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THE FOURIER TRANSFORM FOR HOMOGENEOUS VECTOR BUNDLES OVER QUATERNION UNIT DISK

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Abstract

The Fourier transform for homogeneous vector bundles over quaternion unit disk is studied, and the corresponding inversion formula and Plancherel formula are established.

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§1. Introduction

Let G be a connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup of G, and $X = G/K$ the associated Riemannian symmetric space of noncompact type.

Let (V_τ, τ) be an irreducible unitary representation of K, and E^τ be the homogeneous vector bundle over G/K associated with the given representation τ. It is well known that a cross section $f \in \Gamma(E^\tau)$ may be identified with a vector-valued function $f : G \to V_\tau$ which is right-K-covariant of type τ, i.e.,

$$f(gk) = \tau(k^{-1})f(g). \tag{1.1}$$

We denote by $C_0^\infty(G, \tau)$ the space of compactly supported smooth functions on G that are right-K-covariant of type τ, and $L^2(G, \tau)$ be the Hilbert space of square integrable functions, with the scalar product defined by

$$\langle f_1, f_2\rangle = \int_G \langle f_1(x), f_2(x)\rangle_{V_\tau} dx. \tag{1.2}$$

If τ is the trivial representation of K, then $L^2(G, \tau) = L^2(G/K)$, the corresponding Fourier transform is well-studied, and the inversion formula and Plancherel formula has been established by Harish-Chandra, Gelfand and Helgason. The case $\dim \tau = 1$ is considered by Shimeno[6], which is closely related to the weighted Plancherel formula on the bounded symmetric domain (see [1-5] and [8]).

In this paper, we consider the generalized Fourier transform for vector-valued functions on quaternion unit disk $B = \{|z| < 1 | z \in \mathbb{H}\}$, where $\mathbb{H}$ denotes the set of quaternion numbers. In this case $G = \mathrm{Sp}(1, 1)$ and $K = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

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§2. Preliminaries

Let $\mathbb{H}$ be the set of quaternions. The element of $\mathbb{H}$ can be expressed as $x = x_1 + x_2 i + x_3 j + x_4 k$. $\mathrm{Sp}(1,n)$ is the Lie group of right linear operators on $\mathbb{H}^{n+1}$ which leaves invariant the Hermitian form

$$(x,y) = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n,$$

where $x = (x_0, x_1, \cdots, x_n) \in \mathbb{H}^{n+1}$, $y = (y_0, y_1, \cdots, y_n) \in \mathbb{H}^{n+1}$. In particular, $\mathrm{Sp}(1) := \mathrm{Sp}(1,0) = \{u \in \mathbb{H} \,|\, |u| = 1\}$ is the unit sphere of $\mathbb{H}$, and $\mathrm{Sp}(1,1)$ is a subgroup of $\mathrm{GL}(2,\mathbb{H})$ which is isomorphic to the universal covering group of $\mathrm{SO}_0(1,4)$. It is easy to see that

$$G = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{H}) \,\middle|\, \bar{a}b = \bar{c}d, \quad |a|^2 - |c|^2 = 1, \quad |d|^2 - |b|^2 = 1 \right\},$$

and its maximal compact subgroup

$$K = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \,\middle|\, u, v \in \mathrm{Sp}(1) \right\}.$$

Let $G = \mathrm{Sp}(1,1)$. It has Iwasawa decomposition $G = KAN$, where

$$A = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 - x & x \\ -x & 1 + x \end{pmatrix} \,\middle|\, x = \xi_2 i + \xi_3 j + \xi_4 k, \quad \xi_2, \xi_3, \xi_4 \in \mathbb{R} \right\}.$$

The normalizer group M of A in K is

$$M = \left\{ g = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \,\middle|\, u \in \mathrm{Sp}(1) \right\}.$$

Then $P = MAN$ is the minimal parabolic subgroup of G. We also have Cartan decomposition $G = KAK$.

For $u = x_1 + x_2 i + x_3 j + x_4 k \in \mathrm{Sp}(1)$, let

$$I(u) = \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}.$$

Then the map $I : \mathrm{Sp}(1) \to \mathrm{SU}(2)$ is an isomorphism. For every $n \in \frac{1}{2}\mathbb{Z}^+$, there exists an irreducible unitary representation ρ^n of $\mathrm{SU}(2)$ on the Hilbert space V^n of dimension $2n+1$. Since $K = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ the irreducible unitary representation of K can be realized as $\rho^n \otimes \rho^{n'}$ on $V^n \otimes V^{n'}$. Let $\rho^{n,n'} = \rho^n \otimes \rho^{n'}$, i.e.,

$$\rho^{n,n'}(k) = \rho^n(u) \otimes \rho^{n'}(v) \qquad \text{for } k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in K.$$

If $\tau = \rho^{n,n'}$, then $\tau|_M$ can be decomposed as $\tau|_M = \sum\limits_{k=|n-n'|}^{n+n'} \rho^k$. In this paper, we only consider the case that $\tau = \rho^{0,n}$ or $\tau = \rho^{n,0}$, i.e., $\tau|_M$ is irreducible.

§3. Generalized Fourier Transform for Vector Bundles

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $b \in K/M$, consider the $\mathrm{End}(V_\tau)$-valued function

$$E_{\lambda,b}^{\tau}(g) = e^{-(i\lambda+\rho)(H(g^{-1}k))} \tau(K(g^{-1}k)) \tau(k^{-1}), \qquad b = kM, \tag{3.1}$$

where $H(g)$, $K(g)$ is defined by the Iwasawa decomposition

$$g = K(g) \exp H(g) n(g)$$

with $K(g) \in K$, $H(g) \in \mathfrak{a}$ (the Lie algebra of A) and $n(g) \in N$. For $m \in M$,

$$H(g^{-1}km) = H(g^{-1}k), \quad K(g^{-1}km) = K(g^{-1}k)m,$$

so $E^\tau_{\lambda,b}$ is well-defined. Moreover, we have the following elementary properties

$$E^\tau_{\lambda,b}(kg) = E^\tau_{\lambda,k^{-1}\cdot b}(g)\tau(k^{-1}), \tag{3.2}$$

$$E^\tau_{\lambda,b}(gk) = \tau(k^{-1})E^\tau_{\lambda,b}(g). \tag{3.3}$$

The τ-spherical function can be defined by

$$\phi^\tau_\lambda(g) = \int_{K/M} E^\tau_{\lambda,b}(g)db. \tag{3.4}$$

From (3.2) and (3.3) it is easy to see that

$$\phi^\tau_\lambda(k_1gk_2) = \tau(k_2^{-1})\phi^\tau_\lambda(g)\tau(k_1^{-1}). \tag{3.5}$$

Let $\mathbb{D}(G,\tau)$ be the algebra of G-invariant differential operators mapping sections of E^τ to sections of E^τ, $C^\infty(G,\tau)$ be the space of C^∞ maps from G to V_τ satisfying (1.1). Then $\mathbb{D}(G,\tau)$ acts naturally on $C^\infty(G,\tau)$ in view of the identification of $C^\infty(G,\tau)$ with the space of C^∞ cross sections of E^τ.

Let $C^\infty(G,\mathrm{End}V_\tau;\tau)$ be the space of C^∞ maps from G to $\mathrm{End}V_\tau$ satisfying

$$F(gk) = \tau(k^{-1})F(g).$$

For $F \in C^\infty(G,\mathrm{End}V_\tau;\tau)$, $v \in V_\tau$, the function $f(g) = F(g)v$ is in $C^\infty(G,\tau)$. So we can let $\mathbb{D}(G,\tau)$ act on $C^\infty(G,\mathrm{End}V_\tau;\tau)$ by

$$(DF)(g)v = D(F(g)v), \quad \forall D \in \mathbb{D}(G,\tau), \quad F \in C^\infty(G,\mathrm{End}V_\tau;\tau).$$

Clearly, $DF \in C^\infty(G,\mathrm{End}V_\tau;\tau)$. Moreover, if F is bi-K-covariant of type τ, i.e.,

$$F(k_1gk_2) = \tau(k_2^{-1})F(g)\tau(k_1^{-1}),$$

then DF is also bi-K-covariant.

Theorem 3.1. *For $D \in \mathbb{D}(G,\tau)$, we have*

$$DE^\tau_{\lambda,b} = \chi_{\lambda,\tau}(D)E^\tau_{\lambda,b}, \tag{3.6}$$

$$D\phi^\tau_\lambda = \chi_{\lambda,\tau}(D)\phi^\tau_\lambda, \tag{3.7}$$

where $\chi_{\lambda,\tau}$ is an algebra homomorphism from $\mathbb{D}_\tau$ into $\mathbb{C}$ which is given by

$$\chi_{\lambda,\tau}(D) = \frac{1}{d_\tau}\,\mathrm{Tr}[D\phi^\tau_\lambda(e)] \in \mathbb{C},$$

where d_τ is the dimension of V_τ.

Proof. Let $\tau = \rho^{0,n}$, and K_1 be the subgroup of K defined by

$$K_1 = \left\{ \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \Big| u \in \mathrm{Sp}(1) \right\}.$$

Then $\tau(k) = 1$ for $k \in K_1$. Let $b = k_1M$, and for $g, h \in G$ and $k \in K$,

$$H((gkh)^{-1}k_1) = H(h^{-1}K(h^{-1}g^{-1}k_1)) + H(g^{-1}k_1),$$

$$K((gkh)^{-1}k_1) = K(h^{-1}K(k^{-1}g^{-1}k_1)),$$

$$K(k^{-1}g^{-1}k_1) = k^{-1}(K(g^{-1}k_1)).$$

By (3.1), we have

$$E^\tau_{\lambda,k_1M}(gkh) = E^\tau_{\lambda,K(k^{-1}g^{-1}k_1)M}(h)\tau(k^{-1})E^\tau_{\lambda,k_1M}(g).$$

Since $K/M = \{kM | k \in K_1\}$, by integrating the above equation over K_1, we get

$$\int_{K_1} E^\tau_{\lambda,b}(gkh)dk = \phi^\tau_\lambda(h)E^\tau_{\lambda,b}(g). \tag{3.8}$$

Now we act on both sides with operator $D \in \mathbb{D}(G,\tau)$. Since D commutes with left translations, we have

$$\int_K DE^\tau_{\lambda,b}(gkh)dk = (D\phi^\tau_\lambda)(h)E^\tau_{\lambda,b}(g).$$

For $k \in K_1$, $D\phi^\tau_\lambda(gk) = \tau_\tau(k^{-1})D\phi^\tau_\lambda(g)$, by specializing the argument to $h = e$, we get

$$DE^\tau_{\lambda,b}(g) = D\phi^\tau_\lambda(e)E^\tau_{\lambda,b}(g).$$

This proves (3.6), and (3.7) follows easily from (3.6).

Notice that $D\phi^\tau_\lambda \in C^\infty(G, \mathrm{End}V_\tau; \tau)$ and satisfies

$$D\phi^\tau_\lambda(k_1 g k_2) = \tau(k_2^{-1})D\phi^\tau_\lambda(g)\tau(k_1^{-1}).$$

So $D\phi^\tau_\lambda(e) \in \mathrm{End}_K(V_\tau)$, i.e.,

$$\tau(k)D\phi^\tau_\lambda(e) = D\phi^\tau_\lambda(e)\tau(k).$$

By Schur's Lemma, $D\phi^\tau_\lambda(e) = cI_{d_\tau}$, with $c = \frac{1}{d_\tau}\mathrm{Tr}[D\phi^\tau_\lambda(e)]$.

Finally, we have

$$\begin{aligned}\chi_{\lambda,\tau}(D_1D_2) &= \frac{1}{d_\tau}\mathrm{Tr}[(D_1D_2\phi^\tau_\lambda)(e)] = \frac{1}{d_\tau}\mathrm{Tr}[D_1\big(\chi_{\lambda,\tau}(D_2)\big)\phi^\tau_\lambda)(e)] \\ &= \chi_{\lambda,\tau}(D_1)\chi_{\lambda,\tau}(D_2).\end{aligned}$$

Proposition 3.1. *The τ-spherical function ϕ^τ_λ satisfies*

$$\phi^\tau_\lambda(a_t) = (1-\tanh^2 t)^{-n}\phi^{1,2n+1}_\lambda(t)I_{d_\tau} = (1-\tanh^2 t)^{n+1}\phi^{1,-(2n+1)}_\lambda(t)I_{d_\tau}, \tag{3.9}$$

where

$$a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

and I_{d_τ} is the identity element in $\mathrm{End}(V_\tau)$, *$\phi^{1,2n+1}_\lambda$ and $\phi^{1,-(2n+1)}_\lambda$ are Jacobi functions of order $(1, 2n+1)$ and $(1, -(2n+1))$ respectively.*

Proof. Since ϕ^τ_λ is bi-K-covariant of type τ (see (3.5)), it is determined by its restriction to A, and $\phi^\tau_\lambda(a) \in \mathrm{End}_M(V_\tau)$, $\forall a \in A$. Then $\phi^\tau_\lambda(a) = f(a)I_{d_\tau}$ for some function f on A. f can be calculated explicitly either by using the radial part of the Casimir operator of G, or by using the integral formula of $\phi^\tau_\lambda(a)$. We have (see [7])

$$\begin{aligned} f(a_t) &= \frac{1}{d_\tau}\int_K e^{-(i\lambda+\rho)H(a_t^{-1}k)}\chi^\tau(K(a_t^{-1}k)k^{-1})dk \\ &= (1-\tanh^2 t)^{\frac{3+i\lambda}{2}} F\Big(\frac{2n+3+i\lambda}{2}, \frac{1-2n+i\lambda}{2}; 2; \tanh^2 t\Big).\end{aligned}$$

This proves the proposition.

Proposition 3.2.

$$\phi^\tau_\lambda(x^{-1}y) = \int_K E^\tau_{\lambda,b}(y)(E^\tau_{\lambda,b}(x))^* dk, \tag{3.10}$$

where $*$ denotes adjoint.

Proof. By the definition of ϕ^τ_λ,

$$\phi^\tau_\lambda(x^{-1}y) = \int_K e^{-(i\lambda+\rho)(H(y^{-1}xk))}\tau(K(y^{-1}xk))\tau(k^{-1})dk.$$

By a change of variable $k = K(x^{-1}k_1)$, since $k_1 = K(xk)$, we have

$$\begin{aligned}
&\phi_\lambda^\tau(x^{-1}y)\\
&= \int_K e^{-(i\lambda+\rho)(H(y^{-1}k_1)-H(x^{-1}k_1))}\tau(K(y^{-1}k_1))\tau((K(x^{-1}k_1))^{-1})e^{-2\rho(H(x^{-1}k_1))}dk_1\\
&= \int_K E^\tau_{\lambda,kM}(y)(E^\tau_{\lambda,kM}(x))^* dk.
\end{aligned}$$

Let $\mathcal{D}(G,\tau)$ be the space of compactly supported C^∞-functions on G that are right-K-covariant of type τ, and $\mathcal{D}^\sharp(G,\tau)$ be the space of functions F in $\mathcal{D}(G,\tau)$ which satisfy

$$F(k_1ak_2) = \tau(k_2^{-1})\tau(k_1^{-1})f(a)v, \tag{3.11}$$

where f is a function on A^+ and $v \in V_\tau$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $f \in \mathcal{D}^\sharp(G,\tau)$, the spherical transform is defined by

$$\widehat{F}(\lambda) = \int_G \left(\phi_\lambda^\tau(g)\right)^* F(g)dg. \tag{3.12}$$

For $\lambda \in \mathfrak{a}^*$,$b \in K/M$, $F \in \mathcal{D}(G,\tau)$, the generalized Fourier transform is defined by

$$\widehat{F}(\lambda,b) = \int_G \left(E^\tau_{\lambda,b}(g)\right)^* F(g)dg = \int_G e^{(i\lambda-\rho)(H(g^{-1}k))}\tau(k)\tau(K(g^{-1}k)^{-1})F(g)dg. \tag{3.13}$$

Then we have

Theorem 3.2. *Let $\tau = \rho^{0,n}$. For $F \in \mathcal{D}^\sharp(G,\tau)$, we have the inversion formula*

$$F(g) = \int_0^\infty \phi_\lambda^\tau(g)\widehat{F}(\lambda)\rho_\tau(\lambda)d\lambda + \sum_{m=0}^{k} d_\tau(m)\phi^\tau_{i(2n-1-2m)}(g)\widehat{F}(i(2n-1-2m))$$

and the Plancherel formula

$$\int_G \|f(g)\|^2_{V_\tau}dg = \int_0^\infty \|\widehat{F}(\lambda)\|^2_{V_\tau}\rho_\tau(\lambda)d\lambda + \sum_{m=0}^{k} d_\tau(m)\|\widehat{F}(i(2n-1-2m))\|^2_{V_\tau},$$

where

$$k = \max\Big\{j \in \mathbb{Z}\Big|\ j < \frac{2n-1}{2}\Big\},$$

and the Plancherel measures ρ_τ and d_τ are given by

$$\begin{aligned}
\rho_\tau(\lambda) &= c_G\frac{(2n+1)^2+\lambda^2}{4}\frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda+(-1)^{2n}},\\
d_\tau(m) &= c_G(2n-2m-1)(m+1)(2n-m),
\end{aligned}$$

c_G is a constant only depending on the choice of the Haar measure.

Proof. By Proposition 3.1, for $F \in \mathcal{D}^\sharp(G,\tau)$,

$$(\phi_\lambda^\tau(k_1a_tk_2) * F(k_1a_tk_2) = (1-\tanh^2 t)^{n+1}\phi_\lambda^{1,-(2n+1)}(t)F(a_t).$$

Assume that $F(a) = f(a)v$ for some $v \in V_\tau$. Then

$$\begin{aligned}
\widehat{F}(\lambda) &= c\int_{\mathbb{R}^+}(1-\tanh^2 t)^{n+1}\phi_\lambda^{1,-(2n+1)}(t)f(a_t)(2\sinh t)^3(2\cosh t)^3 dt\cdot v\\
&= c2^{4n+4}\big((1-\tanh^2 t)^{-(n+1)}f(a_t)\big)\tilde{}(\lambda)\cdot v,
\end{aligned} \tag{3.14}$$

where c is a constant depending only on the choice of the Haar measure, and $\tilde{}$ denotes the Jacobi transform, which is defined, for $f \in C_0^\infty(\mathbb{R}^+)$, by

$$\widetilde{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda^{1,-(2n+1)}(t)(2\sinh t)^3(2\cosh t)^{-4n-1}dt.$$

Then by the inversion formula and Plancherel formula of Jacobi transform, we get

$$F(a_t) = \frac{2^{-(4n+4)}}{2\pi c}\int_0^\infty \phi_\lambda^\tau(a_t)\widehat{F}(\lambda)|c(\lambda)|^{-2}d\lambda + \sum_{m=0}^{k}\frac{2^{-(4n+4)}}{c}d(m)\phi^\tau_{i(2n-1-2m)}(a_t)\widehat{F}(i(2n-1-2m)), \tag{3.15}$$

$$c\int_0^\infty \|F(a_t)\|^2_{V_\tau}(2\sinh t)^3(2\cosh t)^3dt = \frac{2^{-(4n+4)}}{2\pi c}\int_0^\infty \|\widehat{F}(\lambda)\|^2_{V_\tau}|c(\lambda)|^{-2}d\lambda + \sum_{m=0}^{k}\frac{2^{-(4n+4)}}{c}d(m)\|\widehat{F}(i(2n-1-2m))\|^2_{V_\tau}, \tag{3.16}$$

where

$$c(\lambda) = \frac{2^{1-2n-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+1-2n))\Gamma(\frac{1}{2}(i\lambda+3+2n))},$$
$$d(m) = -i\operatorname*{Res}_{\lambda=i(2n-1-2m)}(c(\lambda)(-\lambda))^{-1}, \quad 0\le m\le k.$$

By easy calculation, we have

$$\frac{2^{-(4n+4)}}{2\pi c}|c(\lambda)|^{-2} = \frac{(2n+1)^2+\lambda^2}{2^8c}\frac{\lambda\sinh\pi\lambda}{\cosh\pi\lambda+(-1)^{2n}}, \tag{3.17}$$

$$\frac{2^{-(4n+4)}}{c}d(m) = \frac{(2n-2m-1)(m+1)(2n-m)}{2^6c}. \tag{3.18}$$

Taking $c_G = (2^6c)^{-1}$, by (3.15)–(3.18), we prove the theorem.

Theorem 3.3. *Let $\tau=\rho^{0,n}$. For $f\in\mathcal{D}(G,\tau)$, we have the inversion formula*

$$f(g) = \int_0^\infty\int_B E^\tau_{\lambda,b}(g)\widehat{f}(\lambda,b)\rho_\tau(\lambda)dbd\lambda + \sum_{m=0}^{k}d_\tau(m)\int_B E^\tau_{i(2n-1-2m),b}(g)\widehat{F}(i(2n-1-2m),b) \tag{3.19}$$

and the Plancherel formula

$$\int_G\|f(g)\|^2_{V_\tau}dg = \int_0^\infty\int_B\|\widehat{f}(\lambda,b)\|^2_{V_\tau}\rho_\tau(\lambda)dbd\lambda + \sum_{m=0}^{k}d_\tau(m)\int_B\|\langle\widehat{f}(i(2n-1-2m),b),\widehat{f}(-i(2n-1-2m),b)\rangle\|_{V_\tau}db, \tag{3.20}$$

where ρ_τ and d_τ are the same as in Theorem 3.2.

Proof. Let $f\in\mathcal{D}(G,\tau)$, v be an arbitrary fixed vector in V_τ. Let $F: G\mapsto \operatorname{End}V_\tau$ be defined by

$$F(g)(w) = \langle w,v\rangle f(g), \quad w\in V_\tau.$$

Obviously, $f(g)=F(g)v$ and for all $A\in\operatorname{End}V_\tau$,

$$\operatorname{Tr}(AF(g)) = \langle Af(g),v\rangle. \tag{3.21}$$

For $h \in G$, define

$$F_1(g) = \int_K F(hkg)\tau(k)dk.$$

Then F_1 is bi-K-covariant of type τ, and for any $w \in V_\tau$, $F_1(g)w \in \mathcal{D}^\sharp(G,\tau)$. Moreover

$$\mathrm{Tr}F_1(e) = \int_K \mathrm{Tr}(\tau(k^{-1})F(h)\tau(k))dk = \mathrm{Tr}(F(h)) = \langle f(h), v\rangle. \tag{3.22}$$

Let $\{e_i | i = 1, 2, \cdots, d_\tau\}$ be an orthonormal basis of V_τ. Then

$$\begin{aligned}\sum_{i=1}^{d_\tau}\langle (F_1(g)e_i)\widetilde{\ }(\lambda), e_i\rangle_{V_\tau} &= \sum_{i=1}^{d_\tau}\Big\langle \int_G\int_K \left(\phi_{\bar\lambda}^\tau(g)\right)^* F(hkg)\tau(k)dkdge_i, e_i\Big\rangle_{V_\tau}\\ &= \sum_{i=1}^{d_\tau}\Big\langle \int_G\int_K \tau(k^{-1})\left(\phi_{\bar\lambda}^\tau(h^{-1}g)\right)^* F(g)\tau(k)dkdge_i, e_i\Big\rangle_{V_\tau}\\ &= \int_G \mathrm{Tr}\left[\left(\phi_{\bar\lambda}^\tau(h^{-1}g)\right)^* F(g)\right]dg.\end{aligned}$$

By (3.19) and Proposition 3.2, we get

$$\begin{aligned}\sum_{i=1}^{d_\tau}\langle (F_1(g)e_i)\widetilde{\ }(\lambda), e_i\rangle_{V_\tau} &= \Big\langle \int_G \left(\phi_{\bar\lambda}^\tau(h^{-1}g)\right)^* f(g)dg, v\Big\rangle\\ &= \Big\langle \int_G\int_K E_{\lambda,b}^\tau(h)\left(E_{\bar\lambda,b}^\tau(g)\right)^* f(g)dbdg, v\Big\rangle\\ &= \Big\langle \int_K E_{\lambda,b}^\tau(h)\widehat{f}(\lambda,b)db, v\Big\rangle.\end{aligned} \tag{3.23}$$

By (3.22), (3.23) and Theorem 3.2,

$$\begin{aligned}\langle f(h), v\rangle &= \int_0^\infty \Big\langle \sum_{i=1}^{d_\tau}(F_1(g)e_i)\widetilde{\ }(\lambda), e_i\Big\rangle_{V_\tau}\rho_\tau(\lambda)d\lambda\\ &\quad + \sum_{m=0}^{k} d_\tau(m)\Big\langle \sum_{i=1}^{d_\tau}(F_1(e_i))\widetilde{\ }(i(2n-1-2m)), e_i\Big\rangle_{V_\tau}\\ &= \Big\langle \int_0^\infty\int_B E_{\lambda,b}^\tau(h)\widehat{f}(\lambda,b)\rho_\tau(\lambda)dbd\lambda, v\Big\rangle_{V_\tau}\\ &\quad + \Big\langle \sum_{m=1}^{k} d_\tau(m)\int_B E_{i(2n-1-2m),b}^\tau(g)\widehat{f}(i(2n-1-2m),b)db, v\Big\rangle_{V_\tau}.\end{aligned}$$

From this expression we get (3.19).

The Plancherel formula easily follows from (3.19). First, we note that

$$\int_G (f(g))^* E_{\lambda,b}^\tau(g)dg = \Big(\int_G (E_{\lambda,b}^\tau(g))^* f(g)dg\Big)^* = (\widehat{f}(\bar\lambda,b))^*. \tag{3.24}$$

By (3.19),

$$\begin{aligned}\int_G \|f(g)\|_{V_\tau}^2 dg &= \int_G (f(g))^* f(g) dg \\ &= \int_G \int_0^\infty \int_B (f(g))^* E_{\lambda,b}^\tau(g) \widehat{f}(\lambda,b) \rho_\tau(\lambda) db d\lambda dg \\ &\quad + \sum_{m=0}^k d_\tau(m) \int_G \int_B (f(g))^* E_{i(2n-1-2m),b}^\tau(g) \widehat{f}(i(2n-1-2m),b) db dg \\ &= \int_0^\infty \int_B \|\widehat{f}(\lambda,b)\|_{V_\tau}^2 \rho_\tau(\lambda) db d\lambda \\ &\quad + \sum_{m=0}^k d_\tau(m) \int_B \|\langle \widehat{f}(i(2n-1-2m),b), \widehat{f}(-i(2n-1-2m),b)\rangle\|_{V_\tau} db.\end{aligned}$$

This proves (3.20).

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