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SMALLEST g-SUPERSOLUTION FOR BSDE WITH CONTINUOUS DRIFT COEFFICIENTS**

LIN Qingquan* PENG Shige*

Abstract

The authors prove the existence and uniqueness of smallest g-supersolution with an equality constraint on (y,z) for one demensional stochastic differential equations whose drift coefficients are continuous and linearly growing, and whose terminal conditions are square integrable.

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§1. Introduction

Since the result of monotonic limit theorem of BSDE and its application to nonlinear Doob-Meyer decomposition theorem, the smallest g-supersolution with a constraint on (Y,Z) (see [1]), some developments have been done, for example, in [4], where by a penalization method, Chen and Peng discussed nonlinear Doob-Meyer decomposition theorem with the BSDE introduced by Duffie and Epstein[3]. In [5] Lin discussed the smallest g-supersolution to BSDE with a constraint on (y,z) with non-Lipschitz condition imposed on the drift coefficient.

In this work, we suppose that the drift coefficient is linearly growing and continuous in (y,z), and the terminal condition is square integrable, which is the same as in [2]. Under these hypotheses we prove the existence and uniqueness of smallest g-supersolution for a one demensional BSDE with the constraint, $\phi(s,y,z)=0$, with ϕ satisfying Lipschitz condition on (y,z).

Let $(\mathbf{\Omega},\mathcal{F},\mathbf{P})$ be a probability space, $W_t, t>0$ be a d-dimensional standard Brownian motion, $\{\mathcal{F}_t\}_{0\le t\le T}$ be a σ-filtration generated by W_t, where T is a finite constant. We set

$$\mu^2(0,T,R)=\Big\{X:[0,T]\times\Omega\to R;\quad X\in\mathcal{P},\quad \|X\|^2=\mathbf{E}\int_0^T|X_s|^2ds<\infty\Big\},$$

where $\mathcal{P}$ is the set of all predictable processes. Our main result is

Theorem 1.1. *Assume that* $g:\Omega\times[0,T]\times R\times R^d\to R$ *is* $\mathcal{P}\times\mathcal{B}(R^{1+d})$ *measurable and satisfies*

(1) *Linear growth*: $\exists K<\infty,\forall\ \omega,t,y,z\in\Omega\times R\times R^d$,

$$|g(\omega,t,y,z)|\le K(1+|y|+|z|), \tag{H$_1$}$$

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*School of Mathematics and System Science, Shandong University, Jinan 250100, China.
E-mail: Linqingquan@math.sdu.cn; Peng@sdu.edu.cn

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(2) *For fixed* $\omega,\ t, g(\omega,t,\cdot,\cdot)$ *is continuous,* (H_2)

(3) *There exists a g-supersolution* $\widehat{Y}$ *on* $[0,T]$ *with* $\mathbf{E}\sup_{0\le t\le T}|\widehat{Y}_t|^2<\infty$, *terminal condition* $Y_T=\xi$, *and* $(\widehat{Z}_t,\widehat{A}_t)$ *as its decomposition such that* $(\widehat{Y},\widehat{Z})$ *satisfies the constraint*

$$\phi(t,\widehat{Y}_t,\widehat{Z}_t)=0 \quad \text{a.s., a.e.,}$$

i.e. the triple $(\widehat{Y},\widehat{Z},\widehat{A})$ *satisfies the following BSDE*:

$$\widehat{Y}_t=\xi+\int_t^T g(s,\widehat{Y}_s,\widehat{Z}_s)ds+\widehat{A}_T-\widehat{A}_t-\int_t^T \widehat{Z}_s dW_s$$

and $\phi(s,\widehat{Y}_s,\widehat{Z}_s)=0$.

Then there exists a smallest g-supersolution with the constraint (y,z), i.e. there exists an RCLL process Y_t with its decomposition (A_t,Z_t) satisfying following BSDE

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)ds+A_T-A_t-\int_t^T Z_s dW_s \tag{1.1}$$

such that

$$\phi(t,Y_t,Z_t)=0. \tag{1.2}$$

If there exists another solution (Y',Z') for BSDE (1.1) *with the decomposition (A',Z') and* (1.2) *holds, then* $Y'\ge Y$ a.e.

Here, A_t is an RCLL increasing processes with $\mathbf{E}A_T^2<\infty,\phi:\Omega\times[0,T]\times R\times R^d\to R_+$ being a given nonnegative function such that, for each $(y,z)\in R^{1+d},\phi(\cdot,y,z)\in\mu^2(0,T,R_+)$ and ϕ is globally Lipschitz with respect to (y,z).

Remark 1.1. The condition (3) is necessary (cf. Example 3.1 in this paper).

Since the drift coefficient of BSDE (1.1) is only continuous in (y,z), and linearly growing, there is no uniqueness solution for the BSDE in general. An example can be found in [5]. We must extend the concept of g-supersolution and the concept of smallest g-supersolution with the constraint on (y,z) later. The difficulty is that actually there is no comparison theorem and monotonic limit theorem available for such BSDE. To overcome this difficulty, we first construct a sequence of solution for the BSDE, then, in order to get the smallest g-supersolution with the constraint on (y,z), we use a suitable approximation method, by introducing a two index g-supersolution sequence, then obtain an increasing g-supersolution (with only one index). Lastly, by the method of "Weak-convergence" introduced by Peng[1], we have the conclusion. The method with which we study this problem can be viewed as a combination of "strong" convergence and "weak" convergence.

§2. Some Preliminary Results

In this section we prove the the existence of solution for BSDE (1.1) with continuous drift coefficient, where g,A,ξ are given.

Theorem 2.1. *Suppose that* (H_1),(H_2) *hold for a given g, then there exists a solution for BSDE* (1.1), *where A_t is a given RCLL and increasing process, $\xi\in L^2(\Omega,\mathcal{F}_T,P)$. (In fact, the solution is the minimal solution for the BSDE* (1.1).)

For the purpose, we need the following several propositions.

Proposition 2.1. *Let $g:R^p\to R$ be a continuous function with linear growth, that is, there exists a constant $K<\infty$, such that, $\forall x\in R^p,|g(x)|\le K(1+|x|)$. Then the sequence of functions*

$$g_n(x)=\inf_{y\in Q^p}\{g(y)+n|x-y|\} \tag{2.1}$$

is well defined for $n>K$ and it satisfies

- *Linear growth*: $\forall x \in R^p, |g_n(x)| \le K(1+|x|)$;
- *Monotonicity*: $\forall x \in R^p, g_n(x) \uparrow$;
- *Lipschitz condition*: $\forall x \in R^p, |g_n(x) - g_n(y)| \le n|x-y|$;
- *Strong convergence*: *if* $x_n \to x$, *then* $g_n(x_n) \to g(x)$ *as* $n \to \infty$;

where $|x| = \sum_{i=1}^{p} |x_i|$ *(see* [2] *for the proof).*

Proposition 2.2. *Let* g_n *be defined as in* (2.1), *and* A_t, ξ *satisfy the conditions of Theorem* 2.1. *Then there exists a unique pair of processes* $(Y_t^n, Z_t^n) \in \mu^2(0,T,R^{1+d})$, *which is the solution of the following BSDE*

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n)ds + A_T - A_t - \int_0^T Z_s^n dW_s. \tag{2.2}$$

Proof. For fixed n, by Proposition 2.1, the coefficient of BSDE (2.1) g_n satisfies Lipschitz condition with the constant n. By Prosition 1.1 in [1], the result follows.

Proposition 2.3. *There exists a constant* C *depending only on* $K, T, E\xi^2$ *and* EA_T^2, *such that* $\forall n \ge K$, $\|Y^n\| \le C$, $\|Z^n\| \le C$, *where* $\|Y^n\|^2 = E\int_0^T |Y_s^n|^2 ds$.

Proof. Consider $h(t,\omega,y,z) = K(1+|y|+|z|)$, then h is $\mathcal{P}\times\mathcal{B}(R^{1+d})$ measurable function, and satisfies Lipschitz condition. Since $\xi \in L^2(\Omega, \mathcal{F}_T, P), A_t$ is RCLL with $\mathbf{E}A_T^2 < \infty$, we have, by [1], that there exists a unique pair of processes $(U_t, V_t) \in \mu^2(0,T,R^{1+d})$ of solution for the following BSDE

$$U_t = \xi + \int_t^T h(s, U_s, V_s)ds + A_T - A_t - \int_t^T V_s dW_s, \tag{2.3}$$

and there exists a constant C depending only on K, T and $E\xi^2, EA_T^2$, such that $E\sup_{0\le t\le T}|U_t|^2 \le C$, $E\int_t^T |V_t|^2 dt \le C$. By comparison theorem we obtain that, $\forall n \ge m \ge K, Y^m \le Y^n \le U$, so

$$\mathbf{E}\sup_{0\le t\le T}|Y_s^n|^2 \le \mathbf{E}\sup_{0\le t\le T}|U|^2 \le C, \quad \mathbf{E}\int_0^T |Y_s^n|^2 ds \le C.$$

By Itô's formula

$$(Y_T^n)^2 - (Y_t^n)^2 = 2\int_t^T Y_s^n dY_s^n + \int_t^T |Z_s^n|^2 ds,$$

$$\mathbf{E}|Y_t^n|^2 + \mathbf{E}\int_t^T |Z_s^n|^2 ds = \mathbf{E}\xi^2 + 2\mathbf{E}\int_t^T Y_s^n g_n(s, Y_s^n, Z_s^n)ds + 2\mathbf{E}\int_t^T Y_s^n dA_s,$$

$$\Big|2\mathbf{E}\int_t^T Y_s^n g_n(s, Y_s^n, Z_s^n)ds\Big| \le (2K + la^2)\mathbf{E}\Big(\int_t^T |Y_s^n|^2 ds + \frac{2KT}{\lambda^2} + \frac{2K}{\lambda^2}\int_t^T |Z_s^n|^2 ds\Big),$$

$$\Big|2\mathbf{E}\int_t^T Y_s^n dA_s\Big| \le 2\mathbf{E}\sup_{0\le t\le T}|Y_s^n| A_T \le \mathbf{E}\sup_{0\le t\le T}|Y_s^n|^2 + \mathbf{E}A_T^2.$$

So

$$\mathbf{E}\int_t^T |Z_s^n|^2 ds \le \mathbf{E}\xi^2 + \mathbf{E}\sup_{0\le t\le T}|Y_s^n|^2 + \mathbf{E}\Big(A_T^2 + \frac{2K}{\lambda^2}\int_t^T |Y_s^n|^2 ds + \lambda^2\int_t^T |Y_s^n|^2 ds + \frac{2K}{\lambda^2}\int_t^T |Z_s^n|^2 ds\Big) + \frac{2KT}{\lambda^2},$$

$$\Big(1 - \frac{2K}{\lambda^2}\Big)\mathbf{E}\int_t^T |Z_s^n|^2 ds \le \mathbf{E}\xi^2 + \mathbf{E}\Big(\sup_{0\le t\le T}|Y_s^n|^2 + A_T^2 + \Big(\frac{2K}{\lambda^2} + \lambda^2\Big)\int_t^T |Y_s^n|^2 ds\Big) + \frac{2KT}{\lambda^2}.$$

Taking $\lambda^2 > 2K$, we have $E\int_0^T |Z_s^n|^2 ds \le C'$.

Proposition 2.4. *(Y^n, Z^n) converges in $\mu^2(0,T,R^{1+d})$.*

Proof. Take $n_0 \ge K$. Since, by comparision theorem, (Y^n) is increasing and bounded by U, Y^n converges. Denote by Y the limit of Y^n. By dominated convergence theorem, Y^n also converges in $\mu^2(0,T,R)$. Set

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n) ds + A_T - A_t - \int_t^T Z_s^n dW_s,$$
$$Y_t^m = \xi + \int_t^T g_m(s, Y_s^m, Z_s^m) ds + A_T - A_t - \int_t^T Z_s^m dW_s,$$
$$Y_t^n - Y_t^m = \int_t^T [g_n(s, Y_s^n, Z_s^n) - g_m(s, Y_s^m, Z_s^m)] ds - \int_t^T (Z_s^n - Z_s^m) dW_s.$$

Now using Ito's formula and taking $n, m \ge n_0$, we have

$$\begin{aligned}
&\mathbf{E}|Y_0^n - Y_0^m|^2 + \mathbf{E}\int_0^T |Z_s^n - Z_s^m|^2 ds \\
&= 2\mathbf{E}\int_0^T (Y_s^n - Y_s^m)(g_n(s, Y_s^n, Z_s^n) - g_m(s, Y_s^m, Z_s^m)) ds \\
&= 2\Big(\mathbf{E}\int_0^T |Y_s^n - Y_s^m|^2 ds\Big)^{\frac{1}{2}} \Big(\mathbf{E}\int_0^T |g_n(s, Y_s^n, Z_s^n) - g_m(s, Y_s^m, Z_s^m)|^2 ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

Using the uniform linear growth condition on sequence (g_n) and the fact that Y^n, Z^n are bounded in $\mu^2(0,T,R^{1+d})$, we know that $\{Z_s^n\}$ is a Cauchy sequence in $\mu^2(0,T,R^d)$.

By Propositions 2.1 to 2.4 and Theorem 1 in [2], taking limits on m and supremum over t we get

$$\begin{aligned}
\sup_{0\le t\le T} |Y_t^n - Y_t| \le & \int_0^T |g_n(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)| ds \\
& + \sup_{0\le t\le T} \Big| \int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s \Big| \quad P-\text{a.s.}
\end{aligned}$$

From this we deduce that Y^n converges uniformly in t to Y (in particular Y is RCLL process). Remember that Y^n is monotone; therefore we actually have the uniform convergence for the entire sequence and not just for a subsequence. Taking limits in the following BSDE:

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n) ds + A_T - A_t - \int_t^T Z_s^n dW_s, \ \ n \ge K,$$

we deduce that (Y, Z) is an adapted $\mu^2(0,T,R^{1+d})$ solution of (1.1).

Definition 2.1. *Let (Y, Z) be a solution for BSDE (1.1). If for any solution $(\widehat{Y}, \widehat{Z})$ of BSDE (1.1), we have $Y \le \widehat{Y}$, then we call Y the minimal solution for BSDE (1.1).*

Let $(\widehat{Y}, \widehat{Z})$ be any solution of BSDE (1.1) in $\mu^2(0,T,R^{1+d})$. By comparison theorem we have that $\forall n, Y^n \le \widehat{Y}$ and therefore $Y \le \widehat{Y}$. Thus Y is the minimal solution.

Now we introduce the definition of g-supersolution for BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + A_T - A_t - \int_t^T Z_s dW_s.$$

Since the minimal solution is unique for BSDE (1.1) with continuous drift coefficient and linear growth, we can extend the concept of g-supersolution for the case. Also thanks to the existence and uniqueness of minimal solution for the BSDE (1.1) under these condition, we have the proposition of uniqueness of g-supersplution decomposition.

Let BSDE be as follows:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} g(s, Y_s, Z_s)ds + A_\tau - A_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} Z_s dW_s, \quad 0 \le t \le T, \tag{2.4}$$

where τ is a given stopping time, $\xi \in L^2(\Omega, \mathcal{F}_\tau, P), A$ is a given RCLL increasing process with $A_0 = 0$ and $\mathbf{E}A_\tau^2 < \infty$.

Definition 2.2. *If Y_t is a minimal solution for BSDE of* (2.4)*, then we call Y_t a g-supersolution on* $[0,\tau]$*. If $A_t = 0$ in* $[0,\tau]$*, we call Y_t a g-solution on* $[0,\tau]$.

Proposition 2.5. *Given a g-supersolution Y_t on $[0,\tau]$, there exists a unique $Z_t \in \mu^2(0,T,R^d)$ and a unique increasing RCLL process A_t on $[0,\tau]$ with $A_0 = 0$ and $\mathbf{E}A_\tau^2 < \infty$ such that the triple (Y_t, Z_t, A_t) satisfies* (2.3).

Proof. Suppose (Y_t, Z_t, A_t) and (Y_t, Z_t', A_t') satisfy (2.4). We apply Itô's formula to $(Y_t - Y_t)^2 (\equiv 0)$ on $[0,\tau]$ and take expectation

$$\mathbf{E}\int_0^\tau |Z_t - Z_t'|^2 dt + \mathbf{E}\Big[\sum_{t\in(0,\tau]} (\triangle(A_t - A_t'))^2\Big] = 0. \tag{2.5}$$

Thus $Z_t \equiv Z_t'$. From this $A_t \equiv A_t'$.

Definition 2.3. *Let Y_t be a supersolution on $[0,\tau]$ and let (Y_t, Z_t, A_t) be the related unique triple in the sence of Proposition* 2.5*. Then we call (A_t, Z_t) the unique decomposition of Y_t.*

Definition 2.4. *Let Y be a g-supersolution on $[0,\tau]$ with the decomposition (A_t, Z_t), Y' be any g-supersolution on $[0,\tau]$ with the decomposition (A_t', Z_t'). If $Y \le Y'$, we call Y the smallest g-supersolution on $[0,\tau]$.*

Let Y^i be a sequence of g-supersolutions on $[0,T]$,

$$Y_t^i = \xi + \int_t^T g(s, Y_s^i, Z_s^i)ds + A_T^i - A_t^i - \int_t^T Z_s^i dW_s, \quad i = 1, \cdots, \quad 0 \le t \le T, \tag{2.6}$$

where g satisfies the condition in Theorem 2.1, $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, and for every i, A_t^i is a continuous increasing process with $\mathbf{E}{A_T^i}^2 < \infty$. By virtue of Theorem 2.1, there exists a unique minimal solution $(Y^i, Z^i) \in \mu^2(0,T,R^{1+d})$ for BSDE (2.6).

Suppose that Y_t^i converges to Y_t inceasingly with $\mathbf{E}\sup\limits_{0\le t\le T}|Y_t|^2 < \infty$. Obviously

$$\mathbf{E}\sup_{0\le t\le T}|Y_t^i|^2 \le C, \quad \mathbf{E}\int_0^T |Y_s^i - Y_s|^2 ds \to 0,$$

where C is independent of i.

Proposition 2.6. *Let $\{Y_t^i\}$ and $\{A_t^i\}$ be defined as before. Then there exists a constant C, which is independent of i, such that*

$$E\int_t^T |Z_s^i|^2 ds \le C, \quad E|A_T^i|^2 \le C. \tag{2.7}$$

Proof. Since

$$\begin{aligned}
|A_T^i|^2 &= \Big|Y_0^i - \xi - \int_0^T g(s, Y_s^i, Z_s^i)ds + \int_0^T Z_s^i dW_s\Big|^2 \\
&\le C\Big\{|Y_0^i|^2 + |\xi|^2 + \Big|\int_0^T g(s, Y_s^i, Z_s^i)ds\Big|^2 + \Big|\int_0^T Z_s^i dW_s\Big|^2\Big\},
\end{aligned}$$

$$\Big|\int_0^T g(s, Y_s^i, Z_s^i)ds\Big|^2 \le K\int_0^T (1 + |Y_s^i|^2 + |Z_s^i|^2)ds,$$

we have

$$|A_T^i|^2 \le C\Big\{|Y_0^i|^2 + |\xi|^2 + KT + \int_0^T (|Y_s^i|^2 + K|Z_s^i|^2)ds + \Big|\int_0^T Z_s^i dW_s\Big|^2\Big\},$$

$$\mathbf{E}|A_T^i|^2 \le D + C_T\mathbf{E}\int_0^T |Z_s^i|^2 ds, \tag{2.8}$$

where D, C_T are constants. By Itô's formula, we have

$$\begin{aligned}
&\mathbf{E}|Y_t^i|^2 + \mathbf{E}\int_t^T |Z_s^i|^2 ds \\
&\le \mathbf{E}|\xi|^2 + \mathbf{E}\int_0^T (\alpha^{-1}|Y_s^i|^2 + \alpha|g_s^i|^2)ds + 2\mathbf{E}\sup_{0\le t\le T}|Y_s^i||A_T^i| \\
&\le Q + 4C_T\mathbf{E}\sup_{0\le t\le T}|Y_s^i|^2 + \alpha K\mathbf{E}\int_0^T (1 + |Y_s^i|^2 + |Z_s^i|^2)ds + \frac{1}{4C_T}\mathbf{E}|A_T^i|^2 \\
&\le \frac{1}{4C_T}\mathbf{E}|A_T^i|^2 + 2\alpha K\int_0^T |Z_s^i|^2 ds + Q_1.
\end{aligned}$$

Taking $\alpha = \frac{1}{4K}$, we have

$$\frac{1}{2}E\int_0^T |Z_s^i|^2 ds \le \ell_1 + \frac{1}{4C_T}E|A_T^i|^2, \tag{2.9}$$

where Q, ℓ, Q_1 are constants. By (2.7) and (2.8) we have proved this proposition.

Proposition 2.7. *Let* (H_1), (H_2) *hold, and suppose that* A_t^i *is continuous and increasing.* Y_t^i *increasingly converges to* Y_t *with* $\mathbf{E}\sup_{0\le t\le T}|Y_t|^2 < \infty$. *Then there exists a process* $Z \in \mu^2(0, T, R^d)$ *and an RCLL square integrable increasing process* A_t*, such that* (Y_t, Z_t) *satisfies the following equation*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s, \quad 0 \le t \le T, \tag{2.10}$$

where $\{Z_s\}$ *is the limit of* $\{Z_s^i\}, A_t$ *is the limit of* $\{A_t^i\}, dt\times dP$.

Proof. Since $\{Y^i\}, \{Z^i\}$ are bounded in $\mu^2(0, T, R^{1+d})$, and $|g(s, Y^i, Z^i)| \le C(1 + |Y^i| + |Z^i|)$, which implies g is bounded in $\mu^2(0, T, R)$, there exist subsequences which converge to $\{Z, g^o\}$ weakly, for simplicity, we still denote all the corresponding subsequence by $\{Y^i, Z^i, g(\cdot, Y^i, Z^i), A^i\}$. For any t, the following weak convergence holds in $L^2(\Omega, \mathbf{F}_t, P)$,

$$\int_0^t Z_s^i dW_s \to \int_0^t Z_s dW_s, \quad \int_0^t g_s^i ds \to \int_0^t g_s^0 dW_s,$$

$$A_t^i \to A_t \equiv -Y_t + Y_0 - \int_0^t g_s^0 ds + \int_0^t Z_s dW_s,$$

A_t is an increasing process, A_t and Y_t are RCLL. Furthermore, by monotonic limit theorem of BSDE (see [1]) Z^i converges to Z strongly in $\mu^p(0, T, R^d), p \in [1, 2)$ (see [1]).

Since Y^n converges in $\mu^2(0, T, R)$ and $dt \otimes dP$ to $Y \in \mu^2(0, T, R)$, it follows that $G = \sup_n |Y^n|$ is $dt \otimes dP$ integrable.

On the other hand, since $Z^n \to Z$ in $\mu^p(0, T, R^d)$, passing to a subsequence if necessary, we have that $Z^n \to Z$ $dt \otimes dP$ and Z^n is bounded uniformly in $\mu^2(0, T)$. Set $H = \sup_n |Z^n|$, which is $ds \otimes dP$ integrable. Therefore, we get for almost all ω,

$$g(s, Y_t^n, Z_t^n) \to g(s, Y_t, Z_t) \ \ dt - \text{a.e.},$$

$$|g(s, Y_t^n, Z_t^n)| \le K(1 + \sup_n |Y_t^n| + \sup_n |Z_t^n|) = K(1 + G_t + H_t) \in L^1([0, T], dt).$$

Thus, for almost all ω and uniformly in t,

$$\int_t^T g(s, Y_s^n, Z_s^n)ds \to \int_t^T g(s, Y_s, Z_s)ds.$$

From the continuity property of the stochastic integral we get

$$\sup_{0 \le t \le T} \Big| \int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s \Big| \xrightarrow{P} 0.$$

Passing again to a subsequence we can assume that the last convergence is P-a.s. That is, for almost all ω,

$$A_t^i \to A_t \equiv -Y_t + Y^0 - \int_0^t g(s, Y_s, Z_s)ds + \int_0^t Z_s dW_s, \qquad dt \text{ a.e.}$$

The proof is complete.

§3. The Proof of Theorem 1.1

In this section we prove the main result of this paper. Before proving Theorem 1.1, we give an example to indicate that the condition (3) in Theorem 1.1 is needed.

Example 3.1. Suppose that y_{*t}, y_t^* are two given processes and $y_{*t} < y_t^*$. For the following two BSDEs

$$-dY_t = [f_0(Y_t) + g(Z_t)]dt + dA_t - Z_t dW_t, \quad Y_T = \xi, \tag{$*$}$$

$$-d\overline{Y}_t = [f_0(Y_t) + g(\overline{Z}_t)]dt - \overline{Z}_t dW_t, \quad \overline{Y}_0 = y_0^*, \tag{$**$}$$

where $dA_t \ge 0, A_0 = 0, A_T \in L^2(\Omega, \mathcal{F}_T, P)$, and $f_0(t, y), g(t, z) \in L^2(0, T)$ for any $(y, z) \in R^2$ satisfying Lipschitz condition.

Consider the solutions for the two BSDEs in $[y_{*t}, y_t^*]$ and $\overline{Z} \in K = \{Z_t, \phi(Z_t) = 0\}$.

(1) If $\xi \ge \overline{Y}_T$, and $P(\xi > \overline{Y}_T) > 0$, by strict comparison theorem there is no g-supersolution in $[y_{*t}, y_t^*]$ for BSDE($*$).

(2) If $\xi \le \overline{Y}_T$, then there exists a g-supersolution for BSDE($*$) satisfying the constraint.

Proof of Theorem 1.1. We investigate the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + i\int_t^T \phi(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{3.1}$$

where g is as in Theorem 1.1, $\phi(s, \cdot, \cdot)$ is assumed to be a globally Lipschitz condition with respet to (y, z) and $\phi(s, \cdot, \cdot) \ge 0$.

Consider BSDE:

$$Y_t^i(n) = \xi + \int_t^T g_n(s, Y_s^i(n), Z_s^i(n))ds + i\int_t^T \phi(s, Y_s^i(n), Z_s^i(n))ds - \int_t^T Z_s^i(n)dW_s, \tag{3.2}$$

where

$$g_n(x) = \inf_{y \in Q^p} \{g(y) + n|x - y|\}. \tag{3.3}$$

For any n, by Proposition 2.1, g_n satisfies Lipschitz condition and other three properties. From Theorem 2.1 there exists (Y^i, Z^i) satisfying

$$Y_t^i = \xi + \int_t^T g(s, Y_s^i, Z_s^i)ds + i\int_t^T \phi(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dW_s.$$

Notice: Y^i is minimal solution for the above BSDE, and by comparison theorem,

$$\mathbf{E}\sup_{0\le t\le T}|Y_t^i|^2 \le \mathbf{E}\sup_{0\le t\le T}|\widehat{Y}_t|^2 < \infty.$$

Since for any $i, n, Y^i(n) \le Y^{i+1}(n)$, it is obvious that Y^i is increasing and bounded with respect to i. Suppose $Y^i \uparrow Y$. We have

$$\lim_{i\to\infty}\mathbf{E}\int_0^T |Y_s^i - Y_s|^2 ds \to 0, \quad \mathbf{E}\sup_{0\le t\le T}|Y_t|^2 < \infty.$$

Set $A_t^i = i\int_0^t \phi(s, Y_s^i, Z_s^i)ds$. For any fixed i, $\mathbf{E}|A_T^i|^2 < \infty$. From Proposition 2.6, there exists a constant C such that $\mathbf{E}|A_T^i|^2 \le C$, $\mathbf{E}\int_t^T |Z_s^i|^2 ds \le C$. By Proposition 2.7 there exist Z and A, such that (Y, Z, A) satisfy the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s, \tag{3.4}$$

where A_t is an RCLL increasing process with $\mathbf{E}A_T^2 < \infty$. Since $\mathbf{E}A_T^2 \le C$,

$$\mathbf{E}\int_0^T \phi(s, Y_s^i, Z_s^i)ds \le \frac{C}{i}.$$

We also have $Z^i \to Z$ in $\mu^p(0, T, R^d)$, for $p \in [1, 2)$. Then it follows that

$$\phi(s, Y_s^i, Z_s^i) \to \phi(s, Y_s, Z_s), \quad \phi(s, Y_s, Z_s) = 0$$

in strong sense.

Suppose $\overline{Y}$ is any g-supersolution with its decomposition $(\overline{Z}, \overline{A})$, and the following BSDE holds:

$$\overline{Y}_t = \xi + \int_t^T g(s, \overline{Y}_s, \overline{Z}_s)ds + \overline{A}_T - \overline{A}_t - \int_t^T \overline{Z}_s dW_s$$

with $\phi(s, \overline{Y}, \overline{Z}) = 0$. By comparison theorem we have $Y_s^i(n) \le \overline{Y}$, where $(Y^i(n), Z^i(n))$ solves

$$Y_t^i(n) = \xi + \int_t^T g_n(s, Y_s^i(n), Z_s^i(n))ds + i\int_t^T \phi(s, Y_s^i(n), Z_s^i(n))ds - \int_t^T Z_s^i(n)dW_s,$$

$Y_s^i \le \overline{Y}_s$, and finally $Y_s \le \overline{Y}_s$. So Y is the smallest g-supersolution .

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