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THE MINIMAL CLOSED NON-TRIVIAL IDEALS OF TOEPLITZ ALGEBRAS ON DISCRETE GROUPS**

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Abstract

Let G be a discrete group and (G, G_+) an ordered group. Let (G, G_F) be the minimal quasi-ordered group containing (G, G_+). Let $\mathcal{T}^{G_+}(G)$ and $\mathcal{T}^{G_F}(G)$ be the corresponding Toeplitz algebras, and γ^{G_F,G_+} the natural C^*-algebra morphism from $\mathcal{T}^{G_+}(G)$ to $\mathcal{T}^{G_F}(G)$. This paper studies the connection between $\operatorname{Ker}\gamma^{G_F,G_+}$ and the minimal closed ideal of $\mathcal{T}^{G_+}(G)$. It is proved that if G is amenable and $G_F \neq G_+$, then $\operatorname{Ker}\gamma^{G_F,G_+}$ is exactly the minimal closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$. As an application, in the last part of this paper, a character of K-groups of Toeplitz algebras on ordered groups is clarified.

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§1. Introduction

Let G be a discrete (not necessarily abelian) group. For any subset G_+ of G, we say that (G, G_+) is a quasi-partial ordered group if $e \in G_+, G_+ \cdot G_+ \subseteq G_+$ and $G = G_+ \cdot G_+^{-1}$, where e is the unit of G and $G_+^{-1} = \{ g^{-1} \,|\, g \in G_+ \}$; further, (G, G_+) is referred to as a quasi-ordered group if $G = G_+ \cup G_+^{-1}$. Note that when $G_+^0 = G_+ \cap G_+^{-1} = \{e\}$, a quasi-partial ordered group (resp. quasi-ordered group) (G, G_+) is known as a partially ordered (resp. ordered) group.

Let $\{ \delta_g \,|\, g \in G \}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise} \end{cases}$$

for $g, h \in G$. For any $g \in G$, we define a unitary u_g on $\ell^2(G)$ by $u_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any $E \subseteq G$, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{ \delta_g \,|\, g \in E \}$; its projection is denoted by p^E. The C^*-algebra generated by $\{ p^E u_g p^E \,|\, g \in G \}$ is denoted by $\mathcal{T}^E(G)$ and is called the Toeplitz algebra with respect to E.

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Now let (G, G_+) be an ordered group. For any $g \in G_+ \setminus \{e\}$, let G_g be the semigroup of G generated by G_+ and g^{-1}. Let

$$G_F = \bigcap_{g \in G_+ \setminus \{e\}} G_g.$$

Clearly, (G, G_F) is the minimal quasi-ordered group containing (G, G_+) property. Let

$$G_F^0 = G_F \cap G_F^{-1}, \quad F(G_+) = G_F^0 \cap G_+.$$

Let $K(F(G))$ be the closed ideal of $\mathcal{T}^{G_+}(G)$ generated by

$$\{\, 1 - p^{G_+} u_g p^{G_+} u_{g^{-1}} p^{G_+} \,\big|\, g \in F(G_+) \,\}.$$

It is remarkable that when G is abelian, $F(G_+)$ is the positive part of the subgroup of finite elements in G (see [6]), $K(F(G))$ is the minimal closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$ in the sense that every closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$ always contains it (see [1, Theorem 2.11]), and it was proved in [6] that $K(F(G)) = \operatorname{Ker} \gamma^{G_F, G_+}$, where γ^{G_F, G_+} is the natural morphism from $\mathcal{T}^{G_+}(G)$ to $\mathcal{T}^{G_F}(G)$ satisfying

$$\gamma^{G_F, G_+}(p^{G_+} u_g p^{G_+}) = p^{G_F} u_g p^{G_F} \quad \text{for all} \quad g \in G.$$

The purpose of this paper is to generalize the above result to the case when G is a non-abelian group. Let G be a discrete group, (G, G_+) an ordered group. Suppose that $G_+ \subsetneqq G_F$. Let

$$\mathcal{T}^\infty = \operatorname{sp}\{\, p^{G_+} u_g p^{G_+} u_{h^{-1}} p^{G_+} \,\big|\, g, h \in G_+ \,\}.$$

Since G is totally ordered, $\mathcal{T}^\infty$ is a dense $*$-subalgebra of $\mathcal{T}^{G_+}(G)$. Let

$$I_F = \overline{(\mathcal{T}^\infty \cap \operatorname{Ker} \gamma^{G_F, G_+})^{\|\cdot\|}}.$$

Then I_F is a closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$. Obviously, $K(F(G)) \subseteq I_F \subseteq \operatorname{Ker} \gamma^{G_F, G_+}$. In this paper, we will show that $I_F = K(F(G))$, which is the minimal closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$. Moreover, if G is amenable, then $K(F(G)) = \operatorname{Ker} \gamma^{G_F, G_+}$. As an application, in the last part of this paper, we clarify a character of K-groups of Toeplitz algebras on discrete amenable groups.

§2. The Minimal Closed Non-Trivial Ideals

Throughout this section, G is a discrete group, (G, G_+) is an ordered group. We always assume that $G_+ \subsetneqq G_F$. For any $x, y \in G$, by $x \le y$ we mean that $x^{-1} y \in G_+$. From now on, $p^{G_+} u_g p^{G_+}$ will be denoted by T_g for any $g \in G$.

Lemma 2.1.[4,Corollary 2.5] *Let (G, G_+) be an ordered group and (π, H) a unital representation of $\mathcal{T}^{G_+}(G)$. Let $V(g) = \pi(T_g)$ and $L(g) = V(g)V(g)^*$ for g in G_+. Then the following two conditions are equivalent: (1) (π, H) is faithful; (2) $L(g) \neq 1$ for any $g \in G_+ \setminus \{e\}$.*

Proposition 2.1. *Let G be a discrete group and (G, G_+) an ordered group. Then $K(F(G))$ is the minimal closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$.*

Proof. Given any closed non-trivial ideal I of $\mathcal{T}^{G_+}(G)$, let

$$F_I = \{\, g \in G_+ \setminus \{e\} \,\big|\, 1 - T_g T_{g^{-1}} \in I \,\}.$$

Since the quotient map $\rho : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{G_+}(G)/I$ is not faithful, by Lemma 2.1 we know that $F_I \neq \emptyset$. Since for any $g \in G_+ \setminus \{e\}$, $g \in F_I$ if and only if $\rho(T_g)$ is a unitary in $\mathcal{T}^{G_+}(G)/I$, F_I is a semigroup of G_+.

For any $g \in G_+$ and $g_I \in F_I$, if $gg_I^{-1} \notin G_+$, i.e., $g_I = xg$ for some $x \in G_+ \setminus \{e\}$, then

$$\rho(T_x)\,\rho(T_g)\,\rho(T_g)^*\,\rho(T_x)^* = \rho(T_{g_I})\,\rho(T_{g_I})^* = 1. \tag{2.1}$$

Since $\rho(T_x)^*\,\rho(T_x) = 1$, multiplying with $\rho(T_x)^*$ from left and $\rho(T_x)$ from right, by (2.1) we know that $\rho(T_g)\,\rho(T_g)^* = 1$. By (2.1) again we know that $\rho(T_x)\,\rho(T_x)^* = 1$, which implies that $x \in F_I$, thus $gg_I^{-1} = x^{-1} \in (F_I)^{-1}$, so $G_+ \cdot (F_I)^{-1} \subseteq G_+ \cup (F_I)^{-1}$. Similarly

$$(F_I)^{-1} \cdot G_+ \subseteq G_+ \cup (F_I)^{-1}.$$

So if we set $G_I = G_+ \cup (F_I)^{-1}$, then G_I is a semigroup of G. Since G_F is minimal,

$$F(G_+) \subseteq G_I^0 \cap G_+ = F_I \cup \{e\},$$

which implies that $K(F(G)) \subseteq I$. Therefore $K(F(G))$ is minimal.

Next we show that $K(F(G)) = I_F$.

Let (G, G_+) be an ordered group. Let $\mathcal{D} = \operatorname{clos\,sp}\{T_gT_{g^{-1}} \mid g \in G_+\}$. It is a unital abelian C^*-subalgebra of $\mathcal{T}^{G_+}(G)$. Let $\mathcal{D}_0$ be the C^*-subalgebra of $\mathbb{B}\,(\ell^2(G_+))$ consisting of all the operators having diagonal matrix with respect to the canonical basis. It is well-known that there exists a linear and contractive map $E_0 : \mathbb{B}\,(\ell^2(G_+)) \to \mathcal{D}_0$ determined by the following rule: the matrix of $E_0(T)$ (with respect to the canonical basis) is obtained from the one of T by replacing with zero all the entries which are not situated on the principal diagonal.

Lemma 2.2.[2,Section 3.3 and Section 3.6] or [6, Proposition 3.3]

(1) $\mathcal{D} = \{T \in \mathcal{T}^{G_+}(G) \,|\, T$ *has diagonal matrix with respect to the canonical basis of* $\ell^2(G_+)\}$.

(2) *Let* $E = E_0|_{\mathcal{T}^{G_+}(G)}$. *Then* E *is a faithful bounded linear map from* $\mathcal{T}^{G_+}(G)$ *to* $\mathcal{D}$ *such that for any* g, h *in* G_+

$$E(T_gT_{h^{-1}}) = \begin{cases} T_gT_{h^{-1}} & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

Lemma 2.3.[5,Proposition 1.4] *Let* G *be a discrete group,* (G, E_1) *and* (G, E_2) *be two quasi-ordered groups with* $E_1 \subseteq E_2$. *Then there is a* C^**-algebra morphism* γ^{E_2,E_1} *from* $\mathcal{T}^{E_1}(G)$ *to* $\mathcal{T}^{E_2}(G)$ *such that*

$$\gamma^{E_2,E_1}(p^{E_1}u_gp^{E_1}) = p^{E_2}u_gp^{E_2} \quad \text{for all} \quad g \in G.$$

Theorem 2.1. *Let* G *be a discrete group,* (G, G_+) *an ordered group. Then* $K(F(G)) = I_F$.

Proof. Let I be any closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$. Let F_I and G_I be as before.

Step 1. For any $g \in G_+ \setminus (F_I \cup \{e\})$, we prove that $p^{G_+}u_gp^{G_I}u_{g^{-1}}p^{G_+} \notin \mathcal{T}^{G_+}(G)$.

In fact, if

$$T = p^{G_+}u_gp^{G_I}u_{g^{-1}}p^{G_+} - p^{G_+}u_gp^{G_+}u_{g^{-1}}p^{G_+} \in \mathcal{T}^{G_+}(G),$$

then since T has a diagonal matrix with respect to the usual basis $\{\,\delta_g \,\big|\, g \in G_+\}$, by Lemma 2.2, we know that there are $\lambda_0 \in C, \lambda_1, \lambda_2 \cdots, \lambda_n \in C \setminus \{0\}$ and $x_0 = e, x_1, x_2, \cdots, x_n \in G_+$ such that $x_1 < x_2 < \cdots < x_n$ and

$$\Big\| T - \sum_{i=0}^{n} \lambda_i T_{x_i} T_{x_i^{-1}} \Big\| \leq \frac{1}{3}. \tag{2.2}$$

Let $S = \sum_{i=0}^{n} \sigma_i P_{[x_i, x_{i+1})}$, where

$$\sigma_i = \sum_{k=0}^{i} \lambda_k, \quad x_{n+1} = +\infty,$$
$$[x_i, x_{i+1}) = \{ g \in G_+ \,\big|\, x_i \le g < x_{i+1} \}$$

and $P_{[x_i, x_{i+1})}$ is the projection from $\ell^2(G_+)$ to the closed subspace generated by $\{ \delta_g \,\big|\, g \in [x_i, x_{i+1}) \}$. Then $S = \sum_{i=0}^{n} \lambda_i T_{x_i} T_{x_i^{-1}}$. Choose any $h \in F_I$. Since $(G_+ \setminus (F_I \cup \{e\})) \cdot (F_I)^{-1} \subseteq G_+$ and $\|T - S\| \le \frac{1}{3}$, we know that

$$\|(T - S)\, \delta_{gh^{-1}}\| \le \frac{1}{3}. \tag{2.3}$$

Suppose that $gh^{-1} \in [x_i, x_{i+1})$ for some i. Then by (2.3) we know that

$$|\sigma_i - 1| \le \frac{1}{3}. \tag{2.4}$$

By (2.2) again, we know that $\|(T - S)\, \delta_{x_i}\| \le \frac{1}{3}$. By (2.4) we know that $x_i \in g \cdot (F_I)^{-1}$ (otherwise, $|\sigma_i| = \|(T - S)\, \delta_{x_i}\| \le \frac{1}{3}$, which is in contradiction with (2.4)). Let $F = \{ j \,\big|\, 1 \le j \le n,\ x_j \in g \cdot (F_I)^{-1} \}$. Let i_0 be the least number in F. Then $x_{i_0} = gl^{-1}$ for some $l \in F_I$. Now let $y = gl^{-2} \in G_+$. If $y \in [x_j, x_{j+1})$ for some j, then since $\|(T - S)\, \delta_y\| \le \frac{1}{3}$, we know that $|\sigma_j - 1| \le \frac{1}{3}$. It follows that $x_j \in g \cdot (F_I)^{-1}$, which implies that $x_{i_0} \le x_j$. But obviously $x_j \le y < x_{i_0}$. It is a contradiction.

Step 2. Let $h, g_1, g_2, \cdots, g_n \in G_+ \setminus (F_I \cup \{e\})$, such that $g_1 < g_2 < \cdots < g_n$. If

$$\sum_{i=1}^{n} \lambda_i p^{G_+} u_{g_i} p^{G_I} u_{h^{-1}} p^{G_+} \in \mathcal{T}^{G_+}(G),$$

then $\lambda_i = 0$ for $i = 1, 2, \cdots, n$.

In fact, let

$$T = \sum_{i=1}^{n} \lambda_i (p^{G_+} u_{g_i} p^{G_I} u_{h^{-1}} p^{G_+} - p^{G_+} u_{g_i} p^{G_+} u_{h^{-1}} p^{G_+}).$$

Then $T \in \mathcal{T}^{G_+}(G)$, so $T_{hg_n^{-1}} T \in \mathcal{T}^{G_+}(G)$. Since $(G_+ \setminus (F_I \cup \{e\})) \cdot (F_I)^{-1} \subseteq G_+$, we know that

$$\begin{aligned} &T_{hg_n^{-1}} (p^{G_+} u_{g_n} p^{G_I} u_{h^{-1}} p^{G_+} - p^{G_+} u_{g_n} p^{G_+} u_{h^{-1}} p^{G_+}) \\ &= p^{G_+} u_h p^{G_I} u_{h^{-1}} p^{G_+} - p^{G_+} u_h p^{G_+} u_{h^{-1}} p^{G_+}. \end{aligned}$$

So

$$\lambda_n (p^{G_+} u_h p^{G_I} u_{h^{-1}} p^{G_+} - p^{G_+} u_h p^{G_+} u_{h^{-1}} p^{G_+}) = E(T_{hg_n^{-1}} T) \in \mathcal{D}.$$

By Step 1, we know that $\lambda_n = 0$. Similarly, $\lambda_i = 0$ for $i = 1, 2, \cdots, n - 1$.

Step 3. Define a linear operator $\rho : \mathcal{T}^{G_I}(G) \to \mathbb{B}(\ell^2(G_+))$ by $\rho(X) = p^{G_+} X p^{G_+}$ for $X \in \mathcal{T}^{G_I}(G)$. Let $T \in \mathcal{T}^{\infty}$. We prove that if $S = T - \rho\gamma^{G_I, G_+}(T) \in \mathcal{T}^{G_+}(G)$, then $S \in I$. Therefore, $\mathcal{T}^{\infty} \cap \mathrm{Ker}\, \gamma^{G_I, G_+} \subseteq I$.

Let $T = \sum_{i=1}^{n} \lambda_i T_{g_i} T_{h_i^{-1}}$ for some $\lambda_1, \lambda_2, \cdots, \lambda_n \in C \setminus \{0\}$, $g_1, g_2, \cdots, g_n \in G_+$ and

$h_1, h_2, \cdots, h_n \in G_+$. Suppose that $S = T - \rho\gamma^{G_I,G_+}(T) \in \mathcal{T}^{G_+}(G)$. Then

$$S = \sum_{i=1}^{n} \lambda_i (p^{G_+} u_{g_i} p^{G_+} u_{h_i^{-1}} p^{G_+} - p^{G_+} u_{g_i} p^{G_I} u_{h_i^{-1}} p^{G_+}).$$

Let π_I be the quotient map from $\mathcal{T}^{G_+}(G)$ to $\mathcal{T}^{G_+}(G)/I$. Then for any $g \in G_+, h \in F_I$,

$$\pi_I(T_g)\,\pi_I(T_{h^{-1}})\,\pi_I(T_h) = \pi(T_g) = \pi_I(T_{gh^{-1}})\,\pi_I(T_h).$$

Since $\pi_I(T_h)$ is a unitary, we know that $T_g T_{h^{-1}} - T_{gh^{-1}} \in I$, which implies that $T_h T_{g^{-1}} - T_{hg^{-1}} \in I$. So we may assume without loss of generality that $h_1 \le h_2 \le \cdots \le h_n$, $g_i, h_i \notin F_I \cup \{e\}$ for $i = 1, 2, \cdots, n$. If $h_n^{-1} h_{n-1} \notin (F_I)^{-1} \cup \{e\}$, then since $ST_{h_{n-1}} \in \mathcal{T}^{G_+}(G)$, we know that $\lambda_n p^{G_+} u_{g_n} p^{G_I} u_{h_n^{-1} h_{n-1}} p^{G_+} \in \mathcal{T}^{G_+}(G)$, which is a contradiction by Step 2. So $h_n^{-1} h_{n-1} \in (F_I)^{-1} \cup \{e\}$. Let i_0 be the least number such that $h_n^{-1} h_{i_0} \in (F_I)^{-1} \cup \{e\}$. If $h_{i_0} = h_n$, then $h_j = h_n$ for $i_0 < j \le n-1$; otherwise, $h_n^{-1} h_{i_0} \in (F_I)^{-1}$, in this case, for any $i_0 < j < n-1$,

$$h_n^{-1} h_j = (h_n^{-1} h_{i_0})(h_{i_0}^{-1} h_j) \in ((F_I)^{-1} \cdot G_+) \cap G_+^{-1} \subseteq (F_I)^{-1} \cup \{e\}.$$

For any $i_0 \le j \le n-1$, let $h_n^{-1} h_j = x_j^{-1}$ for some $x_j \in F_I \cup \{e\}$. Let $x_n = e$ and

$$\begin{aligned}\tilde{S} = &\sum_{k=1}^{i_0-1} \lambda_k (p^{G_+} u_{g_k} p^{G_+} u_{h_k^{-1}} p^{G_+} - p^{G_+} u_{g_k} p^{G_I} u_{h_k^{-1}} p^{G_+}) \\ &+ \sum_{j=i_0}^{n} \lambda_j (p^{G_+} u_{g_j x_j} p^{G_+} u_{h_n^{-1}} p^{G_+} - p^{G_+} u_{g_j x_j} p^{G_I} u_{h_n^{-1}} p^{G_+}).\end{aligned}$$

Then since

$$T_{g_j} T_{h_j{}^{-1}} - T_{g_j x_j} T_{h_n{}^{-1}} = T_{g_j}(T_{x_j h_n^{-1}} - T_{x_j} T_{h_n^{-1}}) \in I,$$

we know that $\tilde{S} \in \mathcal{T}^{G_+}(G)$. To show that $S \in I$, it suffices to show that $\tilde{S} \in I$. But since $\tilde{S} T_{h_{i_0-1}} \in \mathcal{T}^{G_+}(G)$, we know that

$$\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_I} u_{h_n^{-1} h_{i_0-1}} p^{G_+} \in \mathcal{T}^{G_+}(G).$$

Since $g_j x_j \in (G_+ \setminus (F_I \cup \{e\})) \cdot F_I \subseteq G_+ \setminus (F_I \cup \{e\})$, by Step 2 we know that this can happen only if

$$\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_I} u_{h_n^{-1} h_{i_0-1}} p^{G_+} = 0.$$

Therefore

$$\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_+} = \Big(\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_I} u_{h_n^{-1} h_{i_0-1}} p^{G_+} \Big) T_{h_{i_0-1}{}^{-1} h_n} = 0.$$

So

$$\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_I} u_{h_n^{-1}} p^{G_+} = \rho\,\gamma^{G_I,G_+} \Big(\sum_{j=i_0}^{n} \lambda_j p^{G_+} u_{g_j x_j} p^{G_+} u_{h_n^{-1}} p^{G_+} \Big) = 0.$$

Therefore it is reduced to showing that

$$\sum_{k=1}^{i_0-1} \lambda_k (p^{G_+} u_{g_k} p^{G_+} u_{h_k^{-1}} p^{G_+} - p^{G_+} u_{g_k} p^{G_I} u_{h_k^{-1}} p^{G_+}) \in I.$$

Pursue the former process, eventually we know that $S \in I$.

Step 4. By Step 3, we know that $\mathcal{T}^{\infty} \cap \operatorname{Ker} \gamma^{G_I,G_+} \subseteq I$ for any closed non-trivial ideal I of $\mathcal{T}^{G_+}(G)$. Since $G_F \subseteq G_I$, by Lemma 2.3 we know that there exist two C^*-algebra morphisms $\gamma^{G_I,G_F} : \mathcal{T}^{G_F}(G) \to \mathcal{T}^{G_I}(G)$ and $\gamma^{G_F,G_+} : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{G_F}(G)$ such that

$$\gamma^{G_I,G_F}(p^{G_F} u_g p^{G_F}) = p^{G_I} u_g p^{G_I}, \quad \gamma^{G_F,G_+}(T_g) = p^{G_F} u_g p^{G_F}$$

for all $g \in G$. Obviously, $\gamma^{G_I,G_F} \circ \gamma^{G_F,G_+} = \gamma^{G_I,G_+}$. So $\operatorname{Ker} \gamma^{G_F,G_+} \subseteq \operatorname{Ker} \gamma^{G_I,G_+}$. Therefore, $\mathcal{T}^{\infty} \cap \operatorname{Ker} \gamma^{G_F,G_+} \subseteq I$ for any closed non-trivial ideal I of $\mathcal{T}^{G_+}(G)$.

Remark. At this point, one may ask whether $K(F(G)) = \operatorname{Ker} \gamma^{G_F,G_+}$ for a general discrete ordered group (G, G_+). We are unable to answer this question. However, if G is amenable, then we will show that $K(F(G))$ is exactly equal to $\operatorname{Ker} \gamma^{G_F,G_+}$.

Lemma 2.4.[6,Theorem 3.5] *Let G be an amenable group and (G, E) a quasi-ordered group. Let $V : E \to \mathbb{B}(H)$ be an isometric representation of E (i.e. $V(e) = 1$; $V(g)^*V(g) = 1$, $V(g)V(h) = V(gh)$ for any $g, h \in G_+$; and $V(l)V(l)^* = 1$ for any $l \in E \cap E^{-1}$). Then there is a C^*-algebra morphism $\pi_V : \mathcal{T}^E(G) \to \mathbb{B}(H)$ such that*

$$\pi_V(p^E u_g p^E) = V(g) \quad \textit{for all} \ \ g \in G.$$

Proposition 2.2. *Let G be a discrete amenable group and (G, G_+) an ordered group. Then $\operatorname{Ker} \gamma^{G_F,G_+}$ is the minimal closed non-trivial ideal of $\mathcal{T}^{G_+}(G)$.*

Proof. Since $K(F(G)) \subseteq \operatorname{Ker} \gamma^{G_F,G_+}$, γ^{G_F,G_+} induces a C^*-algebra morphism $\sigma : \mathcal{T}^{G_+}(G)/K(F(G)) \to \mathcal{T}^{G_F}(G)$ such that

$$\sigma([T_g]) = p^{G_F} u_g p^{G_F} \ \ \text{for all} \ \ g \in G.$$

On the other hand, let $V(g) = \pi(T_g)$ for $g \in G_F$, where π is the quotient map from $\mathcal{T}^{G_+}(G)$ to $\mathcal{T}^{G_+}(G)/K(F(G))$. Then since

$$G_F^0 = F(G_+) \cup (F(G_+))^{-1},$$

we know that $V(e) = 1, V(g)^*V(g) = 1$ for any $g \in G_F$ and $V(h)V(h)^* = 1$ for any $h \in G_F^0$. For any $g \in G_+$ and $h \in F(G_+)$,

$$V(g)V(h^{-1})V(h) = V(g) = V(gh^{-1})V(h).$$

Since $V(h)$ is a unitary, we know that $V(g)V(h^{-1}) = V(gh^{-1})$. So V is an isometric representation of G_F. By Lemma 2.4, there is a C^*-algebra morphism $\pi_V : \mathcal{T}^{G_F}(G) \to \mathcal{T}^{G_+}(G)/K(F(G))$ such that $\pi_V(p^{G_F} u_g p^{G_F}) = \pi(T_g)$ for any $g \in G$. Clearly $\pi_V = \sigma^{-1}$, so

$$K(F(G)) = \operatorname{Ker} \gamma^{G_F,G_+}.$$

§3. A Character of K-Groups of Toeplitz Algebras on Ordered Groups

Let G be a discrete group, (G, G_+) an ordered group. When G is a countable infinite abelian group, it is proved in [6] that if

$$K_0(\mathcal{T}^{G_+}(G)) \oplus K_1(\mathcal{T}^{G_+}(G)) \cong \mathbb{Z},$$

then $K_0(\mathcal{T}^{G_+}(G)) \cong \mathbb{Z}$ and $K_1(\mathcal{T}^{G_+}(G)) = 0$. The purpose of this section is to generalize such a result to the case when G is an amenable group.

Lemma 3.1. *Let G be a discrete group, (G, G_+) an ordered group. Then $K(\ell^2(G_+)) \subseteq \mathcal{T}^{G_+}(G)$ if and only if G admits a least positive element, where $K(\ell^2(G_+))$ is the ideal of compact operators on $\ell^2(G_+)$. When G admits a least positive element, $K(\ell^2(G_+)) = K(F(G))$.*

Proof. " $\Longrightarrow$ ". Suppose that $K(\ell^2(G_+)) \subseteq \mathcal{T}^{G_+}(G)$. Then since the quotient map $\pi : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{G_+}(G)\big/K(\ell^2(G_+))$ is not faithful, by Lemma 2.1 we know that there exists $g \in G_+ \setminus \{e\}$ such that $1 - T_g T_{g^{-1}} \in K(\ell^2(G_+))$. So the subset $F_g = \{\, h \in G_+ \,\big|\, h \leq g \,\}$ is finite, therefore G admits a least positive element.

"$\Longleftarrow$ ". For any $T \in \mathbb{B}(\ell^2(G_+))$, if $TT_g = T_g T$ and $TT_{g^{-1}} = T_{g^{-1}} T$ for any $g \in G_+$, then $T = \lambda$ for some $\lambda \in C$, so $\mathcal{T}^{G_+}(G)$ is irreducible. Suppose now that G admits a least positive element g_0. Then $1 - T_{g_0} T_{g_0^{-1}} \in K(\ell^2(G_+))$, so $K(\ell^2(G_+)) \subseteq \mathcal{T}^{G_+}(G)$ since $\mathcal{T}^{G_+}(G)$ is irreducible.

Let g_0 be the least positive element in G. Then $g_0^{-1} g \in G_+$ for any $g \in G_+ \setminus \{e\}$, so $g_0^{-1} = (g_0^{-1} g) g^{-1} \in G_g$ for any $g \in G_+ \setminus \{e\}$, i.e., $g_0 \in F(G_+)$. Since $K(F(G))$ is irreducible (because $\mathcal{T}^{G_+}(G)$ is), we know that $K(\ell^2(G_+)) \subseteq K(F(G))$. By Proposition 2.1 we know that

$$K(\ell^2(G_+)) = K(F(G)).$$

Lemma 3.2. *Let G be an amenable discrete group, (G, G_+) an ordered group. Let $E = G \times \mathbb{Z}$, $E_+ = G_+ \times \mathbb{Z}$. Then $\mathcal{T}^{E_+}(E) \cong \mathcal{T}^{G_+}(G) \otimes C(T)$, where T is the unit circle in C.*

Proof. For any $n \in \mathbb{Z}$, let $\chi_n \in C(T)$, $\chi_n(e^{i\theta}) = e^{in\theta}$ for $\theta \in [0, 2\pi]$. For any $h = (g, n) \in E_+$, let $V(h) = T_g \otimes \chi_n \in \mathcal{T}^{G_+}(G) \otimes C(T)$. Then V is an isometric representation of E_+. By Lemma 2.4, there is a C^*-algebra morphism $\pi_V : \mathcal{T}^{E_+}(E) \to \mathcal{T}^{G_+}(G) \otimes C(T)$ such that

$$\pi_V(p^{E_+} u_{(g,n)} p^{E_+}) = T_g \otimes \chi_n \quad \text{for all} \quad (g, n) \in E.$$

On the other hand, by Lemma 2.4 and [1, Lemma 1.2], we know that there are C^*-algebra morphisms $\rho : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{E_+}(E)$ and $\lambda : C(T) \to \mathcal{T}^{E_+}(E)$ such that

$$\rho(T_g) = p^{E_+} u_{(g,0)} p^{E_+} \text{ for any } g \in G, \quad \lambda(\chi_n) = p^{E_+} u_{(e,n)} p^{E_+} \text{ for any } n \in Z.$$

It is easy to show that for any $g \in G$ and $n \in \mathbb{Z}$,

$$\rho(T_g)\lambda(\chi_n) = \lambda(\chi_n)\rho(T_g).$$

Since $C(T)$ is nuclear, by [3, Corollary T.6.9], we know that there is a C^*-algebra morphism $\rho \otimes \lambda : \mathcal{T}^{G_+}(G) \otimes C(T) \to \mathcal{T}^{E_+}(E)$ such that

$$\big(\rho \otimes \lambda\big)\big(T_g \otimes \chi_n\big) = p^{E_+} u_{(g,0)} p^{E_+} \cdot p^{E_+} u_{(e,n)} p^{E_+} = p^{E_+} u_{(g,n)} p^{E_+}$$

for any $g \in G$ and $n \in \mathbb{Z}$. Clearly, $\rho \otimes \lambda = (\pi_V)^{-1}$, so

$$\mathcal{T}^{E_+}(E) \cong \mathcal{T}^{G_+}(G) \otimes C(T).$$

Theorem 3.1.[6,Theorem 2.2] *Let G be a countable infinite discrete amenable group, (G, G_+) an ordered group. If $K_0(\mathcal{T}^{G_+}(G)) \oplus K_1(\mathcal{T}^{G_+}(G)) \cong \mathbb{Z}$, then*

$$K_0(\mathcal{T}^{G_+}(G)) \cong \mathbb{Z} \quad \textit{and} \quad K_1(\mathcal{T}^{G_+}(G)) = 0.$$

Proof. Let $E = G \times \mathbb{Z}$, $E_+ = G_+ \times \mathbb{Z}$, and (E, E_{lex}) be the lexico-ordered group. By

Lemma 3.1 and Lemma 3.2, we have the following short exact sequence of C^*-algebras

$$0 \longrightarrow K(\ell^2(E_+)) \xrightarrow{i} \mathcal{T}^{E_{\rm lex}}(E) \xrightarrow{\sigma} \mathcal{T}^{G_+}(G) \otimes C(T) \longrightarrow 0$$

where i is the inclusion map and

$$\sigma(p^{E_{\rm lex}} u_{(g,n)} p^{E_{\rm lex}}) = T_g \otimes \chi_n \quad \text{for any} \quad (g,n) \in E.$$

So we have the following periodic six-term exact sequence of K-groups

$$\begin{array}{ccccc} \mathbb{Z} \cong K_0(K) & \xrightarrow{i_*} & K_0(\mathcal{T}^{E_{\rm lex}}(E)) & \xrightarrow{\sigma_*} & K_0(\mathcal{T}^{G_+}(G) \otimes C(T)) \\ \Big\uparrow \text{index map} & & & & \Big\downarrow \delta \\ K_1(\mathcal{T}^{G_+}(G) \otimes C(T)) & \xleftarrow[\sigma_*]{} & K_1(\mathcal{T}^{E_{\rm lex}}(E)) & \xleftarrow[i_*]{} & K_1(K) = 0 \end{array}$$

Since

$$K_0(\mathcal{T}^{G_+}(G) \otimes C(T)) \cong K_1(\mathcal{T}^{G_+}(G) \otimes C(T)) \cong K_0(\mathcal{T}^{G_+}(G)) \oplus K_1(\mathcal{T}^{G_+}(G)) \cong \mathbb{Z},$$

and $\mathcal{T}^{E_{\rm lex}}(E)$ contains a Fredholm operator of index one (which implies that the index map in the exact sequence is an isomorphism), we know that

$$K_0(\mathcal{T}^{E_{\rm lex}}(E)) \cong \mathbb{Z}, \qquad K_1(\mathcal{T}^{E_{\rm lex}}(E)) = 0.$$

Let $V : E_{\rm lex} \to \mathcal{T}^{G_+}(G)$, $\quad V(g,n) = T_g$ for any $(g,n) \in E_{\rm lex}$. Then V is an isometric representation of $E_{\rm lex}$. So there is a C^*-algebra morphism $\rho : \mathcal{T}^{E_{\rm lex}}(E) \to \mathcal{T}^{G_+}(G)$ such that

$$\rho(p^{E_{\rm lex}} u_{(g,n)} p^{E_{\rm lex}}) = T_g \quad \text{for any} \quad (g,n) \in E.$$

Similarly, there is a C^*-algebra morphism $\theta : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{E_{\rm lex}}(E)$ such that

$$\theta(T_g) = p^{E_{\rm lex}} u_{(g,0)} p^{E_{\rm lex}}.$$

Since $\mathcal{T}^{G_+}(G)$ is generated by $\{ T_g \,|\, g \in G_+ \}$, we know that $\rho \circ \theta = id_{\mathcal{T}^{G_+}(G)}$, so

$$\rho_* \circ \theta_* = id_{K_1(\mathcal{T}^{G_+}(G))}.$$

It follows that the map

$$\rho_* : K_1(\mathcal{T}^{E_{\rm lex}}(E)) \to K_1(\mathcal{T}^{G_+}(G))$$

is onto, therefore $K_1(\mathcal{T}^{G_+}(G))= 0$.

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