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ON CRITERION OF THE EXTREMALITY AND CONSTRUCTION OF HAMILTON SEQUENCES FOR A CLASS OF TEICHMÜLLER MAPPINGS***

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Abstract

It is proved that if f is a Teichmüller self-mapping of the unit disk with a holomorphic quadratic differential φ, and φ satisfies the growth condition $m(\varphi, r) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta = o((1-r)^{-s})$, $r \to 1$, for any $s > 1$, then f is extremal, and there exists a sequence $\{t_n\}$, $0 < t_n < 1$, $\lim\limits_{n\to\infty} t_n = 1$, such that $\{\varphi(t_n z)\}$ is a Hamilton sequence. It is the precision of a theorem of Reich-Strebel in 1974, and gives a fairly satisfactory answer to a question of Reich in 1988.

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§1. Introduction

f is a Teichmüller self-mapping of the unit disk $D = \{|z| < 1\}$, if f is a quasiconformal self-mapping of D with a complex dilatation of the form

$$\mu_f(z) = \frac{f_{\bar z}}{f_z} = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad 0 \le k < 1, \quad z \in D, \tag{1.1}$$

where $\varphi(z)$ is holomorphic in D, and k is a constant. Let $Q(f)$ denote the class of all quasiconformal self-mappings of D that agree with f on the boundary ∂D. f will be called extremal if $||\mu_f||_\infty \le ||\mu_g||_\infty$, for any $g \in Q(f)$. Let $B(D)$ denote the class of functions $\varphi(z)$ holomorphic in D, with the additional restriction $0 < ||\varphi|| = \iint\limits_D |\varphi(z)|\, dx\, dy < \infty$. A necessary and sufficient condition that f is an extremal mapping is that there exists a so-called Hamilton sequence[1], namely, a sequence $\varphi_n \in B(D)$, $n = 1, 2, 3, \cdots$, such that

$$\lim_{n\to\infty} \frac{\iint\limits_D \frac{\varphi(z)}{|\varphi(z)|}\varphi_n(z)\, dx\, dy}{||\varphi_n||} = 1.$$

In 1974, E. Reich and K. Strebel[1] proved the following

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Theorem A. *If $\varphi(z)$ satisfies the growth condition*

$$m(\varphi,r)=\frac{1}{2\pi}\int_0^{2\pi}|\varphi(re^{i\theta})|\,d\,\theta=O((1-r)^{-1}),\quad r\to 1,\tag{1.2}$$

then f is extremal, and the extremality of f is no longer implied if $O((1-r)^{-1})$ is replaced by $O((1-r)^{-s})$, for any $s>1$.

In 1988, Reich[2] asked: If $\{t_n\}$ is a sequence of numbers, $0<t_n<1$, and $\lim\limits_{n\to\infty}t_n=1$, does $\{\varphi(t_nz)\}$ constitute a Hamilton sequence? He showed

Theorem B. *If $\varphi(z)$ is holomorphic on $\overline{D}$ except for a finite number of poles on ∂D and f is uniquely extremal, then $\{\varphi(t_nz)\}$ is a Hamilton sequence.*

Theorem C. *If $\varphi(z)$ satisfies the growth condition* (1.2), *then f is uniquely extremal* (*it was deduced in* [3] *in* 1982), *and $\{\varphi(t_nz)\}$ is a Hamilton sequence* (*it was seen in the proof of Theorem* A).

Theorem D. *If f and $\varphi(z)$ in D are corresponding to $Kx+iy$ and* 1 *in the chimney region under a conformal mapping, then f is extremal but not uniquely extremal* (*it was seen in* [4] *in* 1962), *and $\{\varphi(t_nz)\}$ is not a Hamilton sequence.*

In 1995, the authors[5] conjectured that the best possible growth condition for the unique extremality may be

$$m(\varphi,r)=o((1-r)^{-1}\log^s(1-r)^{-1}),\quad r\to 1\ \text{ for any }\ s>1,\tag{1.3}$$

and that the best possible growth condition for the extremality may be

$$m(\varphi,r)=o((1-r)^{-s}),\quad r\to 1\ \text{ for any }\ s>1.\tag{1.4}$$

In view of these and Theorems C, D, it seems that (1.3) should imply that $\{\varphi(t_nz)\}$ is a Hamilton sequence, and that (1.4) should no longer imply that $\{\varphi(t_nz)\}$ is a Hamilton sequence. But our Theorem 2.1 below says that the former is positive, the latter negative, and so the conjecture that (1.4) is the best possible growth condition for the extremality of f is true.

§2. Result and Its Proof

Our result is as follows:

Theorem 2.1. *Suppose f is a Teichmüller self-mapping of D with complex dilatation* (1.1), *and $\varphi(z)$ satisfies the growth condition* (1.4), *then there exists a sequence $\{t_n\}$, $0<t_n<1$, $\lim\limits_{n\to\infty}t_n=1$, such that $\{\varphi(t_nz)\}$ is a Hamilton sequence, and hence f is extremal.*

Proof. For $\frac{1}{2}<t^2<t<1$, set

$$\alpha(t)=\iint\limits_{|z|<t}|\varphi(z)|\,dx\,dy,\quad \beta(t)=\iint\limits_{t<|z|<1}\frac{\overline{\varphi(z)}}{|\varphi(z)|}\varphi(tz)\,dx\,dy,$$

$$\gamma(t)=\iint\limits_{|z|<t}\frac{\overline{\varphi(z)}}{|\varphi(z)|}[\varphi(tz)-\varphi(z)]\,dx\,dy.$$

We have

$$\frac{\iint\limits_{D}\frac{\overline{\varphi(z)}}{|\varphi(z)|}\varphi(tz)\,dx\,dy}{||\varphi_n||}=t^2+t^2\frac{\beta(t)+\gamma(t)}{\alpha(t)}.$$

The proof is completed if we can prove that there exists a sequence $\{t_n\}$, $0<t_n<1$, $\lim\limits_{n\to\infty}t_n=1$, such that

$$\lim_{n\to\infty}\frac{\beta(t_n)}{\alpha(t_n)}=0,\tag{2.1}$$

$$\lim_{n\to\infty}\frac{\gamma(t_n)}{\alpha(t_n)}=0.\tag{2.2}$$

To estimate $\gamma(t)$, we need the following

Lemma 2.1.[4] *$m(\varphi', \rho) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi'(\rho e^{i\theta})|\, d\theta \le \frac{R}{R^2-\rho^2} m(\varphi, R)$, where $R = \frac{1+\rho}{2}$.*

Set

$$s(t) = \int_{\frac{1}{2}}^{t} m(\varphi, r)\, dr, \quad \sigma(t) = \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2R}{1-R} m(\varphi, R)\, dR, \quad \delta(t) = \sigma(t) - \sigma(t^2).$$

By Lemma 2.1, one finds that

$$\begin{aligned}
|\gamma(t)| &\le \iint\limits_{|z|<t} |\varphi(tz) - \varphi(z)|\, dx\, dy \\
&\le \int_0^t r\, dr \int_0^{2\pi} d\theta \int_{tr}^{r} |\varphi'(\rho e^{i\theta})|\, d\rho \le 2\pi t \int_0^t dr \int_{tr}^{r} \frac{R}{R^2-\rho^2} m(\varphi, R)\, d\rho \\
&\le 4\pi \int_0^t dr \int_{\frac{1+tr}{2}}^{\frac{1+r}{2}} \frac{R}{(3R-1)(1-R)} m(\varphi, R)\, dR \le 4\pi \int_0^t dr \int_{\frac{1+tr}{2}}^{\frac{1+r}{2}} \frac{m(\varphi, R)}{1-R}\, dR \\
&= 4\pi \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2R}{1-R} m(\varphi, R)\, dR - 4\pi \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2R}{t(1-R)} m(\varphi, R)\, dR \\
&\le 4\pi[\sigma(t) - \sigma(t^2)] = 4\pi\delta(t),
\end{aligned} \tag{2.3}$$

and that

$$\alpha(t) = 2\pi \int_0^t r m(\varphi, r)\, dr \ge \pi s(t), \tag{2.4}$$

$$|\beta(t)| \le \frac{2\pi}{t^2} \int_{t^2}^{t} r m(\varphi, r)\, dr \le \frac{4\pi}{t}[s(t) - s(t^2)]. \tag{2.5}$$

To prove (2.1) and (2.2), we need the following lemmas

Lemma 2.2. *There exists a sequence $\{t_n\}$, $0 < t_n < 1$, $\lim\limits_{n\to\infty} t_n = 1$, such that*

$$\lim_{n\to\infty} \frac{\delta(t_n)}{s(1+t_n^2/2)} = 0. \tag{2.6}$$

Proof. We first show that

$$\lim_{t\to 1} \frac{\sigma(t)}{\sigma(t^2)} = 1. \tag{2.7}$$

Otherwise, we may suppose that the right side of (2.7) is a constant $c > 1$. Hence there exists t_0, $\frac{1}{2} < t_0 < 1$ and c_0, $1 < c_0 < c$, $1 < c_0 < 2$ such that for any t, $t_0 \le t < 1$, it holds that

$$\sigma(t) > c_0 \sigma(t^2). \tag{2.8}$$

Define $t_n^2 = t_{n-1}, n = 1, 2, 3, \cdots$, then $\lim\limits_{n\to\infty} t_n = 1$. Making use of (2.8), one derives $\sigma(t_n) > c_0^n \sigma(t_0)$, $n = 1, 2, 3, \cdots$. Since $t_0 = t_n^{2^n}$, evidently

$$c_0^n = \Big(\frac{\log t_0}{\log t_n}\Big)^{\frac{1}{\log_{c_0} 2}}, \quad \sigma(t_n) > c_0^n \sigma(t_0) \sim c_1 \Big(\frac{1}{1-t_n}\Big)^{c_2}, \quad n \to \infty, \tag{2.9}$$

where $c_1 = \sigma(t_0)\big(\log \frac{1}{t_0}\big)^{c_2}$, $c_2 = \frac{1}{\log_{c_0} 2} > 0$. On the other hand, it follows from (1.4) that

$$\sigma(t_n) \le 2t_n \int_{\frac{1}{2}}^{\frac{1+t_n}{2}} m(\varphi, r)\, dr = o((1-t_n)^{1-s}), \quad n \to \infty$$

for any $s > 1$, which contradicts (2.9). Therefore, (2.7) holds.

Since $\sigma(t^2) \le 2t^2 \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} m(\varphi, r)\, dr = 2t^2 s((1+t^2)/2)$, we have

$$\frac{\delta(t)}{s((1+t^2)/2)} = \frac{\sigma(t^2)\left[\frac{\sigma(t)}{\sigma(t^2)} - 1\right]}{s((1+t^2)/2)} \le 2t^2\Big[\frac{\sigma(t)}{\sigma(t^2)} - 1\Big].$$

In view of (2.7), we obtain (2.6).

Lemma 2.3. *There exists a sequence* $\{t_n\}$, $0 < t_n < 1$, $\lim\limits_{n\to\infty} t_n = 1$, *such that*

$$\lim_{n\to\infty} \frac{s((1+t_n^2)/2) - s(t_n^2)}{s((1+t_n^2)/2)} = 0, \tag{2.10}$$

$$\lim_{n\to\infty} \frac{s(t_n^2)}{s((1+t_n^2)/2)} = 1. \tag{2.11}$$

Proof. Set

$$K(t) = \int_{t^2}^{\frac{1+t^2}{2}} \frac{1+t-2R}{1-R} m(\varphi,R)\,dR - \int_{t^2}^{\frac{1+t^2}{2}} \frac{1+t^2-2R}{1-R} m(\varphi,R)\,dR.$$

It is clear that $K(t) \le \delta(t)$, and that

$$K(t) = \int_{t^2}^{\frac{1+t^2}{2}} \frac{t(1-t)}{1-R} m(\varphi,R)\,dR \ge \frac{t}{1+t}\int_{t^2}^{\frac{1+t^2}{2}} m(\varphi,R)\,dR = \frac{t}{1+t}[s((1+t^2)/2) - s(t^2)].$$

Hence $\frac{s((1+t^2)/2)-s(t^2)}{s((1+t^2)/2)} \le \frac{(1+t)K(t)}{ts((1+t^2)/2)} \le \frac{(1+t)\delta(t)}{ts((1+t^2)/2)}$. It follows from (2.6) that (2.10), and then (2.11) holds.

Lemma 2.4. *There exists a sequence* $\{t_n\}$, $0 < t_n < 1$, $\lim\limits_{n\to\infty} t_n = 1$, *such that*

$$\lim_{n\to\infty} \frac{s(t_n) - s(t_n^2)}{s(t_n)} = 0, \tag{2.12}$$

$$\lim_{n\to\infty} \frac{\delta(t_n)}{s(t_n)} = 0. \tag{2.13}$$

Proof. Since $s(t)$ is increasing, we have $\lim\limits_{n\to\infty} \frac{s(t)-s(t^2)}{s(t)} \le \frac{s((1+t^2)/2)-s(t^2)}{s((1+t^2)/2)}$. (2.12) comes by (2.10). Note that $\frac{\delta(t)}{s(t)} \le \frac{\delta(t)}{s((1+t^2)/2)} \cdot \frac{s((1+t^2)/2)}{s(t^2)}$, and (2.13) follows from (2.6) and (2.11).

Combining (2.3)–(2.5), (2.12) and (2.13), now we see easily that (2.1) and (2.2) hold. The proof of Theorem 2.1 is completed.

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