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ON MULTIPLE POSITIVE SOLUTIONS FOR A NONLINEAR ELLIPTIC PROBLEM IN R^N**

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Abstract

By making use of variational method, the authors obtain some results about existence of multiple positive solutions and their asymptotic behavior as the parameter $\lambda \to +\infty$ for a semilinear elliptic problem in R^N.

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§1. Introduction

Consider the semilinear elliptic problem

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = u^p, \quad x \in R^N, \\ u \in H^1(R^N), \quad u > 0 \text{ in } \ R^N, \end{cases} \tag{1.1}$$

where $p \in (1, \frac{N+2}{N-2})$ for $N \geq 3$, and $p \in (1, +\infty)$ for $N = 1, 2$, $0 \leq a(x) \in C(R^N)$, $\lambda > 0$ is a real parameter.

The existence and uniqueness of solution for such problems have been considered by many authors recently (see [1–4] and references therein). In [3], T. Bartsch and Wang, Z. Q. proved that with more genaral nonlinearities (1.1) has at least one solution for λ large under some conditions on $a(x)$, one of which is that $a^{-1}(0)$ has nonempty interior, and they put forward a question whether or not one can get rid of this assumption.

In this paper, we study the existence of multiple solutions for (1.1) and the asymptotic behavior of the solutions as $\lambda \to +\infty$. We only assume that $a(x)$ has some flatness at its zero points, i.e.

(a_1) $a \in C(R^N, R)$ satisfies $a \geq 0, a^{-1}(0) = \bigcup\limits_{i=1}^{K} \{a^i\}$ and $D^\alpha a(a^i) = 0, 1 \leq |\alpha| \leq k-1$, $a(x + a^i) = a(x + a^j), i \neq j$, x near zero point. Here K is a positive integer and $k \geq 2$ will be determined later.

(a_2) There exists $a_\infty > 0$ such that $\liminf\limits_{|x| \to \infty} a(x) \geq a_\infty > 0$.

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In [5], D. M. Cao and E. S. Noussair gave a result about existence of multiple positive solutions of (1.1) with $a(x) = 0$ and nonlinearity $Q(\lambda x)|u|^{p-2}u$ instead of u^p. They were mainly interested in the effects of the "shape" of $Q(x)$ while our focus is on $a(x)$. It is easily seen that if $\inf a(x) > 0$ and $a(x)$ is bounded, one can get a similiar result completely parallel to [5]. But under conditions $(a_1), (a_2)$, it seems impossible to prove that the corresponding energy functional satisfies (PS) or local (PS) conditions by using concentration-compactness principle as in [5], and this gives rise to some difficulties in the proof of the existence of multiple solutions. By some delicate estimations, we noticed that the minimizing sequence concentrates at the zero points of $a(x)$ for λ large enough (see Section 2), and then by making use of Ekeland's variational principle, we obtain the following results about the existence of multiple solutions of (1.1).

Theorem 1.1. *Suppose conditions* (a_1), (a_2) *hold, and*

$$k > \max\Big\{k', 4\Big(\frac{p+1}{p-1} - \frac{N+1}{2}\Big)\Big\}, \tag{1.2}$$

where

$$k' = \begin{cases} \frac{2}{p-1} & \text{if } N \geq 3, \\ \frac{4}{p-1} & \text{if } N = 2, \\ \frac{p^2+7}{2(p-1)} & \text{if } N = 1, \end{cases}$$

k is as in (a_1). *Then there exists $\lambda_0 \geq 1$ such that problem* (1.1) *has at least K positive solutions $u^i (i = 1, \cdots, K)$ for each $\lambda \geq \lambda_0$.*

By the proof of Theorem 1.1, we immediately know that the solutions concentrate at the zero points of $a(x)$ in L^{p+1} norm as $\lambda \to +\infty$. Moreover, the solutions also concentrate in L^∞ norm. More precisely, we have

Corollary 1.1. *Under the conditions of Theorem* 1.1, *the solutions $u^i (i = 1, \cdots, K)$ obtained in Theorem* 1.1 *concentrate at a^i in the following sense*: *For any small $\delta > 0$, as $\lambda \to +\infty$,*

$$\begin{cases} \max\limits_{\mathrm{dist}(x,a^i) \geq \delta} u^i \to 0, \\ \max\limits_{\mathrm{dist}(x,a^i) < \delta} u^i \to +\infty. \end{cases} \tag{1.3}$$

The paper is organized as follows. Section 2 contains some preliminary lemmas. The proofs of Theorem 1.1 and Corollary 1.1 are given in Section 3.

Throughout this paper, $C, C_0, C_1, C_2, \cdots$ denote (possibly different) positive constants.

§2. Notations and Some Lemmas

As usual, we consider the functional

$$I_\lambda(u) = \frac{1}{2}\int_{R^N} (|\nabla u|^2 + (\lambda a(x) + 1)u^2)dx - \frac{1}{p+1}\int_{R^N} |u|^{p+1}dx, \quad \lambda \geq 1$$

on space

$$E = \Big\{u \in H^1(R^N) : \|u\|^2 = \int_{R^N} (|\nabla u|^2 + (a(x)+1)u^2)dx < +\infty\Big\}.$$

Obviously, $I_\lambda \in C^1(E, R^1)$ and a positive critical point of I_λ is a solution of Equation (1.1) (see [1,3]).

Let $\delta_0 > 0, L > 0$ be such that

$$a^i_{\delta_0} \cap a^j_{\delta_0} = \emptyset, \quad i \neq j \quad \text{and} \quad \bigcup_{i=1}^{K} a^i_{\delta_0} \subset \prod_{i=1}^{N} (-L, L),$$

where $a^i_{\delta_0} = \{x \in R^N | \text{dist}(x, a^i) < \delta_0\}$, and a^i is as in the condition (a_1). Let $\varphi \in C(R^1, R^1)$ be such that

$$\varphi(t) = \begin{cases} 2L, & t > 2L, \\ t, & -L \leq t \leq L, \\ -2L, & t < -2L \end{cases} \tag{2.1}$$

and define $g \in C(E, R^N)$ by

$$g(u) = (g^i(u))_{1 \leq i \leq N} = \left(\frac{\int_{R^N} \varphi(x_i) |u|^{p+1} dx}{\int_{R^N} |u|^{p+1} dx} \right)_{1 \leq i \leq N}. \tag{2.2}$$

Define

$$M_\lambda = \{u \in E : u \not\equiv 0, \ \langle I'_\lambda(u), u \rangle = 0\},$$
$$M^i_\lambda = \{u \in M_\lambda : g(u) \in a^i_{\delta_0}\},$$
$$O^i_\lambda = \{u \in M_\lambda : g(u) \in \partial a^i_{\delta_0}\},$$

where $1 \leq i \leq K$, and $M_\lambda \neq \emptyset, O^i_\lambda \neq \emptyset$ (see [5]). Let

$$m^i_\lambda = \inf_{M^i_\lambda} I_\lambda, \quad \overline{m^i_\lambda} = \inf_{O^i_\lambda} I_\lambda.$$

By (a_1), it is not hard to prove that for $i \neq j$,

$$m^i_\lambda = m^j_\lambda \geq c_1 > 0, \quad \overline{m^i_\lambda} = \overline{m^j_\lambda} \geq c_2 > 0, \tag{2.3}$$

where c_1, c_2 are independent of λ.

In this section, we give the estimations of m^i_λ and $\overline{m^i_\lambda}$ for λ large, and then by making use of Ekeland-variational principle, we obtain a (PS) sequence.

Lemma 2.1. *Suppose the condition* (a_1) *holds. Then for* $\lambda \geq 1$ *large enough,*

$$m^i_\lambda \leq C \lambda^{\frac{2}{k+2}(-\frac{p+1}{p-1} - \frac{N}{2})},$$

where the constant $C > 0$ *is independent of* λ*, and* k *is as in* (a_1)*.*

Proof. Let

$$v = \begin{cases} 1 - \lambda^\sigma |x - a^i|, & |x - a^i| < \lambda^{-\sigma}, \\ 0, & |x - a^i| \geq \lambda^{-\sigma}, \end{cases}$$

where $\sigma > 0$ is a constant to be determined later.

It is easily seen that supp$v \subset a^i_{\delta_0}$ for λ large enough and there exists $t_0 > 0$ such that $u_0 = t_0 v \in M^i_\lambda$ and $g(u_0) \in a^i_{\delta_0}$. By computing, we have

$$\int_{R^N} |v|^s = K_1 \lambda^{-\sigma N}, \tag{2.4}$$

where $K_1 = \int_0^1 \int_{|\omega|=1} (1-\rho)^s \rho^{N-1} d\rho d\omega$, and

$$\int_{R^N} |\nabla v|^2 = \omega_N \lambda^{2\sigma - N\sigma}, \tag{2.5}$$

where ω_N is the area of the unit sphere in R^N, and

$$\int_{R^N} |x - a^i|^k v^s = K_2 \lambda^{-\sigma N - k\sigma}, \tag{2.6}$$

where

$$K_2 = \int_0^1 \int_{|\omega|=1} \rho^k (1-\rho)^s \rho^{N-1} d\rho d\omega.$$

Let $\sigma = \frac{1}{k+2}$. By (a_1), using Taylor expansion of $a(x)$ about a^i and (2.4)–(2.6), we get

$$\begin{aligned} m_\lambda^i &\le I_\lambda(u_0) = I_\lambda(t_0 v) \\ &\le \max_{t\ge 0} \Big[\frac{t^2}{2}\int_{R^N} (|\nabla v|^2 + C_1\lambda|x-a^i|^k v^2 + v^2)dx - \frac{t^{p+1}}{p+1}\int_{R^N} |v|^{p+1}dx\Big] \\ &\le \max_{t\ge 0}[C_2\lambda^{2\sigma - N\sigma}t^2 - C_3\lambda^{-N\sigma}t^{p+1}] \\ &\le C\lambda^{\frac{2}{k+2}(\frac{p+1}{p-1}-\frac{N}{2})} \end{aligned} \tag{2.7}$$

for $\lambda \ge 1$ large enough, where $C_i(i=1,2,3)$ and C are positive constants independent of λ.

Lemma 2.2. *Under the conditions of Theorem* 1.1, *for* λ *large enough,* $\overline{m_\lambda^i} > m_\lambda^i$.

Proof. Let $u_n \in O_\lambda^i$ be a minimizing sequence of $\overline{m_\lambda^i}$, i.e. $g(u_n) \in \partial a_{\delta_0}^i$,

$$\frac{1}{2}\int_{R^N} |\nabla u_n|^2 + (\lambda a(x)+1)u_n^2 = \frac{1}{p+1}\int_{R^N} |u_n|^{p+1} + \overline{m_\lambda^i} + o(1) \quad \text{as } n\to\infty, \tag{2.8}$$

$$\int_{R^N} |\nabla u_n|^2 + (\lambda a(x)+1)u_n^2 = \int_{R^N} |u_n|^{p+1}. \tag{2.9}$$

By (2.8), (2.9), we get

$$\int_{R^N} |\nabla u_n|^2 + (\lambda a(x)+1)u_n^2 \le \frac{2(p+1)}{p-1}\overline{m}_\lambda^i + o(1) \quad \text{as } n\to\infty. \tag{2.10}$$

Using Sobolev inequality in (2.9), one can prove that

$$|u_n|^{p+1}_{L^{p+1}(R^N)} \ge C^* > 0, \tag{2.11}$$

where C^* is independent of λ, n and δ_0.

Let $\delta < \delta_0$ and

$$a_{\frac{\delta}{4}}^{-1}(0) = \bigcup_{i=1}^K a_{\frac{\delta}{4}}^i = \bigcup_{i=1}^K \Big\{x\in R^N \Big| \operatorname{dist}(x,a^i) < \frac{\delta}{4}\Big\}.$$

By (a_2) and (2.10), we have

$$|u_n|^2_{L^2\left(R^N\backslash a_{\frac{\delta}{4}}^{-1}(0)\right)} \le C\frac{\overline{m}_\lambda^i}{\lambda}. \tag{2.12}$$

If $N\ge 3$, by setting $\theta = \frac{N(p-1)}{2(p+1)}$, the Gagliardo-Nirenberg inequality, (2.10) and (2.12) yield

$$|u_n|^{p+1}_{L^{p+1}\left(R^N\backslash a_{\frac{\delta}{4}}^{-1}(0)\right)} \le C|\nabla u_n|^{\theta(p+1)}_{L^2\left(R^N\backslash a_{\frac{\delta}{4}}^{-1}(0)\right)}|u_n|^{(1-\theta)(p+1)}_{L^2\left(R^N\backslash a_{\frac{\delta}{4}}^{-1}(0)\right)} \le C\left(\frac{\overline{m}_\lambda^i}{\lambda^{1-\theta}}\right)^{\frac{p+1}{2}}. \tag{2.13}$$

Similiarly, if $N=1,2$, let $s_{\varepsilon_0} > p$ be such that

$$\frac{1}{p+1} = \frac{\theta_{\varepsilon_0}}{s_{\varepsilon_0}+1} + \frac{1-\theta_{\varepsilon_0}}{2},$$

where $\varepsilon_0 > 0$ is a small constant, and $\theta_{\varepsilon_0} = 2(\frac{1}{2} - \frac{1}{p+1}) + \varepsilon_0$. Then we also have

$$|u_n|^{p+1}_{L^{p+1}\left(R^N\backslash a_{\frac{\delta}{4}}^{-1}(0)\right)} \le \left(\frac{\overline{m}_\lambda^i}{\lambda^{1-\theta_{\varepsilon_0}}}\right)^{\frac{p+1}{2}}. \tag{2.14}$$

Now, we prove this lemma by contradiction. Suppose that $\overline{m}_\lambda^i \le m_\lambda^i$. Then by Lemma 2.1, (2.13), (2.14) and the condition (1.2), it can be shown that as $\lambda \to +\infty$,

$$|u_n|^{p+1}_{L^{p+1}\left(R^N \backslash a^{-1}_{\frac{\delta}{4}}(0)\right)} = o(1) \tag{2.15}$$

and then using (2.11), we get

$$|u_n|^{p+1}_{L^{p+1}\left(a^{-1}_{\frac{\delta}{4}}(0)\right)} \ge \frac{C^*}{2} > 0 \tag{2.16}$$

for λ large enough. Since $g(u_n) \in \partial a^i_{\delta_0}$, (2.15), (2.16) imply that there is at least two points $a^{j_1}, a^{j_2} \in a^{-1}(0)$ and $C_1^* > 0, C_2^* > 0$ (independent of n, λ, δ) such that

$$|u_n|^{p+1}_{L^{p+1}\left(a^{j_1}_{\frac{\delta}{4}}\right)} \ge C_1^* > 0, \quad |u_n|^{p+1}_{L^{p+1}\left(a^{j_2}_{\frac{\delta}{4}}\right)} \ge C_2^* > 0$$

for λ large enough. In fact, if $\{|u_n|^{p+1}_{L^{p+1}}\}$ concentrates at any one zero point of $a(x)$ in the sense of (2.15) and (2.16), then by the definition of g, we know that $g(u_n) \in a^i_{\frac{\delta_0}{2}}$ or $g(u_n) \in R^N \backslash \overline{a}^i_{\delta_0}$ for λ large enough, a contradiction.

By (2.3), we assume (without loss of generality) that $\{|u_n|^{p+1}_{L^{p+1}}\}$ concentrates at a^i and a^j, i.e.

$$|u_n|^{p+1}_{L^{p+1}\left(a^i_{\frac{\delta}{4}}\right)} \ge C_1^* > 0, \quad |u_n|^{p+1}_{L^{p+1}\left(a^j_{\frac{\delta}{4}}\right)} \ge C_2^* > 0, \tag{2.17}$$

$$|u_n|_{L^{p+1}\left(R^N \backslash \left(a^i_{\frac{\delta}{4}} \cup a^j_{\frac{\delta}{4}}\right)\right)} = o(1) \quad \text{as} \quad \lambda \to +\infty. \tag{2.18}$$

Let $v_n^i = u_n \psi_i, v_n^j = u_n \psi_j$, where $\psi_i, \psi_j \in C^1(R^N)$ satisfy $0 \le \psi_i \le 1, 0 \le \psi_j \le 1$ and

$$\psi_i(x) = \begin{cases} 1, & x \in a^i_{\frac{\delta}{2}}, \\ 0, & x \in R^N \backslash a^i_\delta, \end{cases} \qquad \psi_j(x) = \begin{cases} 1, & x \in a^j_{\frac{\delta}{2}}, \\ 0, & x \in R^N \backslash a^j_\delta. \end{cases}$$

According to (2.10), Lemma 2.1 and the condition (1.2), we have

$$\int_{R^N \backslash a^{-1}_{\frac{\delta}{2}}(0)} |u_n \nabla u_n| \le C \frac{\overline{m_\lambda^i}}{\lambda^{\frac{1}{2}}} \le C \frac{m_\lambda^i}{\lambda^{\frac{1}{2}}} \le C \frac{\lambda^{\frac{2}{k+2}\left(\frac{p+1}{p-1} - \frac{N}{2}\right)}}{\lambda^{\frac{1}{2}}} = o(1) \ \text{ as } \lambda \to +\infty. \tag{2.19}$$

Then by (2.9),(2.18) and (2.19), one can easily prove that as $\lambda \to +\infty$,

$$A_i + A_j + B_i + B_j + C_{ij} = o(1), \tag{2.20}$$

where

$$\begin{aligned}
A_i &= \int_{R^N} |\nabla v_n^i|^2 + (\lambda a(x) + 1)|v_n^i|^2 - \int_{R^N} |v_n^i|^{p+1}, \\
A_j &= \int_{R^N} |\nabla v_n^j|^2 + (\lambda a(x) + 1)|v_n^j|^2 - \int_{R^N} |v_n^j|^{p+1}, \\
B_i &= 2\int_{R^N} \psi_i(1-\psi_i)|\nabla u_n|^2 + 2\int_{R^N} (\lambda a(x) + 1)\psi_i(1-\psi_j)u_n^2 \ge 0, \\
B_j &= 2\int_{R^N} \psi_j(1-\psi_j)|\nabla u_n|^2 + 2\int_{R^N} (\lambda a(x) + 1)\psi_j(1-\psi_j)u_n^2 \ge 0, \\
C_{ij} &= \int_{R^N} (1-\psi_i-\psi_j)^2(|\nabla u_n|^2 + (\lambda a(x) + 1)u_n^2)dx \ge 0.
\end{aligned}$$

If $A_i < o(1)$, there exists $t \in (0,1)$ such that $tv_n^i \in M_\lambda$, then $tv_n^i \in M_\lambda^i$. So by (2.18)–(2.20), we have that as $n \to \infty, \lambda \to +\infty$,

$$\begin{aligned} m_\lambda^i \geq \overline{m_\lambda^i} &= I_\lambda(u_n) + o(1) \\ &= \frac{1}{2}\int_{R^N} |\nabla u_n|^2 + (\lambda a(x)+1)u_n^2 - \frac{1}{p+1}\int_{R^N} |u_n|^{p+1} + o(1) \\ &= I_\lambda(v_n^i) + I_\lambda(v_n^j) + \frac{1}{2}B_i + \frac{1}{2}B_j + \frac{1}{2}C_{ij} + o(1) \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{R^N} |\nabla v_n^i|^2 + (\lambda a(x)+1)|v_n^i|^2 + \frac{1}{p+1}A_i \\ &\quad + \frac{1}{2}A_j + \frac{1}{2}B_i + \frac{1}{2}B_j + \frac{1}{2}C_{ij} + o(1) \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{R^N} |\nabla (tv_n^i)|^2 + (\lambda a(x)+1)|tv_n^i|^2 + \Big(\frac{1}{p+1} - \frac{1}{2}\Big)A_i + o(1) \\ &> m_\lambda^i; \end{aligned}$$

A contradiction is obtained.

If $A_j < o(1)$, we can prove that $m_\lambda^i \geq \overline{m_\lambda^i} > m_\lambda^j$ by above arguments; this contradicts (2.3).

If $A_i \geq o(1), A_j \geq o(1)$, by (2.20) we know that

$$A_i = o(1), \quad A_j = o(1), \quad B_i = o(1), \quad B_j = o(1), \quad C_{ij} = o(1).$$

Then there exists $t_\lambda = 1 + o(1)$ as $\lambda \to +\infty$, such that $t_\lambda v_n^i \in M_\lambda^i$. So

$$\begin{aligned} I_\lambda(v_n^i) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{R^N} |\nabla v_n^i|^2 + (\lambda a(x)+1)|v_n^i|^2 + o(1) \\ &\geq \frac{1}{t_\lambda^2} m_\lambda^i + o(1) \quad \text{as } \lambda \to +\infty. \end{aligned}$$

Similarly, $I_\lambda(v_n^j) \geq \frac{1}{t_\lambda^2} m_\lambda^j + o(1)$. Hence

$$\begin{aligned} m_\lambda^i \geq \overline{m_\lambda^i} &= I_\lambda(u_n) + o(1) = I_\lambda(v_n^i) + I_\lambda(v_n^j) + o(1) \\ &\geq \frac{m_\lambda^i + m_\lambda^j}{t_\lambda^2} + o(1) > m_\lambda^i \quad \text{as } \lambda \to +\infty, \end{aligned}$$

a contradiction. The proof of Lemma 2.2 is completed.

Now, one can apply Ekeland Variational principle to the closed set $M_\lambda^i \cup O_\lambda^i$ to set a minimizing sequence of m_λ^i which is a (PS) sequence.

Lemma 2.3. *Assume the conditions of Theorem* 1.1 *hold. Then for* $i = 1, \cdots, K$, *there exists* $\lambda_0 \geq 1$ *such that for each* $\lambda > \lambda_0$, m_λ^i *has a minimizing sequence* $\{u_n\} \subset M_\lambda^i$ *satisfing* $u_n \geq 0$ *and*

$$I_\lambda(u_n) \to m_\lambda^i, \quad I'_\lambda(u_n) \to 0 \quad \text{in } H^{-1} \quad \text{as } n \to \infty.$$

Proof. The proof of this lemma is similiar to [5], so it is omitted.

§3. Proof of the Results

Proof of Theorem 1.1 and Corollary 1.1. Let $\{u_n^i\}$ be the minimizing sequence of $m_\lambda^i (1 \leq i \leq K)$ obtained in Lemma 2.3, i.e. $\{u_n^i\} \subset M_\lambda^i$ satisfying

$$\begin{cases} I_\lambda(u_n^i) \to m_\lambda^i, \\ I'_\lambda(u_n^i) \to 0 \ \text{ in } \ H^{-1} \end{cases} \quad \text{as } \ n \to +\infty. \tag{3.1}$$

So as (2.10), we have

$$\int_{R^N} |\nabla u_n^i|^2 + (\lambda a(x)+1)|u_n^i|^2 \leq \frac{2(p+1)}{p-1} m_\lambda^i + o(1) \quad \text{as } n \to \infty. \tag{3.2}$$

Then there exists $u^i \in E$ and a subsequence of $\{u_n^i\}$ (still denoted by u_n^i) such that

$$u_n^i \to u^i \quad \text{weakly in} \quad E.$$

Thus one may check that u^i is a solution of Equation (1.1), and the result of Theorem 1.1 follows if we can show that $u^i \not\equiv 0$ and $u^i \neq u^j$ for $i \neq j, 1 \leq i,j \leq K$ for λ large enough. In fact, by the same arguments as in Lemma 2.2, we know that $|u_n^i|_{L^{p+1}}$ concentrate at a^i, i.e. for any $0 < \delta < \frac{\delta_0}{4}$, as $\lambda \to +\infty$

$$\begin{cases} |u_n^i|^{p+1}_{L^{p+1}(R^N \backslash a_\delta^i)} = o(1), \\ |u_n^i|^{p+1}_{L^{p+1}(a_\delta^i)} \geq \frac{C^*}{2} > 0, \end{cases} \tag{3.3}$$

where C^* is independent of n, λ and δ, a_δ^i is defined in Section 2. This implies that $u^i \not\equiv 0$ and $u^i \neq u^j$ for λ large enough by making use of the Sobolev imbeding theorem. The proof of Therem 1.1 is completed.

Forthermore, by using Brezis-Lieb Lemma[7] to (3.2),(3.3), we get

$$|u^i|^{p+1}_{L^{p+1}(R^N \backslash a_\delta^i)} = o(1) \quad \text{as } \lambda \to +\infty, \tag{3.4}$$

$$\int_{R^N} |\nabla u^i|^2 + (\lambda a(x)+1)|u^i|^2 \leq \frac{2(p+1)}{p-1} m_\lambda^i. \tag{3.5}$$

Then by Lemma 2.1 and (2.19), we have

$$\int_{R^N \backslash a^i_{\frac{\delta}{2}}} |\nabla u^i \cdot u^i| = o(1) \quad \text{as } \lambda \to +\infty. \tag{3.6}$$

Let ψu^i be a test-function for Equation (1.1), where $0 \leq \psi \leq 1$, $\psi = 1$ in $R^N \backslash a^i_{\frac{3\delta}{4}}$, $\psi = 0$ in $a^i_{\frac{\delta}{2}}$ and $|\nabla \psi| < \frac{4}{\delta}$. Then by (3.4),(3.6), we get

$$\int_{R^N \backslash a^i_{\frac{3\delta}{4}}} |\nabla u^i|^2 + |u^i|^2 = o(1) \quad \text{as } \lambda \to +\infty.$$

Note that u^i is a subsolution of $\Delta u + c(x)u = 0$ with $c(x) = |u^i|^{p-1}$. By the one-sided Harnack inequality and the Sobolev inequality (see [2, 8]), we have

$$\max_{R \backslash a_\delta^i} |u^i| \leq C \Big(\int_{R^N \backslash a^i_{\frac{3\delta}{4}}} |u^i|^{2^*} \Big)^{\frac{1}{2^*}} \leq C \int_{R^N \backslash a^i_{\frac{3\delta}{4}}} |\nabla u^i|^2 + |u^i|^2 = o(1) \quad \text{as } \lambda \to +\infty,$$

where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* \in (1, +\infty)$ for $N = 1, 2$. So

$$\max_{R^N \backslash a_\delta^i} |u^i| \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

Since C^* in (3.3) is independent of δ,

$$\max_{a_\delta^i} u^i \to +\infty \quad \text{as} \quad \lambda \to \infty.$$

This completes the proof of Corollary 1.1.

Remark. Suppose (a_2) holds and $a \geq 0, a^{-1}(0) = B_1(0)$. Then by the proof of Theorem 1.1 and Corollary 1.1, we know that

$$m_\lambda^i \leq C < +\infty,$$

and $u_\lambda \to u_0$ in $C^2_{\rm loc}(R^N)$ as $\lambda \to +\infty$. Here $u_0 \equiv 0$ in $R^N \backslash B_1(0)$ and satisfies

$$\begin{cases} -\Delta u_0 + u_0 = u_0^p, \\ u_0|_{\partial B_1(0)} = 0, \quad u_0 > 0 \quad \text{in}\, B_1(0). \end{cases}$$

Moreover, by the results of [2], u_λ has only one local maximum point x_λ such that

$$x_\lambda \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

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