R}L_2L_4^{-1}\Big\|_\sigma \le \frac{1}{R}, \quad \Big\|-L_4^{-1}L_3\frac{1}{R}\Big\|_\sigma \le \frac{1}{R}, \quad \Big\|L_4^{-1}L_3\frac{1}{R}L_2L_4^{-1} + L_4^{-1}\Big\|_\sigma \le \frac{1}{R} + 2,$$

which implies

$$\|L^{-1}\|_\sigma \le \frac{1}{R} + \Big(\frac{1}{R} + 2\Big) = 2\Big(\frac{1}{R} + 1\Big).$$

The proof of this lemma is completed.

Noting that $M = \|\cos s \cos j_0 x\|_{\sigma_0}$, we have

(u.b0) $\|u_0(\Omega^+) - u_0(\Omega^-)\|_{\sigma_0} = M|\epsilon^+ - \epsilon^-|$.

Because

$$\begin{aligned}
&F_{\Omega^+}(u_0(\Omega^+)) - F_{\Omega^-}(u_0(\Omega^-)) \\
&= [J_{\Omega^+} u_0(\Omega^+) - J_{\Omega^-} u_0(\Omega^-)] - [u_0(\Omega^+)^3 - u_0(\Omega^-)^3] \\
&= [J_{\Omega^+} - J_{\Omega^-}]u_0(\Omega^+) + J_{\Omega^-}[u_0(\Omega^+) - u_0(\Omega^-)] - [u_0(\Omega^+)^3 - u_0(\Omega^-)^3],
\end{aligned}$$

$$\begin{aligned}
\|[J_{\Omega^+} - J_{\Omega^-}]u_0(\Omega^+)\|_{\sigma_0} &= \|[-(\Omega^+)^2 + (\Omega^-)^2]\epsilon^+ \cos s \cos j_0 x\|_{\sigma_0} \\
&\le c|\Omega^+ - \Omega^-| \le c|\epsilon^+ - \epsilon^-|, \\
\|J_{\Omega^-}[u_0(\Omega^+) - u_0(\Omega^-)]\|_{\sigma_0} &= \|(-(\Omega^-)^2 + j_0^2 + a)(\epsilon^+ - \epsilon^-) \cos s \cos j_0 x\|_{\sigma_0} \\
&\le c|\epsilon^+ - \epsilon^-|, \\
\|u_0(\Omega^+)^3 - u_0(\Omega^-)^3\|_{\sigma_0} &= \|u_0(\Omega^+) - u_0(\Omega^-)\|_{\sigma_0} \|u_0(\Omega^+)^2 + u_0(\Omega^+)u_0(\Omega^-) \\
&\quad + u_0(\Omega^-)^2]\|_{\sigma_0} \le c|\epsilon^+ - \epsilon^-|,
\end{aligned}$$

we have

(F.b0) $F_{\Omega^+}(u_0(\Omega^+)) - F_{\Omega^-}(u_0(\Omega^-)) \le c_0|\epsilon^+ - \epsilon^-|$.

In the following, we prove by induction the following Lipschitz estimates.

(F'.bn)

$$\begin{aligned}
\|(F'_{n,\Omega^+})^{-1}w - (F'_{n,\Omega^-})^{-1}w\|_\sigma \le c\tau_n^{-2-8\beta}(\epsilon^+\epsilon^-)^{-2}|\Omega^+ - \Omega^-|\|w\|_{\sigma+3\tau_n}, \\
\forall \sigma \le \sigma_n - \tau_n, \quad \forall w \in H_{\sigma+3\tau_n},
\end{aligned}$$

(v.bn) $\|v_n(\Omega^+) - v_n(\Omega^-)\|_{\sigma_n-\tau_n} \le ce^{-(\frac{3}{2})^n}|\Omega^+ - \Omega^-|$,

(F.bn) $\|F_{\Omega^+}(u_n(\Omega^+)) - F_{\Omega^-}(u_n(\Omega^-))\|_{\sigma_n-\tau_n} \le c(\epsilon^+ + \epsilon^-)^3 e^{-2(\frac{3}{2})^n}|\Omega^+ - \Omega^-|$,

(u.bn) $\|u_n(\Omega^+) - u_n(\Omega^-)\|_{\sigma_n-\tau_n} \le c|\epsilon^+ - \epsilon^-|$,

where $\epsilon^+ = \epsilon(\Omega^+), \epsilon^- = \epsilon(\Omega^-)$. Here, to simplify the statement, we treat only the case where $f(x,u) \equiv 0$. The proof for the general case $f(x,u) \not\equiv 0$ is similar.

Proof. (i) **(u.b0)** and **(v.b1)–(v.bn)$\Longrightarrow$(u.bn)**.

Because $|\Omega^+ - \Omega^-| = c|(\epsilon^+)^2 - (\epsilon^-)^2| \le c|\epsilon^+ - \epsilon^-|$.

(ii) **(v.bn)$\Longrightarrow$(F.bn)**.

Because

$$\begin{aligned}
&F_{\Omega^+}(u_n(\Omega^+)) - F_{\Omega^-}(u_n(\Omega^-)) \\
&= -(3u_{n-1}(\Omega^+) + v_n(\Omega^+))v_n(\Omega^+)^2 + (3u_{n-1}(\Omega^-) + v_n(\Omega^-))v_n(\Omega^-)^2 \\
&= [3(u_{n-1}(\Omega^-) - u_{n-1}(\Omega^+)) + v_n(\Omega^-) - v_n(\Omega^+)]v_n(\Omega^-)^2 \\
&\quad + (3u_{n-1}(\Omega^+) + v_n(\Omega^+))(v_n(\Omega^-)^2 - v_n(\Omega^+)^2), \\
&\|[3(u_{n-1}(\Omega^-) - u_{n-1}(\Omega^+)) + v_n(\Omega^-) - v_n(\Omega^+)]v_n(\Omega^-)^2\|_{\sigma_n-\tau_n} \\
&\le [3\|u_{n-1}(\Omega^-) - u_{n-1}(\Omega^+)\|_{\sigma_n-\tau_n} + \|v_n(\Omega^-) - v_n(\Omega^+)\|_{\sigma_n-\tau_n}]\|v_n(\Omega^-)\|_{\sigma_n}^2 \\
&\le c|\epsilon^+ - \epsilon^-|((\epsilon^-)^2 e^{-(\frac{3}{2})^n})^2 \le c(\epsilon^-)^3 e^{-2(\frac{3}{2})^n}|(\epsilon^+)^2 - (\epsilon^-)^2| \\
&= \tilde{c}(\epsilon^-)^3 e^{-2(\frac{3}{2})^n}|\Omega^+ - \Omega^-|,
\end{aligned}$$

$$
\begin{aligned}
&\|(3u_{n-1}(\Omega^+)+v_n(\Omega^+))(v_n(\Omega^-)^2-v_n(\Omega^+)^2)\|_{\sigma_n-\tau_n}\\
&\le \|3u_{n-1}(\Omega^+)+v_n(\Omega^+)\|_{\sigma_n}\|v_n(\Omega^-)+v_n(\Omega^+)\|_{\sigma_n}\|v_n(\Omega^-)-v_n(\Omega^+)\|_{\sigma_n-\tau_n}\\
&\le c(\epsilon^++\epsilon^-)^3e^{-2(\frac{3}{2})^n}|\Omega^+-\Omega^-|,
\end{aligned}
$$

(u.bn) follows.

(iii) **(F.b(n-1))** and **(F'.bn)**$\Longrightarrow$**(v.bn)**.

According to the definition,

$$
\begin{aligned}
v_n(\Omega^+)-v_n(\Omega^-)&=(F'_{n,\Omega^+})^{-1}F(u_{n-1}(\Omega^+))-(F'_{n,\Omega^-})^{-1}F(u_{n-1}(\Omega^-))\\
&=[(F'_{n,\Omega^+})^{-1}-(F'_{n,\Omega^-})^{-1}]F(u_{n-1}(\Omega^+))\\
&\quad+(F'_{n,\Omega^-})^{-1}[F(u_{n-1}(\Omega^+))-F(u_{n-1}(\Omega^-))].
\end{aligned}
$$

Without loss of generality, we may assume that $\epsilon^+<\epsilon^-$. Then **(v.bn)** follows from

$$
\begin{aligned}
&\|[(F'_{n,\Omega^+})^{-1}-(F'_{n,\Omega^-})^{-1}]F(u_{n-1}(\Omega^+))\|_{\sigma_n-\tau_n}\\
&\le c\tau_n^{-2-8\beta}(\epsilon^+\epsilon^-)^{-2}|\Omega^+-\Omega^-|\|F(u_{n-1}(\Omega^+))\|_{\sigma_{n-1}}\\
&\le c\tau_n^{-2-8\beta}(\epsilon^+\epsilon^-)^{-2}|\Omega^+-\Omega^-|(\epsilon^+)^{\frac{9}{2}}e^{-2(\frac{3}{2})^{n-1}}\le e^{-(\frac{3}{2})^n}|\Omega^+-\Omega^-|,\\
&\|(F'_{n,\Omega^-})^{-1}[F(u_{n-1}(\Omega^+))-F(u_{n-1}(\Omega^-))]\|_{\sigma_n-\tau_n}\\
&\le c(\epsilon^{-2})\tau_n^{-4\beta}\|F(u_{n-1}(\Omega^+))-F(u_{n-1}(\Omega^-))\|_{\sigma_n+\tau_n}\\
&\le c(\epsilon^-)^{-2}\tau_n^{-4\beta}(\epsilon^++\epsilon^-)^3e^{-2(\frac{3}{2})^{n-1}}|\Omega^+-\Omega^-|\le e^{-(\frac{3}{2})^n}|\Omega^+-\Omega^-|.
\end{aligned}
$$

(iv) **(u.b(n-1))**$\Longrightarrow$**(F'.bn)**.

Firstly, we have

$$
\|(J_{\Omega^+}-J_{\Omega^-})w\|_\sigma\le c\tau_n^{-2}|\Omega^+-\Omega^-|\|w\|_{\sigma+\tau_n},\quad\forall\sigma>0,\quad\forall w\in H_{\sigma+\tau_n},
$$

since J_Ω is diagonal with

$$
\begin{aligned}
|J_{\Omega^+}(z,z)-J_{\Omega^-}(z,z)|&=|[k^2(\Omega^+)^2-j^2-a]-[k^2(\Omega^-)^2-j^2-a]|=k^2|(\Omega^+)^2-(\Omega^-)^2|\\
&=k^2|\Omega^+-\Omega^-||\Omega^++\Omega^-|\le ck^2|\Omega^+-\Omega^-|,\quad\forall z=(j,k)\in Z^2.
\end{aligned}
$$

Secondly, we have

$$
\begin{aligned}
&\|u_{n-1}(\Omega^+)^2-u_{n-1}(\Omega^-)^2\|_{\sigma_{n-1}}\\
&\le\|u_{n-1}(\Omega^+)-u_{n-1}(\Omega^-)\|_{\sigma_{n-1}}\|u_{n-1}(\Omega^+)+u_{n-1}(\Omega^-)\|_{\sigma_{n-1}}\\
&\le c(\epsilon^++\epsilon^-)|\epsilon^+-\epsilon^-|\le c|\Omega^+-\Omega^-|.
\end{aligned}
$$

Therefore, for any given $\sigma\le\sigma_n$, we have

$$
\|(F'_{n,\Omega^+}-F'_{n,\Omega^-})w\|_{\sigma+\tau_n}\le c\tau_n^{-2}|\Omega^+-\Omega^-|\,\|w\|_{\sigma+\tau_n},\quad\forall w\in H_{\sigma+\tau_n}.
$$

Then **(F'.bn)** follows from **(F'.an)**, since

$$
(F'_{n,\Omega^+})^{-1}-(F'_{n,\Omega^-})^{-1}=(F'_{n,\Omega^+})^{-1}(F'_{n,\Omega^-}-F'_{n,\Omega^+})(F'_{n,\Omega^-})^{-1}.
$$

Proof of Proposition 3.2. It follows from

$$
\begin{aligned}
\|u_n(\Omega^+)^2-u_n(\Omega^-)^2\|_{\sigma_n-\tau_n}&\le\|u_n(\Omega^+)+u_n(\Omega^-)\|_{\sigma_n}\|u_n(\Omega^+)-u_n(\Omega^-)\|_{\sigma_n}\\
&\le c(\epsilon^++\epsilon^-)|\epsilon^+-\epsilon^-|=c|(\epsilon^+)^2-(\epsilon^-)^2|=\tilde{c}|\Omega^+-\Omega^-|
\end{aligned}
$$

that the first estimate is valid.

According to the definition, $h_{\Omega^\pm,u_{n-1}(\Omega^\pm)}(z) = L_2^\pm(L_4^\pm)^{-1}L_3^\pm$, where

$$\begin{aligned}
L_2^\pm &= X_1F'_{n,\Omega^\pm}X_2 = -3X_1\langle u_{n-1}(\Omega^\pm)^2\rangle X_2,\\
L_3^\pm &= X_2F'_{n,\Omega^\pm}X_1 = -3X_2\langle u_{n-1}(\Omega^\pm)^2\rangle X_1,\\
L_4^\pm &= X_2F'_{n,\Omega^\pm}X_2 = X_2J_\Omega X_2 - 3X_2\langle u_{n-1}(\Omega^\pm)^2\rangle X_2,
\end{aligned}$$

and $X_1 = \{z\}$, $X_2 = B^o(z)$. Therefore

$$\begin{aligned}
&|h_{\Omega^+,u_{n-1}(\Omega^+)}(z) - h_{\Omega^-,u_{n-1}(\Omega^-)}(z)| = \|L_2^+(L_4^+)^{-1}L_3^+ - L_2^-(L_4^-)^{-1}L_3^-\|_{\sigma_{n-1}}\\
\le{}& \|(L_2^+ - L_2^-)(L_4^+)^{-1}L_3^+\|_{\sigma_{n-1}} + \|L_2^-[(L_4^+)^{-1} - (L_4^-)^{-1}]L_3^+\|_{\sigma_{n-1}}\\
&+ \|L_2^-(L_4^-)^{-1}(L_3^+ - L_3^-)\|_{\sigma_{n-1}}.
\end{aligned}$$

It follows from **(u.a(n-1))** and **(u.b(n-1))** that

$$\begin{gathered}
\|L_2^\pm\|_{\sigma_{n-1}} \le c\epsilon(\Omega^\pm)^2, \quad \|L_3^\pm\|_{\sigma_{n-1}} \le c\epsilon(\Omega^\pm)^2,\\
\|L_2^+ - L_2^-\|_{\sigma_{n-1}} \le c|\Omega^+ - \Omega^-|, \quad \|L_3^+ - L_3^-\|_{\sigma_{n-1}} \le c|\Omega^+ - \Omega^-|.
\end{gathered}$$

And, noting that $|J_{\Omega^\pm}(p,p)| \ge d_0$ for any given $p \in B^o(z)$, we have

$$\|(L_4^\pm)^{-1}\|_{\sigma_{n-1}} \le 2d_0^{-1}, \quad \|L_4^+ - L_4^-\|_{\sigma_{n-1}} \le ck^2|(\Omega^+)^2 - (\Omega^-)^2| \le c|\Omega^+ - \Omega^-|.$$

Therefore

$$\begin{aligned}
\|(L_2^+ - L_2^-)(L_4^+)^{-1}L_3^+\|_{\sigma_{n-1}} &\le c\epsilon(\Omega^+)^2|\Omega^+ - \Omega^-|,\\
\|L_2^-(L_4^-)^{-1}(L_3^+ - L_3^-)\|_{\sigma_{n-1}} &\le c\epsilon(\Omega^-)^2|\Omega^+ - \Omega^-|,\\
\|L_2^-[(L_4^+)^{-1} - (L_4^-)^{-1}]L_3^+\|_{\sigma_{n-1}} &= \|L_2^-(L_4^-)^{-1}(L_4^- - L_4^+)(L_4^+)^{-1}L_3^+\|_{\sigma_{n-1}}\\
&\le c\epsilon(\Omega^+)^2\epsilon(\Omega^-)^2k^2|\Omega^+ - \Omega^-|,
\end{aligned}$$

which implies the second estimate in the proposition.

Acknowledgements. The author wishes to express his hearty thanks to Professor Li Tatsien and Professor Qin Tiehu for their patient guidance, invalueable help and warm encouragement.

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