Chin. Ann. of Math.
21B: 3(2000),323-338.

STOCHASTIC LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS WITH RANDOM COEFFICIENTS***

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Abstract

This paper studies a stochastic linear quadratic optimal control problem (LQ problem, for short), for which the coefficients are allowed to be random and the cost functional is allowed to have a negative weight on the square of the control variable. The authors introduce the stochastic Riccati equation for the LQ problem. This is a backward SDE with a complicated nonlinearity and a singularity. The local solvability of such a backward SDE is established, which by no means is obvious. For the case of deterministic coefficients, some further discussions on the Riccati equations have been carried out. Finally, an illustrative example is presented.

Keywords Stochastic LQ problem, Riccati equation, Backward stochastic differential equation, Malliavin calculus
1991 MR Subject Classification 93E20, 49K45, 49N10, 60H10
Chinese Library Classification O231.3, O232, O211.63 **Document Code** A
Article ID 0252-9599(2000)03-0323-16

§1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $w(\cdot)$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $w(\cdot)$, augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$. We consider the following state equation

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dw(t), \quad t \in [\tau, T], \\ x(\tau) = \xi, \end{cases} \tag{1.1}$$

where $\tau \in \mathcal{T}[0,T]$, the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times taking values in $[0,T]$, $\xi \in \mathcal{X}_\tau \stackrel{\Delta}{=} L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$; A, B, C, D are matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes. In the above, $u(\cdot) \in \mathcal{U}[\tau, T] \stackrel{\Delta}{=} L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m)$ is a control process and $x(\cdot)$ is the corresponding state process.

Manuscript received June 25, 1999. Revised January 18, 2000.

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***Project supported by the National Natural Science Foundation of China (No.79790130), the National Distinguished Youth Science Foundation of China (No.19725106), and the Natural Science Foundation of the Ministry of Education of China (No.97024607).

Clearly, for any $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau, T]$, there exists a unique (strong) solution $x(\cdot) \in L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^n)$ to (1.1). Thus, we can define a cost functional as follows:

$$J(\tau, \xi; u(\cdot)) = E\Big\{ \int_\tau^T \big[\langle Q(t)x(t), x(t)\rangle + \langle R(t)u(t), u(t)\rangle\big] dt + \langle Gx(T), x(T)\rangle \big| \mathcal{F}_\tau \Big\}, \quad (1.2)$$

where $Q(\cdot)$ and $R(\cdot)$ are symmetric matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes and G is a symmetric matrix-valued $\mathcal{F}_T$-measurable bounded random variable. Note that, as in [6], we do not assume the nonnegativity of $R(\cdot)$.

Now, we state the stochastic linear quadratic optimal control problem as follows:

Problem (LQ). *For each $\tau \in \mathcal{T}[0, T]$ and $\xi \in \mathcal{X}_\tau$, find a $\overline{u}(\cdot) \in \mathcal{U}[\tau, T]$, such that*

$$J(\tau, \xi; \overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau, \xi; u(\cdot)) \stackrel{\Delta}{=} V(\tau, \xi), \quad \text{a.s. } \omega \in \Omega. \quad (1.3)$$

Function V is called the value function of Problems (LQ).

In this paper, we are going to continue the study in [6]. We solve Problem (LQ) via solving corresponding Riccati equation. To avoid some unnecessary repetition, we refer the readers to [6] for notations that we omit the definitions here.

Now, we recall the following basic assumption (see [6]).

(S) Let

$$\begin{cases} A, C \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n}), \quad B \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n\times m}), \quad D \in C_{\mathcal{F}}([0, T]; \mathbb{R}^{n\times m}), \\ Q \in C_{\mathcal{F}}([0, T]; \mathcal{S}^n), \quad R \in C_{\mathcal{F}}([0, T]; \mathcal{S}^m), \quad G \in L^\infty_{\mathcal{F}_T}(\Omega; \mathcal{S}^n). \end{cases} \quad (1.4)$$

In addition, we also introduce the following interesting special cases of the above.

(DV) Let

$$\begin{cases} A, C \in L^\infty(0, T; \mathbb{R}^{n\times n}), \quad B \in L^\infty(0, T; \mathbb{R}^{n\times m}), \quad D \in C([0, T]; \mathbb{R}^{n\times m}), \\ Q \in C([0, T]; \mathcal{S}^n), \quad R \in C([0, T]; \mathcal{S}^m), \quad G \in \mathcal{S}^n. \end{cases} \quad (1.5)$$

(DI) Let

$$A, C \in \mathbb{R}^{n\times n}, \quad B, D \in \mathbb{R}^{n\times m}, \quad Q, G \in \mathcal{S}^n, \quad R \in \mathcal{S}^m. \quad (1.6)$$

We see that under (DV) (resp. (DI)), the control system (1.1) is of time-varying (resp. time-invariant) deterministic coefficients and under (S), (1.1) is of random coefficients.

The following definition is copied from [6].

Definition 1.1. *Problem* (LQ) *is said to be* (i) *partially finite at* $(\tau, \xi) \in \Delta[0, T]$ *if*

$$\mathbf{P}(V(\tau, \xi) > -\infty) > 0. \quad (1.7)$$

(ii) (*uniquely*) *partially solvable at* $(\tau, \xi) \in \Delta[0, T]$ *if there exists a* (*unique*) *control* $\overline{u}(\cdot) \in \mathcal{U}[\tau, T]$, *such that*

$$J(\tau, \xi; \overline{u}) = V(\tau, \xi), \qquad \text{a.s. } \omega \in (V(\tau, \xi) > -\infty). \quad (1.8)$$

If one has

$$\mathbf{P}\big(V(\tau, \xi) > -\infty\big) = 1, \quad (1.9)$$

we omit the word "partially" in the above notions.

If for $\tau \in \mathcal{T}[0, T]$, Problem (LQ) is finite (resp. (uniquely) solvable) at all (τ, ξ) with $\xi \in \mathcal{X}_\tau$, we say that Problem (LQ) is finite (resp. (uniquely) solvable) at τ. If Problem (LQ) is finite (resp. (uniquely) solvable) at all $\tau \in \mathcal{T}[0, T]$, we say that Problem (LQ) is finite (resp. (uniquely) solvable).

The following result was proved in [6].

Theorem 1.2. *Let* (S) *hold. Suppose Problem* (LQ) *is partially finite at some* $(\tau, \xi) \in \Delta[0, T)$. *Then*

$$R(T) + D(T)^T G D(T) \geq 0, \qquad \text{a.s. } \omega \in (V(\tau, \xi) > -\infty). \quad (1.10)$$

Moreover, it was shown in [6] by an example that if the equality holds in (1.10), Problem (LQ) might be not finite. We will assume a little stronger assumption than (1.10) later.

We now introduce the Riccati equation corresponding to Problem (LQ):

$$\begin{cases} dP = -\big\{PA + A^TP + C^TPC + \Lambda C + C^T\Lambda + Q - \big(PB + C^TPD + \Lambda D\big) \\ \qquad \times\big(R + D^TPD\big)^{-1}\big(B^TP + D^TPC + D^T\Lambda\big)\big\}dt + \Lambda dw(t), \quad t \in [\tau, T], \\ P(T) = G, \\ \det[R(t) + D(t)^TP(t)D(t)] \neq 0, \quad t \in [\tau, T], \quad \text{a.s. } \omega \in \Omega. \end{cases} \tag{1.11}$$

This is a backward stochastic differential equation (BSDE, for short). An adapted solution of (1.11) is a pair (P, Λ) of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{S}^n$-valued processes. Once an adapted solution (P, Λ) is obtained, one can construct an optimal state feedback control (see §2). Thus, we hope to solve (1.11).

Note that the drift of (1.11) is quadratic in Λ and has a singularity in P. These bring some essential difficulties in proving the existence of an adapted (local) solution. We emphasize that unlike the forward SDEs, the local existence of an adapted solution to BSDE (1.11) is by no means obvious. Some recent efforts devoted to the study of BSDEs with local Lipschitz drift can be found in [11,12] (for one-dimensional case, global solutions) and [15] (for higher dimensional case, local and global solutions). However, those works do not cover the case that we have here, in particular, the singularity in P was not considered. One of main contributions of this paper is the establishment of local existence of an adapted solution to (1.11).

We refer the readers to [8,1,3,4] for finite-dimensional deterministic LQ problems, to [10,13] for infinite-dimensional counterpart, to [16,2,5,17], especially to [6] for stochastic LQ problems. Other relevant results about BSDEs can be found in [14,7,17].

§2. Optimal State Feedback Control

If $(P(\cdot), \Lambda(\cdot))$ is a pair of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, $\mathcal{S}^n$-valued processes satisfying (1.11) for some $\tau \in \mathcal{T}[0, T)$, we call $(P(\cdot), \Lambda(\cdot), \tau)$ a local adapted solution of (1.11). A local adapted solution $(P(\cdot), \Lambda(\cdot), \tau)$ is called a global adapted solution of (1.11) if in (1.11), $[\tau, T]$ is replaced by $(\tau, T]$ with

$$\tau = \sup\{t \leq T \mid \det[R(t) + D(t)^TP(t)D(t)] = 0\}, \tag{2.1}$$

and such a τ happens to be a stopping time. We see that τ is a part of unknowns if we are looking for a global adapted solution of (1.11). From this point of view, (1.11) on $(\tau, T]$ with (2.1) is a free boundary problem for a nonlinear BSDE.

Note that (1.11) is a nonlinear BSDE with the coefficient being random, quadratic in Λ, and having singularities in P (the place where $\det[R + D^TPD] = 0$). The existence of local adapted solutions to such BSDEs is unknown so far. Recall that in [15] several local existence results were obtained for nonlinear BSDEs under some technical conditions, which do not cover the case that we have here. Let us postpone the study of existence of local adapted solutions of (1.11), and first present the following result, which motivates the study of (1.11). See [5] for a related result.

Theorem 2.1. *Let* (S) *hold. Let*

$$\begin{cases} \Omega_+ \stackrel{\Delta}{=} \{R(T) + D(T)^TGD(T) \geq \delta_0\} = \{R(T) + D(T)^TGD(T) \geq 0\} \in \mathcal{F}_\tau, \\ \mathbf{P}(\Omega_+) > 0 \end{cases} \tag{2.2}$$

for some $\delta_0 > 0$ *and* $\tau \in \mathcal{T}[0, T)$. *Suppose* (1.11) *admits an adapted solution* $(P(\cdot), \Lambda(\cdot))$ *on*

$[\tau, T]$ such that

$$\begin{cases} B\Psi, D\Psi \in L^\infty_{\mathcal{F}}(\tau, T; \mathbb{R}^{n\times n}), \\ \Psi \stackrel{\Delta}{=} [R + D^T PD]^{-1}[B^T P + D^T PC + D^T \Lambda]. \end{cases} \tag{2.3}$$

Then Problem (LQ) *is partially solvable at τ with the partially optimal control $\overline{u}(\cdot)$ being of state feedback form*

$$\overline{u}(t) = -\Psi(t)x(t), \qquad t \in [\tau, T], \tag{2.4}$$

$$V(\tau, \xi) = \langle P(\tau)\xi, \xi\rangle, \quad \text{a.s. } \omega \in (V(\tau, \xi) > -\infty) = \Omega_+, \ \forall \xi \in \mathcal{X}_\tau. \tag{2.5}$$

Furthermore, if $\mathbf{P}(\Omega_+) = 1$, then Problem (LQ) *is solvable at τ with the optimal control $\overline{u}(\cdot)$ of form* (2.4)*, and the optimal cost being*

$$V(\tau, \xi) = \langle P(\tau)\xi, \xi\rangle, \qquad \forall \xi \in \mathcal{X}_\tau. \tag{2.6}$$

Proof. By (S) and (2.3), we see that the following SDE

$$\begin{cases} d\overline{x}(t) = \big[A(t) - B(t)\Psi(t)\big]\overline{x}(t)dt + \big[C(t) - D(t)\Psi(t)\big]\overline{x}(t)dw(t), \\ \overline{x}(\tau) = \xi, \end{cases} \tag{2.7}$$

admits a unique strong solution $\overline{x}(\cdot)$ satisfying $E \sup\limits_{\tau\le s\le T} |\overline{x}(t)|^2 \le KE|\xi|^2$, where K is a constant. Then the control $\overline{u}(\cdot)$ defined by (2.4) is admissible, and, by taking such a control, $\overline{x}(\cdot)$ is the corresponding state process. We claim that $\overline{u}(\cdot)$ is a partially optimal control. To this end, we take any $u(\cdot) \in \mathcal{U}[\tau, T]$, and let $x(\cdot)$ be the corresponding state process. Applying Itô's formula to $\langle P(\cdot)x(\cdot), x(\cdot)\rangle$, we have

$$\begin{aligned} &\langle P(T)x(T), x(T)\rangle = \langle P(\tau)\xi, \xi\rangle \\ &+ \int_\tau^T \Big\{\langle [(PB + C^T PD + \Lambda D)(R + D^T PD)^{-1}(B^T P + D^T PC + D^T\Lambda) - Q]x(t), x(t)\rangle \\ &+ 2\langle [B^T P + D^T PC + D^T \Lambda]x(t), u(t)\rangle + \langle D^T PDu(t), u(t)\rangle\Big\}dt + \int_\tau^T [\cdots]dw(t). \end{aligned}$$

Hence, it follows that (note (1.11) and $\Omega_+ \in \mathcal{F}_\tau$)

$$\begin{aligned} &J(\tau, \xi; u(\cdot))I_{\Omega_+} - \langle P(\tau)\xi, \xi\rangle I_{\Omega_+} \\ &= E\Big\{I_{\Omega_+}\Big[\int_\tau^T \big[\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle\big]dt + \langle P(T)x(T), x(T)\rangle - \langle P(\tau)\xi, \xi\rangle\Big]\Big|\mathcal{F}_\tau\Big\} \\ &= E\Big\{I_{\Omega_+}\int_\tau^T \Big[\langle (PB + C^T PD + \Lambda D)(R + D^T PD)^{-1}(B^T P + D^T PC + D^T\Lambda)x(t), x(t)\rangle \\ &\quad + \langle [R + D^T PD]u(t), u(t)\rangle + 2\langle [B^T P + D^T PC + D^T \Lambda]x(t), u(t)\rangle\Big]dt\Big|\,\mathcal{F}_\tau\Big\} \\ &= E\Big\{I_{\Omega_+}\int_\tau^T |[R + D^T PD]^{\frac{1}{2}}\{[R + D^T PD]^{-1}[B^T P + D^T PC + D^T\Lambda]x(t) + u(t)\}|^2 dt\Big|\,\mathcal{F}_\tau\Big\}. \end{aligned}$$

This yields

$$J(\tau, \xi, \overline{u}(\cdot))I_{\Omega_+} = \langle P(\tau)\xi, \xi\rangle I_{\Omega_+} \le J(\tau, \xi; u(\cdot))I_{\Omega_+}, \tag{2.8}$$

which implies that $\Omega_+ \subseteq (V(\tau, \xi) > -\infty)$. By Theorem 1.1, it is necessary that $\Omega_+ = (V(\tau, \xi) > -\infty)$, and $\overline{u}(\cdot)$ is a partially optimal control. The last assertion is clear.

The first part of the above theorem gives a sufficient condition for the partial solvability of Problem (LQ) . This is a unique feature of LQ problems with random coefficients.

Note that (2.2) holds with $\mathbf{P}(\Omega_+) = 1$, for some $\delta_0 > 0$, is equivalent to (1.10).

§3. Local Solvability of the Riccati Equation—the Case D=0

From Theorem 2.1, we see that when (S) and (1.10) hold, provided one can find an adapted solution (P,Λ) to the Riccati equation (1.11) on $[\tau,T]$, an optimal control of state feedback form can be obtained (see (2.4)). We now look at the local solvability of (1.11). Let us start with a special case: $D=0$. Note that in the present case (1.11) becomes

$$\begin{cases} dP=-\big\{PA+A^TP+C^TPC+\Lambda C+C^T\Lambda+Q-PBR^{-1}B^TP\big\}dt+\Lambda dw(t),\ t\in[\tau,T],\\ P(T)=G, \end{cases} \tag{3.1}$$

which is still a nonlinear BSDE. The advantage of this case is that the drift is at most linear in Λ and there is no singularities (in P). This allows us to have the following result, without additional technical conditions.

Theorem 3.1. *Let* (S) *hold. Let* $\delta_0>0$ *and* $t_0\in[0,T)$ *such that*

$$\big|\det R(t)\big|\ge\delta_0,\qquad t\in[t_0,T],\ \text{a.s. }\ \omega\in\Omega. \tag{3.2}$$

Then (3.1) *admits a local adapted solution* (P,Λ,τ) *and* (P,Λ) *is unique on* $[\tau,T]$.

The main idea of proving the above result is to use the contraction mapping theorem. Due to the "backward" nature of the equation (3.1), we need to be careful in taking care of the pair (P,Λ) (not just P, which is much easier). The following Gronwall type inequality involving conditional expectations will be useful.

Lemma 3.1. *Let* $\tau\in\mathcal{T}[0,T)$. *Let* $\varphi(\cdot)\in C_{\mathcal{F}}([\tau,T];\mathbb{R})$ *be nonnegative,* $f(\cdot)\in L^1_{\mathcal{F}}(\tau,T;\mathbb{R})$, $g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$, *and* $K_0>0$, *such that*

$$\varphi(s\vee\tau)\le E\Big\{K_0\int_{s\vee\tau}^T\varphi(r)dr+\int_{s\vee\tau}^T f(r)dr+g\Big|\,\mathcal{F}_{s\vee\tau}\Big\},\quad\forall s\in[0,T]. \tag{3.3}$$

Then

$$\varphi(s\vee\tau)\le E\Big\{e^{K_0(T-s\vee\tau)}g+\int_{s\vee\tau}^T e^{K_0(r-s\vee\tau)}f(r)dr\Big|\,\mathcal{F}_{s\vee\tau}\Big\},\quad\forall s\in[0,T]. \tag{3.4}$$

Proof. Let $(\overline{\varphi},\overline{\psi})$ be the adapted solution of the following linear BSDE

$$\begin{cases} d\overline{\varphi}(t)=-\big[K_0\overline{\varphi}(t)+f(t)\big]dt+\overline{\psi}(t)dw(t),\\ \overline{\varphi}(T)=g. \end{cases} \tag{3.5}$$

Then $\widehat{\varphi}(t)\stackrel{\Delta}{=}\varphi(t)-\overline{\varphi}(t)$ satisfies

$$E\big\{\widehat{\varphi}(s\vee\tau)\big|\mathcal{F}_{s\vee\tau}\big\}\le E\Big\{K_0\int_{s\vee\tau}^T\widehat{\varphi}(r)dr\Big|\,\mathcal{F}_{s\vee\tau}\Big\},\quad\text{a.s. }\ \omega\in\Omega. \tag{3.6}$$

Thus, for any $F\in\mathcal{F}_{s\vee\tau}$, we have

$$\begin{aligned}\int_F\widehat{\varphi}(s\vee\tau)d\mathbf{P}&\le K_0\int_F\int_{s\vee\tau}^T\widehat{\varphi}(r)drd\mathbf{P}\\ &\le K_0\int_{s\vee\tau}^T\Big[\int_F\widehat{\varphi}(r)d\mathbf{P}\Big]dr,\quad\forall s\in[0,T],\ \text{a.s. }\ \omega\in\Omega.\end{aligned} \tag{3.7}$$

Here, we have used the usual Fubini's Thereom, noting that all the integrals involved in the above are usual ones (not Itô's type integral). Now, by the proof of usual Gronwall's inequality, we see easily that

$$\int_F\widehat{\varphi}(s\vee\tau)d\mathbf{P}\le0,\qquad\forall s\in[0,T],\ \text{a.s. }\ \omega\in\Omega,\ F\in\mathcal{F}_{s\vee\tau}. \tag{3.8}$$

Consequently

$$\varphi(s\vee\tau)\le\overline{\varphi}(s\vee\tau)=E\Big\{e^{K_0(T-s\vee\tau)}g+\int_{s\vee\tau}^{T}e^{K_0(r-s\vee\tau)}f(r)dr\Big|\mathcal{F}_{s\vee\tau}\Big\},$$

which proves (3.4).

Note that in the above lemma, $f(\cdot)$ and g are not necessarily to be nonnegative.

Proof of Theorem 3.1. Take any $p(\cdot)\in L^\infty_{\mathcal{F}}(\tau,T;\mathcal{S}^n)$. We solve the following linear nonhomogenuous BSDE

$$\begin{cases} dP=-\big\{PA+A^TP+C^TPC+\Lambda C+C^T\Lambda+Q-pBR^{-1}B^Tp\big\}dt\\ \qquad +\Lambda dw(t),\quad t\in[t_0,T],\\ P(T)=G.\end{cases}\tag{3.9}$$

By [14], (3.9) admits a unique adapted solution (P,Λ) over $[t_0,T]$. By Itô's formula, we have (note (S))

$$\begin{aligned}&E\Big\{|P(s\vee t_0)|^2+\int_{s\vee t_0}^{T}|\Lambda(r)|^2dr\Big|\mathcal{F}_{s\vee t_0}\Big\}\\ &=E\Big\{|G|^2-2\int_{s\vee t_0}^{T}\mathrm{tr}\,\big\{P\big[PA+A^TP+C^TPC+\Lambda C\\ &\quad+C^T\Lambda+Q-pBR^{-1}B^Tp\big]\big\}dr\Big|\mathcal{F}_{s\vee t_0}\Big\}\\ &\le K_0E\Big\{1+\int_{s\vee t_0}^{T}\big\{|P(r)|^2+|p(r)|^4\big\}dr\Big|\mathcal{F}_{s\vee t_0}\Big\}.\end{aligned}\tag{3.10}$$

It follows from Lemma 3.1 that

$$E\Big\{|P(s\vee t_0)|^2+\int_{s\vee t_0}^{T}|\Lambda(r)|^2dr\Big|\mathcal{F}_{s\vee t_0}\Big\}\le KE\Big\{1+\int_{s\vee t_0}^{T}|p(r)|^4dr\Big|\mathcal{F}_{s\vee t_0}\Big\},$$
$$\forall 0\le s\le T,\ \text{a.s. }\ \omega\in\Omega\tag{3.11}$$

for some constant K (depending only on the bounds of A,B,C,Q,R,R^{-1},G and T). In particular,

$$|P(t)|^2\le KE\Big\{1+\int_t^T|p(r)|^4dr\Big|\mathcal{F}_t\Big\},\qquad\forall t_0\le t\le T,\ \text{a.s. }\ \omega\in\Omega.\tag{3.12}$$

Consequently, for any $0<\varepsilon<T-t_0$,

$$\operatorname*{esssup}_{\substack{T-\varepsilon\le t\le T\\ \omega\in\Omega}}|P(t,\omega)|^2\le K+K\operatorname*{esssup}_{\substack{T-\varepsilon\le t\le T\\ \omega\in\Omega}}|p(t,\omega)|^4\varepsilon.\tag{3.13}$$

Now, we choose $\varepsilon>0$ small enough such that $K+K(K+1)^4\varepsilon\le(K+1)^2$, and let $\mathcal{B}$ be the closed ball of radius $K+1$ centered at 0 in $L^\infty_{\mathcal{F}}(\tau,T;\mathcal{S}^n)$. Then the map $p\mapsto P$ is from $\mathcal{B}$ to itself. Using a similar argument as above, we can show that this map is contractive (we might need to properly shrink $\varepsilon>0$ a little further). Thus, by Contraction Mapping Theorem, we obtain a unique fixed point P, which together with the corresponding Λ and $\tau\stackrel{\Delta}{=}T-\varepsilon$ gives a local adapted solution. It is clear that for the obtained τ, (P,Λ) is unique on $[\tau,T]$.

§4. Local Solvability of the Riccati Equation—the General Case

Now, we look at the case that D is not necessarily 0. The essential difficulty is that the drift term in (1.11) not only depends on Λ quadratically, but also contains some singularities in P. Thus, we need to bound Λ from above and bound $(R+D^TPD)$ from below. In order to bound Λ, we need, roughly speaking, to "differentiate" the equation to obtain an equation for Λ. To achieve this, we need some results involving Malliavin calculus. Let us make some preliminaries first.

Let Ξ be the set of all (scalar) random variables ξ of form

$$\xi = f\Big(\int_0^T h(s)dw(s)\Big), \tag{4.1}$$

where $f \in C_b^1(\mathbb{R}^k)$, $h(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^k)$. For any $\xi \in \Xi$ of form (4.1), we define

$$\mathcal{D}_\theta \xi = \langle f_x\Big(\int_0^T h(s)dw(s)\Big), h(\theta)\rangle, \qquad 0 \le \theta \le T, \tag{4.2}$$

called the Malliavin derivative of ξ. Next, for any $\xi \in \Xi$, we define

$$\|\xi\|_{1,q} \stackrel{\Delta}{=} \begin{cases} \Big\{E\Big(|\xi|^q + \int_0^T |\mathcal{D}_t\xi|^q dt\Big)\Big\}^{1/q}, & 1 \le q < \infty, \\ \operatorname*{esssup}_{[0,T]\times\Omega}\big\{|\xi| + |\mathcal{D}_t\xi|\big\}, & q = \infty, \end{cases} \tag{4.3}$$

and let $\mathbb{D}_{1,q}$ be the completion of Ξ under the norm $\|\cdot\|_{1,q}$ defined by (4.3). Clearly, operator $\mathcal{D}$ admits a closed extension on $\mathbb{D}_{1,q}$. It is important that for given $t \in [0,T)$,

$$\xi \text{ is } \mathcal{F}_t\text{-measurable} \quad \Rightarrow \quad \mathcal{D}_\theta\xi = 0, \quad \forall \theta \in (t,T].$$

We let $\mathbb{D}^n_{1,q}$ and $\mathbb{D}^{m\times n}_{1,q}$ be the set of all $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$-valued random variables with each component belonging to $\mathbb{D}_{1,q}$, respectively. For any k-dimensional random vector η and any random field f defined on $[0,T]\times\mathbb{R}^k\times\Omega$, we will distinguish $\mathcal{D}_\cdot\big(f(t,\eta(\omega),\omega)\big)$ (the Malliavin derivative of the stochastic process $f(t,\eta(t)))$ from $\mathcal{D}_\cdot f(t,\eta(\omega),\omega) \stackrel{\Delta}{=} [\mathcal{D}_\cdot f](t,x,\omega)\big|_{x=\eta(\omega)}$. We let $L^q_{\mathcal{F}}(\tau,T;\mathbb{D}_{1,q})$ be the set of all $\{\mathcal{F}_t\}_{t\ge 0}$-adapted processes y defined on $[\tau,T]$, extending to be zero on $[0,\tau)$, such that the following is finite:

$$\|y\|_{L^q_{\mathcal{F}}(\tau,T;\mathbb{D}_{1,q})} \stackrel{\Delta}{=} \begin{cases} \Big(\int_0^T \|y(t)\|^q_{\mathbb{D}_{1,q}} dt\Big)^{1/q}, & 1 \le q < \infty, \\ \operatorname*{esssup}_{[0,T]} \|y(t)\|_{\mathbb{D}_{1,\infty}}, & q = \infty. \end{cases} \tag{4.4}$$

We can accordingly define $L^q_{\mathcal{F}}(\tau,T;\mathbb{D}^{n\times m}_{1,q})$, etc. In what follows, we will only use the case $q = 2, \infty$.

The following additional assumption will be crucial for our local existence of adapted solution (P,Λ,τ) to (1.11).

(SS) Let

$$\begin{cases} A, C, Q \in L^\infty_{\mathcal{F}}(0,T;\mathbb{D}^{n\times n}_{1,\infty}), & B, D \in L^\infty_{\mathcal{F}}(0,T;\mathbb{D}^{n\times m}_{1,\infty}), \\ R \in L^\infty_{\mathcal{F}}(0,T;\mathbb{D}^{m\times m}_{1,\infty}), & G \in \mathbb{D}^{n\times n}_{1,\infty}. \end{cases} \tag{4.5}$$

Moreover, let $(\xi(\cdot),\zeta(\cdot))$ be the unique adapted solution of BSDE:

$$d\xi(t) = \zeta(t)dw(t), \quad \xi(T) = G. \tag{4.6}$$

Suppose the following holds:

$$|\det[R(t) + D(t)^T\xi(t)D(t)]| \ge \delta_0, \quad t \in [t_0,T], \text{ a.s. } \omega \in \Omega \tag{4.7}$$

for some $\delta_0 > 0$, $t_0 \in [0,T)$.

The above assumption imposes some "smoothness" conditions on the coefficients in ω. Also, (4.17) is stronger than (1.10). Note that when (DV) or (DI) holds, (SS) holds automatically.

Theorem 4.1. *Let* (S) *and* (SS) *hold. Then* (1.11) *admits a local adapted solution* (P,Λ,τ)*, and on* $[\tau,T]$*,* (P,Λ) *is unique. Moreover*

$$P \in L^\infty_{\mathcal{F}}(\tau,T;\mathbb{D}^{n\times n}_{1,\infty}), \quad \Lambda \in L^2_{\mathcal{F}}(\tau,T;\mathbb{D}^{n\times n}_{1,2}), \tag{4.8}$$

$$|P(t)| + |\mathcal{D}_\theta P(t)| + E\Big\{ \int_t^T \big[|\Lambda(r)|^2 + |\mathcal{D}_t \Lambda(r)|^2\big] dr \Big| \mathcal{F}_t \Big\} \le K, \quad \forall t, \theta \in [\tau, T], \text{ a.s. } \omega \in \Omega \tag{4.9}$$

for some constant $K > 0$, and $\mathcal{D}_t P(t)$ is a version of $\Lambda(t)$.

Proof. The main idea is to use the contraction mapping theorem. Thus, let us first introduce a Banach space $X[\tau, T]$, which is a subspace of $L^\infty_{\mathcal{F}}(\tau, T; \mathcal{S}^n)^2$, whose norm is given by

$$\|(p,l)\|_{X[\tau,T]} \overset{\Delta}{=} \operatorname*{esssup}_{\substack{t,s\in[\tau,T]\\ \omega\in\Omega}} \Big\{ |p(s)| + |l(s)| + |\mathcal{D}_t p(s)| + \Big[E\Big(\int_t^T |\mathcal{D}_t l(r)|^2 dr \Big| \mathcal{F}_t \Big) \Big]^{1/2} \Big\}, \quad \forall (p,l) \in X[\tau, T], \tag{4.10}$$

where $\tau \in \mathcal{T}[T-\varepsilon, T)$ undetermined with $0 < \varepsilon < T - t_0$ small. Next, for any $K \ge 1$, $0 < \delta \le 1$ we let $\mathcal{B}_{K,\delta}$ be the closed subset in $X[\tau, T]$ which consists of all the pairs (p, l) such that

$$\|(p,l)\|_{X[\tau,T]} \le K, \tag{4.11}$$

$$|\det[R(t) + D(t)^T p(t) D(t)]| \ge \delta, \qquad \forall t \in [\tau, T], \text{ a.s.} \tag{4.12}$$

Note that the set $\mathcal{B}_{K,\delta}$ is a kind of "annulus" in $X[\tau, T]$. Now, for any $(p, l) \in \mathcal{B}_{K,\delta}$, we consider the following BSDE

$$\begin{cases} dP = -\Theta(t, p(t), \lambda(t))dt + \Lambda dw(t), & t \in [\tau, T], \\ P(T) = G, \end{cases} \tag{4.13}$$

where

$$\Theta(t,p,l) = pA + A^T p + C^T pC + lC + C^T l + Q - \big(pB + C^T pD + lD\big) \times \big(R + D^T pD\big)^{-1} \big(B^T p + D^T pC + D^T l\big). \tag{4.14}$$

By [14], there exists a unique adapted solution $(P(\cdot), \Lambda(\cdot))$ to (4.13). We want to show that for a suitable choice of τ, K and δ, $(P, \Lambda) \in X[\tau, T]$. Then we have defined a solution map $(p, l) \mapsto (P, \Lambda)$, from $\mathcal{B}_{K,\delta}$ to itself. Further, a similar argument will show that this map is contractive and thus admits a unique fixed point, which gives the unique local adapted solution.

We now split the proof into several steps.

Step 1. For suitable $\tau \in \mathcal{T}[T-\varepsilon, T)$, $K \ge 1$ and $0 < \delta \le 1$, (4.12) holds for (P, Λ).

Applying Itô's formula to $|P(\cdot) - \xi(\cdot)|^2$ (recall (4.6)), we have ($\forall s \in [0, T]$)

$$\begin{aligned} &|P(s\vee\tau) - Y(s\vee\tau)|^2 \\ &\le \int_{s\vee\tau}^T |P(r) - Y(r)|^2 dr + \int_{s\vee\tau}^T \big\{ |\Theta(r, p(r), l(r))|^2 - |\Lambda(r) - Z(r)|^2 \big\} dr \\ &\quad - 2\int_{s\vee\tau}^T \operatorname{tr}\big\{ \big[P(r) - Y(r)\big]\big[\Lambda(r) - Z(r)\big] \big\} dw(r). \end{aligned}$$

Thus, by Lemma 3.1, we obtain

$$\begin{aligned} &|P(t\vee\tau) - Y(t\vee\tau)|^2 + E\Big(\int_{t\vee\tau}^T |\Lambda(r) - Z(r)|^2 dr \big| \mathcal{F}_{t\vee\tau} \Big) \\ &\le E\Big\{ \int_{t\vee\tau}^T e^{r - t\vee\tau} |\Theta(r, p(r), l(r))|^2 dr \big| \mathcal{F}_{t\vee\tau} \Big\} \\ &\le K_0\Big(1 + \frac{K^4}{\delta^2}\Big)(T - t\vee\tau), \qquad t \in [0, T], \text{ a.s. } \omega \in \Omega. \end{aligned} \tag{4.15}$$

Hereafter, $K_0 > 0$ represents a generic constant (independent of K and δ, and thus uniform in $(p,l) \in \mathcal{B}_{K,\delta}$), which can be different at different places. Now, we observe the following

$$\begin{aligned}
&\big|R(t) + D(t)^T \xi(t) D(t) - R(t) - D(t)^T P(t) D(t)\big| \\
&\le |D(t)|^2 |\xi(t) - P(t)| \\
&\le K_0 \Big(1 + \frac{K^4}{\delta^2}\Big)(T-t) \stackrel{\Delta}{=} \eta(t), \quad \forall t \in [\tau, T], \text{ a.s. } \omega \in \Omega, \qquad (4.16)
\end{aligned}$$

where K_0 is an absolute constant. Thus, for $0 < \varepsilon < T - t_0$ small, by defining $\tau = T - \varepsilon$, we have

$$\eta(t) \le \frac{\delta_0}{2} \stackrel{\Delta}{=} \delta, \quad \forall t \in [\tau, T], \quad \text{a.s. } \omega \in \Omega,$$

which leads to (4.11). It is important to note that this τ only depends on K, δ and is independent of particular (p, l).

Step 2. For suitable choice of $\tau \in \mathcal{T}[T-\varepsilon, T)$, $K \ge 1$ and $0 < \delta \le 1$, $(P, \Lambda) \in \mathcal{B}_{K,\delta}$.

Let $\tau \in \mathcal{T}[T-\varepsilon, T]$ for some small $\varepsilon > 0$. Applying Itô's formula to $|P(\cdot)|^2$, we have

$$\begin{aligned}
|P(s \vee \tau)|^2 \le |G|^2 + \frac{1}{\mu} \int_{s\vee\tau}^T |P(r)|^2 dr + \int_{s\vee\tau}^T \big\{\mu |\Psi(r, p(r), l(r))|^2 - |\Lambda(r)|^2\big\} ds \\
- 2 \int_{s\vee\tau}^T \operatorname{tr}\big[P(r)\Lambda(r)\big] dw(r) \qquad (4.17)
\end{aligned}$$

for any $\mu \in (0, 1]$. Thus, by Lemma 3.1, we obtain

$$\begin{aligned}
&|P(t)|^2 + E\Big(\int_t^T |\Lambda(r)|^2 dr \Big| \mathcal{F}_t\Big) \le E\Big\{e^{\frac{T-t}{\mu}} |G|^2 + \mu \int_t^T e^{\frac{r-t}{\mu}} |\Theta(r, p(r), l(r))|^2 dr \Big| \mathcal{F}_t\Big\} \\
&\le K_0 e^{\frac{\varepsilon}{\mu}} \Big\{1 + \mu\Big(1 + \frac{K^4}{\delta^2}\Big)\varepsilon\Big\}, \qquad t \in [\tau, T], \text{ a.s. } \omega \in \Omega. \qquad (4.18)
\end{aligned}$$

Next, $(p, l) \in \mathcal{B}_{K,\delta}$, together with (SS), implies that $\Theta(\cdot, p(\cdot), l(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{D}^{n\times n}_{1,2})$ with

$$\begin{aligned}
|\mathcal{D}_\theta \Theta(r, p(r), l(r))| &\le K_0 \big[1 + |p(r)| + |l(r)|\big] + \frac{K_0}{\delta}\big[1 + |p(r)| + |l(r)|\big]\big[|\mathcal{D}_\theta p(r)| + |\mathcal{D}_\theta l(r)|\big] \\
&\le K_0(1 + K) + \frac{K_0}{\delta}(1 + K)\big[|\mathcal{D}_\theta p(r)| + |\mathcal{D}_\theta l(r)|\big]. \qquad (4.19)
\end{aligned}$$

Then by [7], $(P, \Lambda) \in L^2_{\mathcal{F}}(\tau, T; \mathbb{D}^{n\times n}_{1,2})$, and a version of it satisfies

$$\begin{cases} \mathcal{D}_\theta P(t) = \mathcal{D}_\theta \Lambda(t) = 0, \quad \tau \le \theta < t \le T; \\ \mathcal{D}_\theta P(t) = \mathcal{D}_\theta G - \int_t^T \mathcal{D}_\theta \Theta(s, p(s), \lambda(s)) ds - \int_t^T \mathcal{D}_\theta \Lambda dw(s), \quad \tau \le \theta \le t \le T. \end{cases} \qquad (4.20)$$

Moreover, $\{\mathcal{D}_t P(t) : \tau \le t \le T\}$ is a version of $\{\Lambda(t) : \tau \le t \le T\}$.

Similarly to (4.17)–(4.18), we have

$$\begin{aligned}
&|\mathcal{D}_\theta P(t)|^2 + E\Big(\int_t^T |\mathcal{D}_\theta \Lambda(r)|^2 dr \Big| \mathcal{F}_t\Big) \\
&\le E\Big\{e^{\frac{T-t}{\mu}} |\mathcal{D}_\theta G|^2 + \mu \int_t^T e^{\frac{r-t}{\mu}} |\mathcal{D}_\theta \Theta(r, p(r), l(r))|^2 dr \Big| \mathcal{F}_t\Big\} \\
&\le e^{\frac{\varepsilon}{\mu}} K_0 + \mu e^{\frac{\varepsilon}{\mu}} \varepsilon K_0 (1 + K^2) + \mu e^{\frac{\varepsilon}{\mu}} \frac{K_0}{\delta^2}(1 + K^2) E\Big(\int_t^T |\mathcal{D}_\theta p(r)|^2 + |\mathcal{D}_\theta l(r)|^2 dr \Big| \mathcal{F}_t\Big) \\
&\le \overline{K}_0 e^{\frac{\varepsilon}{\mu}} \Big\{1 + \mu\varepsilon(1 + K^2) + \frac{\mu}{\delta^2}(1 + K^2)K^2\Big\}, \quad \tau \le \theta \le t \le T, \text{ a.s.} \qquad (4.21)
\end{aligned}$$

Adding (4.18) to (4.21), and taking $\mu = \sqrt{\varepsilon}$, we obtain

$$\begin{aligned}&|P(t)|^2 + |\mathcal{D}_\theta P(t)|^2 + E\Big(\int_t^T |\Lambda(r)|^2 dr\Big|\mathcal{F}_t\Big) + E\Big(\int_t^T |\mathcal{D}_\theta \Lambda(r)|^2 dr\Big|\mathcal{F}_t\Big)\\ &\le K_0 e^{\sqrt{\varepsilon}}\Big\{1 + \varepsilon^{3/2}\Big(1 + \frac{K^4}{\delta^2}\Big) + \frac{\sqrt{\varepsilon}}{\delta^2}(1+K^2)K^2\Big\}\\ &\le \overline{K}_0\Big\{1 + \sqrt{\varepsilon}\Big(1 + \frac{K^4}{\delta^2}\Big)\Big\}. \end{aligned}\tag{4.22}$$

We note that $\overline{K}_0$ is an absolute constant. Now, let us take $K = \overline{K}_0 + 1$. We may choose $\varepsilon > 0$ small enough so that

$$\overline{K}_0\Big\{1 + \sqrt{\varepsilon}\Big(1 + \frac{K^4}{\delta^2}\Big)\Big\} \le (\overline{K}_0 + 1)^2. \tag{4.23}$$

For such a choice of K, recalling the fact that $\mathcal{D}_t P(t)$ is a version of $\Lambda(t)$, we see that $(P, \Lambda) \in \mathcal{B}_{K,\delta}$.

Step 3. Completion of the proof. Now, for any $(p, l), (\bar{p}, \bar{l}) \in \mathcal{B}_{K,\delta}$, let (P, Λ) and $(\bar{P}, \bar{\Lambda})$ be the corresponding adapted solutions of (4.13). Then by a similar argument used above, we can prove that

$$\|(P, \Lambda) - (\bar{P}, \bar{\Lambda})\|_{X[\tau,T]} \le \alpha \|(p, l) - (\bar{p}, \bar{l})\|_{X[\tau,T]} \tag{4.24}$$

for some $\alpha \in (0, 1)$. Here, we probably need to shrink $\varepsilon > 0$ a little further. By the Contraction Mapping Theorem, we obtain a unique fixed point of the map $(p, l) \mapsto (P, \Lambda)$, which gives a local adapted solution to (1.11) having the properties stated in the theorem.

Corollary 4.1. *Let* (S) *hold. Let either* $D = 0$ *and* (3.2) *hold, or* (SS) *hold. Then there exists a* $\tau \in \mathcal{T}[0, T)$*, such that Problem* (LQ) *is solvable at* τ*.*

To conclude this section, let us point out the following. The proof of Theorem 4.1 can be modified so that it applies to the local solvability of more general nonlinear BSDEs of form

$$\begin{cases} dY(t) = b(t, Y(t), Z(t))dt + Z(t)dw(t), & t \in [\tau, T],\\ Y(T) = \eta, \end{cases} \tag{4.25}$$

where $b(t, y, z) : [0, T] \times \mathcal{O} \times \mathbb{R}^n \to \mathbb{R}^n$, with $\mathcal{O} \subseteq \mathbb{R}^n$ being a domain, is allowed to have any order of growth in (y, z). Since $\mathcal{O}$ is only a domain in $\mathbb{R}^n$, b is allowed to have very bad behavior as $y \to \partial\mathcal{O}$. What we need to assume is the following

(N) For any $(y, z) \in \mathcal{O} \times \mathbb{R}^n$, $b(t, y, z)$ is $\{\mathcal{F}_t\}_{t\ge 0}$-progressively measurable and continuous, $[\mathcal{D}_\theta b](t, y, z)$, $b_y(t, y, z)$, $b_z(t, y, z)$, $b_{yy}(t, y, z)$, $b_{yz}(t, y, z)$, $b_{zz}(t, y, z)$, $[\mathcal{D}_\theta b_y](t, y, z)$ and $[\mathcal{D}_\theta b_z](t, y, z)$ are continuous. Moreover, for any compact set $F \subseteq \mathcal{O} \times \mathbb{R}^n$, there exists a constant $K > 0$, such that

$$\begin{aligned}&|b(t, y, z)| + |[\mathcal{D}_\theta b](t, y, z)| + |b_y(t, y, z)| + |b_z(t, y, z)| + |b_{yy}(t, y, z)|\\ &+ |b_{yz}(t, y, z)| + |b_{zz}(t, y, z)| + |[\mathcal{D}_\theta b_y](t, y, z)| + |[\mathcal{D}_\theta b_z](t, y, z)| \le K,\\ &\forall t, \theta \in [0, T],\ (y, z) \in F,\ \text{a.s. } \omega \in \Omega.\end{aligned} \tag{4.26}$$

Proposition 4.1. *Let* (N) *hold. Let* $\eta \in \mathbb{D}^n_{1,\infty}$ *such that*

$$d(\xi(t), \partial\mathcal{O}) \stackrel{\Delta}{=} \inf_{y \in \partial\mathcal{O}} |\xi(t) - y| \ge \delta, \qquad \text{a.s. } \omega \in \Omega,\ \forall t \in [t_0, T] \tag{4.27}$$

for some $\delta > 0$ *and* $t_0 \in [0, T)$*, where* $(\xi(\cdot), z(\cdot))$ *is the adapted solution of the following BSDE*

$$d\xi(t) = \zeta(t)dw(t), \quad \xi(T) = \eta. \tag{4.28}$$

Then (4.25) *admits a local solution* $(Y(\cdot), Z(\cdot), \tau)$*, with* $\tau \in \mathcal{T}[0, T)$*, and on* $[\tau, T]$*,* $(Y(\cdot), Z(\cdot))$ *is unique.*

We omit the proof here.

§5. Deterministic Coefficient Case

In this section, we consider the case of deterministic coefficient. For the simplicity of presentation, we will assume (DI) (The situation under (DV) can be discussed in a similar fashion). We introduce the following terminal value problem of differential equation for matrix-valued function $P(\cdot)$:

$$\begin{cases} \dot{P}(t) = -P(t)A - A^T P(t) - Q \\ \qquad +[P(t)B + C^T P(t)D][R + D^T P(t)D]^{-1}[B^T P(t) + D^T P(t)C], \quad t \in [\tau, T], \\ P(T) = G, \end{cases} \tag{5.1}$$

where $\tau \in [0, T)$. In the present case, (4.25) can be replaced by

$$R + D^T GD > 0. \tag{5.2}$$

Then one can always find some $\tau \in [0, T)$, such that (5.1) admits a solution $P(\cdot)$ satisfying

$$R + D^T P(t)D > 0, \qquad t \in [\tau, T]. \tag{5.3}$$

Clearly, $(P(\cdot), 0, \tau)$ is also a local adapted solution of (1.10). On the other hand, if (DI) holds, then both (S) and (SS) hold. Thus, by Theorem 4.1, $(P(\cdot), 0)$ is the unique local adapted solution of (1.11) on $[\tau, T]$. Hence, (5.1) actually coincides with (1.11) if condition (DI) is assumed. Consequently, we have the following result.

Proposition 5.1. *Suppose for $\tau \in [0, T)$, there exists a solution $P(\cdot) : [\tau, T] \to \mathcal{S}^n$ to* (5.1) *satisfying* (5.3). *Then Problem* (LQ) *is solvable at τ with the optimal control $\overline{u}(\cdot)$ being of state feedback form*

$$\overline{u}(t) = -[R + D^T P(t)D]^{-1}[B^T P(t) + D^T P(t)C]x(t), \qquad t \in [\tau, T], \tag{5.4}$$

and (2.9) *holds.*

From the above result, we see that the interval on which the Riccati equation (5.1) admits a solution $P(\cdot)$ satisfying Condition (5.3) is closely related to the solvability of our LQ problems. Let us now define (see [6])

$$\begin{cases} I_R \stackrel{\Delta}{=} \{\tau \in [0, T) \bigm| \text{(5.1) admits a solution } P(\cdot) \text{ satisfying (5.3)}\}, \\ I_S \stackrel{\Delta}{=} \{\tau \in [0, T) \bigm| \forall t \in [\tau, T], \xi \in \mathcal{X}_t,\ \exists \overline{u} \in \mathcal{U}[t, T],\ V(t, \xi) = J(t, \xi; \overline{u}(\cdot))\}, \\ I_F \stackrel{\Delta}{=} \{\tau \in [0, T) \bigm| \forall t \in [\tau, T], \xi \in \mathcal{X}_t,\ V(\tau, \xi) > -\infty, \quad \text{a.s.} \ \}. \end{cases} \tag{5.5}$$

By Proposition 5.1, we see that $I_R \subseteq I_S \subseteq I_F$. Now, we define

$$\sigma = \inf I_R. \tag{5.6}$$

Thus, $(\sigma, T]$ is the maximum interval on which (5.1) admits a solution $P(\cdot)$ such that (5.3) holds on $(\sigma, T]$. From the definition of σ, and Proposition 5.1, we see that for any $\tau \in (\sigma, T]$, Problem (LQ) is solvable at τ. The following result is found in [6].

Theorem 5.1. *Let* (S) *and* (5.2) *hold. Then the following are equivalent*:

(i) $\sigma \in I_F$.

(ii) *There exists a sequence $\tau_k \downarrow \sigma$ and $P_\sigma \in \mathcal{S}^n$, such that*

$$\lim_{k \to \infty} P(\tau_k) = P_\sigma. \tag{5.7}$$

In this case, it is necessary that

$$V(\sigma, \xi) = \langle P_\sigma \xi, \xi \rangle, \qquad \text{a.s.} \ \ \omega \in \Omega,\ \forall \in \mathcal{X}_\sigma. \tag{5.8}$$

(iii) *There exists a $P_\sigma \in \mathcal{S}^n$, such that*

$$\lim_{\tau \downarrow \sigma} P(\tau) = P_\sigma. \tag{5.9}$$

From (5.1), as well as Theorem 5.1, we see that if $\sigma \in I_F$, which implies that $P(\tau)$ stays bounded as $\tau\downarrow\sigma$, it is necessary that

$$\lim_{\tau\downarrow\sigma}\det\left[R + D^T P(\tau)D\right] = 0. \tag{5.10}$$

We now would like to look at when $\sigma \in I_S$.

Theorem 5.2. *Let* (S), (5.2) *and* (5.9) *hold. Then* $\sigma \in I_S$ *if and only if*

$$\Psi \stackrel{\Delta}{=} [R + D^T PD]^{-1}[B^T P + D^T PC] \in L^2(\sigma, T; \mathbb{R}^{m\times n}). \tag{5.11}$$

Proof. $\Leftarrow$: From (5.11), we know that for any $\xi \in \mathcal{X}_\sigma$, the following SDE

$$\begin{cases} dx(t) = [A - B\Psi(t)]x(t)dt + [C - D\Psi(t)]x(t)dw(s), & t \in [\sigma, T],\\ x(\sigma) = \xi \end{cases}$$

admits a unique strong solution $x(\cdot)$ satisfying

$$E\big\{|x(t)|^k\big|\mathcal{F}_\sigma\big\} \le K_0(1 + |\xi|^k), \qquad \forall t \in [\sigma, T],\ \text{a.s. } \omega \in \Omega,\ k \ge 1. \tag{5.12}$$

Since $\Psi(\cdot)$ is a deterministic function, we then obtain

$$\overline{u}(\cdot) \stackrel{\Delta}{=} -\Psi(\cdot)x(\cdot) \in \mathcal{U}[\tau, T].$$

Clearly $x(\cdot)$ is the state process of the system under control $\overline{u}(\cdot)$. Hence, similarly to the proof of Theorem 5.1, we obtain

$$J(\sigma, \xi; u(\cdot)) \ge J(\sigma, \xi; \overline{u}(\cdot)) = \langle P_\sigma \xi, \xi\rangle,$$

proving $\sigma \in I_S$.

$\Rightarrow$: For any $\xi \in \mathcal{X}_\sigma$, let $(\overline{x}(\cdot), \overline{u}(\cdot))$ be an optimal pair of Problem (LQ) at (σ, ξ). Then, by dynamic programming principle, $\overline{u}\big|_{[\tau,T]} \in \mathcal{U}[\tau, T]$ is an optimal control of Problem (LQ) at $(\tau, \overline{x}(\tau))$, for any $\tau \in (\sigma, T]$. Using Proposition 5.1, we must have

$$\overline{u}(t) = -\Psi(t)\overline{x}(t), \qquad \forall t \in (\sigma, T]. \tag{5.13}$$

Thus, it follows that

$$\Psi(\cdot)\widehat{\Phi}(\cdot)\xi \in L^2_{\mathcal{F}}(\sigma, T; \mathbb{R}^m), \qquad \forall \xi \in \mathcal{X}_\sigma, \tag{5.14}$$

where $\widehat{\Phi}(\cdot)$ is the solution of

$$\begin{cases} d\widehat{\Phi}(t) = [A - B\Psi(t)]\widehat{\Phi}(t)dt + [C - D\Psi(t)]\widehat{\Phi}(t)dw(t), & t \in [\sigma, T],\\ \widehat{\Phi}(\sigma) = I. \end{cases}$$

It is clearly that $E\big\{|\widehat{\Phi}(t)^{-1}|\big|\mathcal{F}_\sigma\big\} \le K_0, \quad \forall t \ge \sigma$. Hence, (5.13) implies (5.10) (note again that $\Psi(\cdot)$ is deterministic).

The following example shows that sometimes

$$\sigma \stackrel{\Delta}{=} \inf I_R = \inf I_S. \tag{5.15}$$

Example 5.1. Consider the following one-dimensional control system

$$\begin{cases} dx(t) = \frac{1}{2}x(t)dt + u(t)dw(t), & t \in [\tau, T],\\ x(\tau) = \xi \end{cases} \tag{5.16}$$

with the cost functional

$$J(\tau, \xi; u(\cdot)) = E\Big\{\int_\tau^T eu(t)^2 dt - x(T)^2\Big|\mathcal{F}_\tau\Big\}. \tag{5.17}$$

Thus, we have the case $A = \frac{1}{2}$, $B = C = Q = 0$, $D = 1$, $G = -1$ and $R = e$. Consequently, $R + D^T GD = e - 1 > 0$, and the Riccati equation (5.1) reads

$$\begin{cases} \dot{P}(t) = -P(t), & t \in [\tau, T],\\ P(T) = -1, \end{cases} \tag{5.18}$$

whose solution is given by

$$P(t) = -e^{T-t}, \qquad t \in [\tau, T]. \tag{5.19}$$

Then (5.3) becomes $R + D^T P(t) D = e - e^{T-t} > 0, \qquad t \in [\tau, T]$, which leads to (by (5.6)) $\sigma = T - 1$. A direct computation shows that for any $u(\cdot) \in \mathcal{U}[\tau, T]$,

$$J(\tau, \xi; u(\cdot)) = -e^{T-\tau}\xi^2 + eE\Big\{ \int_\tau^T u(t)^2 [1 - e^{T-1-r}] dr \big| \mathcal{F}_\tau \Big\}.$$

Thus

$$V(\tau, \xi) = \begin{cases} -\infty, & \forall (\tau, \xi) \in \Delta[0, T-1), \\ -e^{T-\tau}\xi^2, & \forall (\tau, \xi) \in \Delta[T-1, T]. \end{cases}$$

Hence, (5.15) holds.

We note that (5.19) coincides with (5.1) only on $(\sigma, T]$ since the later is at most defined on $(\sigma, T]$. At this moment, we do not know if (5.15) holds in general.

§6. One-Dimensional Cases

We now look at the one-dimensional time-invariant deterministic cases, for which we will solve our LQ problem more completely via Riccati equation.

Note that the case $D = 0$ reduces Riccati Equation (5.1) to a standard one (for deterministic LQ problems). Thus, let us assume that $D = 1$, after scaling. Then, we can write our Riccati Equation (5.1) and Condition (5.3) as follows

$$\begin{cases} \dot P = -(2A + C^2)P - Q + \frac{(B+C)^2 P^2}{P+R}, & \tau \in [\tau, T], \\ P(T) = G, \\ P(t) + R > 0, \qquad t \in [\tau, T]. \end{cases} \tag{6.1}$$

Now, by making a change of variable:

$$y(t) = P(T-t) + R, \qquad t \le T, \tag{6.2}$$

we can further write (6.1) as follows:

$$\begin{cases} \dot y = \frac{ay^2 + by + c}{y}, & t \ge 0, \\ y(0) = g, \\ y(t) > 0, & t \ge 0, \end{cases} \tag{6.3}$$

where

$$\begin{cases} a = (2A + C^2) - (B+C)^2, \quad b = Q - R\big[(2A + C^2) - 2(B+C)^2\big], \\ c = -R^2 (B+C)^2 \le 0, \quad g = G + R > 0. \end{cases} \tag{6.4}$$

The last (strict) inequality in (6.4), which is an analog of (5.2), will be assumed in the sequel. It is easy to see the following

(i) (6.1) admits a solution $P(\cdot)$ on $[\tau, T]$ if and only if (6.3) admits a solution $y(\cdot)$ on $[0, T-\tau]$, which implies the solvability of Problem (LQ) over $[\tau, T]$.

(ii) $\lim\limits_{t \downarrow \sigma} P(t)$ exists if and only if $\lim\limits_{t \uparrow T-\sigma} y(t)$ exists.

(iii) $\Psi \equiv \frac{(B+C)P}{P+R} \in L^2(\sigma, T)$ if and only if $\frac{c}{y} \in L^2(0, T-\sigma)$.

In what follows, we let

$$\theta = \sup\{ t > 0 \mid (6.3) \text{ admits a solution on } [0, t] \} = T - \sigma. \tag{6.5}$$

Then θ determines the maximal interval in which our LQ problems are solvable through the Riccati equation. The following result gives the explicit description of θ in terms of a, b, c and g defined by (6.4).

Proposition 6.1. *Let* (DI) *hold with* $n = m = 1$*, and* a, b, c, g *be defined by* (6.4)*. Then the following hold.*

(i) *When* $b = c = 0$*,* $\theta = +\infty$*.*

(ii) *When* $b = 0$ *and* $c \neq 0$*, we have* $\theta < +\infty$ *if and only if* $c < -a^+g^2$*. In this case,*

$$\theta = \begin{cases} \frac{g^2}{2|c|}, & a = 0, \\ \frac{1}{2a} \ln \frac{c}{ag^2+c}, & a \neq 0. \end{cases} \tag{6.6}$$

(iii) *When* $b \neq 0$ *and* $c = 0$*, we have* $\theta < +\infty$ *if and only if* $b < -a^+g$*. In this case,*

$$\theta = \begin{cases} \frac{g}{|b|}, & a = 0, \\ \frac{1}{a} \ln \frac{b}{ag+b}, & a \neq 0. \end{cases} \tag{6.7}$$

(iv) *When* $a = 0$ *and* $bc \neq 0$*, we have* $\theta < +\infty$ *if and only if* $b < 0$*. In this case,*

$$\theta = \frac{|c|}{b^2} \Big\{ \frac{bg}{c} - \ln \Big(1 + \frac{bg}{c} \Big) \Big\}. \tag{6.8}$$

(v) *When* $abc \neq 0$*, let* $\Delta = b^2 - 4ac$*.*

(a) *If* $\Delta > 0$*, then* $\theta < +\infty$ *if and only if one of the following holds:*

$$\begin{cases} a > 0, & b < \sqrt{\Delta} - 2ag, \\ a < 0, & b \notin \big[0, \big(2|a|g - \sqrt{\Delta}\big)^+\big]. \end{cases}$$

In either of the above cases,

$$\theta = \frac{1}{\sqrt{\Delta}} \Big\{ y_+ \ln \frac{y_+}{y_+ - g} - y_- \ln \frac{y_-}{y_- - g} \Big\}, \tag{6.9}$$

where $y_\pm = \frac{-b \pm \sqrt{\Delta}}{2a}$*.*

(b) *If* $\Delta = 0$*, then* $\theta < +\infty$ *if and only if* $b \notin \big[0, 2|a|g\big]$*. In this case,*

$$\theta = \frac{1}{a} \Big\{ \ln \frac{b}{2ag+b} + \frac{2ag}{2ag+b} \Big\}. \tag{6.10}$$

(c) *If* $\Delta < 0$*, one always has* $\theta < +\infty$ *and*

$$\theta = \frac{1}{2a} \Big\{ \ln \frac{c}{ag^2+bg+c} - \frac{2b}{\sqrt{-\Delta}} \Big[\tan^{-1} \frac{b}{\sqrt{-\Delta}} - \tan^{-1} \frac{2ag+b}{\sqrt{-\Delta}} \Big] \Big\}. \tag{6.11}$$

The proof is pretty straightforward and we omit it here.

From the above result, we can determine the maximum interval $(\sigma, T]$ on which the Riccati equation (6.1) is solvable, which leads to the solvability of our LQ problems at any $\tau \in (\sigma, T]$. Also, we see that in the present case,

$$\lim_{\tau \downarrow \sigma} P(\tau) = -R. \tag{6.12}$$

Thus, when $\sigma \in [0, T)$, we always have $\sigma \in I_F$. Finally, we have the following result.

Proposition 6.2. *Let* $\sigma \in [0, T)$ *be defined by* (5.6)*. Then* $\sigma \in I_S$ *if and only if*

$$R(B + C) = 0. \tag{6.13}$$

Proof. By Theorem 5.2, we know that $\sigma \in I_S$ if and only if $\Psi \equiv \frac{(B+C)P}{P+R} \in L^2(\sigma, T)$.

In the case $R(B+C) = 0$, $\Psi(t) = (B+C) \in L^2(\sigma, T)$. On the other hand, if $R(B+C) \neq 0$, $\Psi \in L^2(\sigma, T)$ is equivalent to $\frac{c}{y} \in L^2(0, \theta)$ with $c \equiv -R^2(B+C)^2 < 0$. Now, from (6.3), we have

$$y(t)^2 = -\frac{1}{2} \int_t^\theta [ay(s)^2 + by(s) + c] ds = \frac{|c|(\theta - t)}{2} + o(\theta - t).$$

Thus, $\frac{c}{y(t)} = \frac{\sqrt{2|c|}}{\sqrt{\theta - t + o(\theta - t)}} \notin L^2(0,\theta)$. This means that $\Psi \notin L^2(\sigma, T)$, proving $\sigma \notin I_S$.

§7. An Example

In this section, we present an example for which our theory applies (less non-trivially). Consider the following state equation

$$\begin{cases} d\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dw(t), \\ \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{cases} \tag{7.1}$$

with the cost functional

$$\begin{aligned} J(\tau, \xi, \eta; u(\cdot)) = E\Big\{ \int_\tau^T & \big[- u(t)^2 + x(t)^2 + 2(t - T - 1)x(t)y(t) \big] dt \\ & + x(T)^2 + G_3 y(T)^2 \Big| \mathcal{F}_\tau \Big\}. \end{aligned} \tag{7.2}$$

We assume that

$$G_3 \in \mathbb{D}_{1,\infty}, \quad G_3 - 1 \geq \delta_0 > 0, \quad \text{a.s.} \tag{7.3}$$

In terms of the general framework, we have

$$\begin{cases} A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ B = 0, \ C = 0, \\ Q = \begin{pmatrix} 1 & t - T - 1 \\ t - T - 1 & 0 \end{pmatrix}, \ G = \begin{pmatrix} 1 & 0 \\ 0 & G_3 \end{pmatrix}, \ R = -1. \end{cases} \tag{7.4}$$

Note that

$$R(T) + D(T)^T G D(T) = G_3 - 1 \geq \delta_0 > 0. \tag{7.5}$$

Thus, condition (1.11) holds. The corresponding Riccati equation takes the following form $\left(P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}, \Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_3 \end{pmatrix}\right)$:

$$\begin{cases} d\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} = \Big\{ - \begin{pmatrix} 0 & P_1 \\ P_1 & 2P_2 \end{pmatrix} + \frac{1}{P_3 - 1}\begin{pmatrix} \Lambda_2^2 & \Lambda_2\Lambda_3 \\ \Lambda_2\Lambda_3 & \Lambda_3^2 \end{pmatrix} - Q \Big\} dt \\ \qquad\qquad + \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_3 \end{pmatrix} dw(t), \\ P(T) = G. \end{cases} \tag{7.6}$$

This is equivalent to the following

$$\begin{cases} dP_1 = \Big[\frac{\Lambda_2^2}{P_3 - 1} - 1 \Big] dt + \Lambda_1 dw(t), \quad P_1(T) = 1, \\ dP_2 = \Big[- P_1 + \frac{\Lambda_2\Lambda_3}{P_3 - 1} + T + 1 - t \Big] dt + \Lambda_2 dw(t), \quad P_2(T) = 0, \\ dP_3 = \Big[- 2P_2 + \frac{\Lambda_3^2}{P_3 - 1} \Big] dt + \Lambda_3 dw(t), \quad P_3(T) = G_3. \end{cases} \tag{7.7}$$

By Theorem 4.1, we know that (7.6) admits a local solution (P, Λ, τ) and on $[\tau, T]$, (P, Λ) is unique. By uniqueness, this local solution satisfies the following on $[\tau, T]$:

$$\Lambda_1 = \Lambda_2 = P_2 = 0, \quad P_1 = 1 + T - t, \tag{7.8}$$

and (P_3, Λ_3) is the adapted solution of

$$\begin{cases} dP_3 = \frac{\Lambda_3^2}{P_3-1}dt + \Lambda_3 dw(t), \\ P_3(T) = G_3. \end{cases} \tag{7.9}$$

A direct computation (using Itô's formula) gives

$$E\Big(\frac{1}{P_3(t)-1}\Big)^2 \leq E\Big(\frac{1}{P_3(t)-1}\Big)^2 + E\int_t^T \frac{\Lambda_3(s)^2}{[P_3(s)-1]^4}ds = E\Big(\frac{1}{G_3-1}\Big)^2 \leq \frac{1}{\delta_0^2}. \tag{7.10}$$

Also

$$EP_3(t)^2 + 3E\int_t^T \Lambda_3(s)^2 ds = EG_3^2. \tag{7.11}$$

Hence, (7.9) admits a unique adapted solution on $[0, T]$. Consequently, (7.6) admits a unique adapted solution (P, Λ) over $[0, T]$, and the corresponding Problem (LQ) is solvable (on $[0, T]$).

Acknowledgement. The authors would like to thank an anonymous referee for detecting an error in an earlier version of this paper.

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