Chin. Ann. of Math.
21B: 3(2000),375-380.

THE REALIZATION OF MULTIPLIER HILBERT BIMODULE ON BIDUAL SPACE AND TIETZE EXTENSION THEOREM**

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Abstract

The multiplier bimodule of Hilbert bimodule is introduced in a way similar to [1], and its realization on a quotient of bidual space and Tietze extension theorem are obtained similar to that in C^*-algebra case. As a result, the multiplier bimodule here is also a Hilbert bimodule.

Keywords Realization on bidual space, Multiplier Hilbert bimodule, Tietze extension theorem

1991 MR Subject Classification 46L40

Chinese Library Classification O177.1 **Document Code** A

Article ID 0252-9599(2000)03-0375-06

§1. Introduction and Preliminary

The multiplier bimodule of imprimitive bimodule and its application to coaction system were introduced in [1], but it is generally not an imprimitive bimodule as noted in [1] by an example. In this paper, the same concept is introduced for Hilbert bimodules which properly contain imprimitive bimodules (for example, the same example as above), and we obtain the realization of multiplier bimodule on a quotient of bidual space and Tietze extension theorem similar to that in C^*-algebra case. As a result, our multiplier bimodule is also a Hilbert bimodule.

Let A,B be C^*-algebras with bounded approximate units $\{E_\lambda\}$ and $\{F_\lambda\}$ respectively (For simplicity, we use the same index λ both for $\{E_\lambda\}$ and $\{F_\lambda\}$. In fact, for $\{E_{\lambda_1}\}$ and $\{F_{\lambda_2}\}$, we can let $\lambda=(\lambda_1,\lambda_2)$ with $\lambda'=(\lambda'_1,\lambda'_2)\le\lambda$ iff $\lambda'_1\le\lambda_1$ and $\lambda'_2\le\lambda_2$, and let $E_\lambda=E_{\lambda_1}$, $F_\lambda=F_{\lambda_2}$), X be an A-B bimodule with A-valued and B-valued inner products ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_B$ such that X is a left A Hilbert module and right B Hilbert module in the sense of [3]. We call X an A-B Hilbert bimodule if $x\langle y,z\rangle_B={}_A\langle x,y\rangle z$,

$$\langle ax,ax\rangle_B\le\|a\|^2\langle x,x\rangle_B,\quad {}_A\langle xb,xb\rangle\le\|b\|^2\ {}_A\langle x,x\rangle.$$

So an A-B Hilbert bimodule is an A-B Banach bimodule with norm

$$\|x\|^2=\|{}_A\langle x,x\rangle\|=\|\langle x,x\rangle_B\|.$$

Manuscript received December 3, 1998. Revised October 9, 1999.

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**Project supported by the National Natural Science Foundation of China (No.19601029).

For an A-B Hilbert bimodule X, we have canonical isomorphisms $A \otimes_A X \cong X \cong X \otimes_B B$ and that $I_A = \overline{\text{span}}\{{}_A\langle x, y\rangle | x, y \in X\}$ is an ideal of A, $I_B = \overline{\text{span}}\{\langle x, y\rangle_B | x, y \in X\}$ is an ideal of B. The last fact implies ${}_A\langle xb, y\rangle = {}_A\langle x, yb^*\rangle$ and $\langle ax, y\rangle_B = \langle x, a^*y\rangle_B$ by applying [6, 6.12] and Cohen (Factorization) Lemma to I_A-I_B imprimitive bimodule X. It is well known that X is also an $M(A)$-$M(B)$ Banach bimodule with

$$k(ax) = (ka)x, \quad (xb)l = x(bl), \quad k \in M(A), \quad l \in M(B).$$

In a way similar to [1], we call a multiplier of X a pair (m_A, m_B), where m_A (m_B) is a left A (right B) module homomorphism from A (B) to X with $m_A(a)b = am_B(b)$. So $\|m_A\| = \|m_B\| < \infty$, and m_A, m_B have (Hilbert module) adjoints

$$m_A^*(x) = \lim_{\lambda\to\infty} {}_A\langle x, m_A(E_\lambda)\rangle, \quad m_B^*(x) = \lim_{\lambda\to\infty} \langle m_B(F_\lambda), x\rangle_B$$

(the limits exist because we can write x as yb and az with $a \in A$, $b \in B$, $y, z \in X$ by Cohen Lemma), i.e.

$$m_A^*(xb) = {}_A\langle x, m_B(b^*)\rangle, \quad m_B^*(ax) = \langle m_A(a^*), x\rangle_B,$$

and so $m_A \in L_A(A, X)$, $m_B \in L_B(B, X)$.

Let $M(X)$ be the set of all multipliers of X. Then $X \subseteq M(X)$ with $x_A(a) = ax$, $x_B(b) = xb$. For every $k \in M(A)$, $l \in M(B)$, $m, n \in M(X)$, let

$$(km)_A(a) = m_A(ak), \quad (km)_B(b) = km_B(b);$$
$$(ml)_A(a) = m_A(a)l, \quad (ml)_B(b) = m_B(lb);$$
$$a({}_{M(A)}\langle m, n\rangle) = n_A^* m_A(a), \quad {}_{M(A)}\langle m, n\rangle a = (a^*({}_{M(A)}\langle n, m\rangle))^*;$$
$$(\langle m, n\rangle_{M(B)})b = m_B^* n_B(b), \quad b\langle m, n\rangle_{M(B)} = ((\langle n, m\rangle_{M(B)})b^*)^*.$$

It is easy to check that

$$((km)_A, (km)_B) \in M(X), \quad ((nl)_A, (nl)_B) \in M(X),$$
$${}_{M(A)}\langle m, n\rangle \in M(A), \quad \langle m, n\rangle_{M(B)} \in M(B),$$
$$\|m_A\|^2 = \|{}_{M(A)}\langle m, m\rangle\|, \quad \|m_B\|^2 = \|\langle m, m\rangle_{M(B)}\|,$$

and these define $M(X)$ to be an $M(A)$-$M(B)$ Banach bimodule with norm $\|m\| = \|m_A\|$ and with X as a submodule. Let $\{m_\lambda\}$ be a net in $M(X)$. We say m_λ strictly converges to $m \in M(X)$ (denoted by $m_\lambda \xrightarrow{s} m$), if for every $a \in A$, $b \in B$, $(m_\lambda)_A(a)$ and $(m_\lambda)_B(b)$ converge to $m_A(a)$ and $m_B(b)$ in norm respectively.

§2. Main Theorem

Let X be an A-B Hilbert bimodule. Then X can be viewed as an $M(A)$-$M(B)$ Banach bimodule. So the (Banach space) dual X^* of X is an $M(B)$-$M(A)$ Banach bimodule with $(l\phi k)(x) = \phi(kxl)$, where $k \in M(A)$, $l \in M(B)$, $\phi \in X^*$, $x \in X$. Moreover the $M(B)$-$M(A)$ Banach bimodual X^* induces the $M(A)$-$M(B)$ Banach bimodule X^{**}. Let

$$N = \{x^{**} \in X^{**} | ax^{**}b = 0, \ \ \forall a \in A, \ \ b \in B\}.$$

It is easy to see that N is an ω^* closed $M(A)$-$M(B)$ submodule of X^{**} and $X^{**}/N = (\overline{\text{span}\{BX^*A\}})^*$. Since for every $x \in X$,

$$\|x\| = \sup_{\|a\|\le 1, \|b\|\le 1} \|axb\| = \sup_{\|\phi\|\le 1, \|a\|\le 1, \|b\|\le 1} \|(b\phi a)(x)\| \le \|\tilde{x}\| \le \|x\|,$$

X can be viewed as an $M(A)$-$M(B)$ Banach submodule not only of X^{**} but also of X^{**}/N. Let

$$M_1(X) = \{\widetilde{x^{**}} \in X^{**}/N, a\widetilde{x^{**}}, \widetilde{x^{**}}b \in X, \forall a \in A, b \in B\}.$$

It is clear that $M_1(X)$ is an $M(A)$-$M(B)$ Banach bimodule with X as a submodule.

A net $\{\omega_\lambda\}$ in X^{**}/N is called strictly convergent to $\omega \in X^{**}/N$ (denoted by $\omega_\lambda \xrightarrow{s} \omega$), if the nets $\{a\omega_\lambda\}$ and $\{\omega_\lambda b\}$ are convergent to $a\omega$ and ωb in norm topology respectively for every $a \in A$, $b \in B$.

For every $\omega \in M_1(X)$, since the mappings from $A \times X \to B$ defined by $(a, x) \to \langle a^*\omega, x\rangle_B$ and $(a, x) \to \langle x, a^*\omega\rangle_B$ are A-balanced and $A \otimes X \cong X$, we can define the mappings $\langle \omega, \cdot\rangle_B$ and $\langle \cdot, \omega\rangle_B$ from X to B with $\langle \omega, ax\rangle_B = \langle a^*\omega, x\rangle_B$ and $\langle ax, \omega\rangle_B = \langle x, a^*\omega\rangle_B$. Similarly we have two mappings ${}_A\langle \omega, \cdot\rangle$ and ${}_A\langle \cdot, \omega\rangle$ with ${}_A\langle \omega, xb\rangle = {}_A\langle \omega b^*, x\rangle$ and ${}_A\langle xb, \omega\rangle = {}_A\langle x, \omega b^*\rangle$. Then if at least one of the elements ω,ν in $M_1(X)$ is in X,

$${}_A\langle \omega, \nu\rangle^* = {}_A\langle \nu, \omega\rangle, \quad \langle \omega, \nu\rangle_B^* = \langle \nu, \omega\rangle_B,$$

and

$$\|{}_A\langle \omega, \nu\rangle\| \le \|\omega\|\ \|\nu\|, \quad \|\langle \omega, \nu\rangle_B\| \le \|\omega\|\ \|\nu\|$$

by Cohen lemma.

For every $\omega, \nu \in M_1(X)$, we define ${}_{M(A)}\langle \omega, \nu\rangle$ and $\langle \omega, \nu\rangle_{M(B)}$ to be the elements of $M(A)$ and $M(B)$ respectively with

$$({}_{M(A)}\langle \omega, \nu\rangle)a = {}_A\langle \omega, a^*\nu\rangle, \quad a({}_{M(A)}\langle \omega, \nu\rangle) = {}_A\langle a\omega, \nu\rangle,$$
$$(\langle \omega, \nu\rangle_{M(B)})b = \langle \omega, \nu b\rangle_B, \quad b(\langle \omega, \nu\rangle_{M(B)}) = \langle \omega b^*, \nu\rangle_B.$$

Directly checking shows the definition is well defined.

Main Theorem. *With the inner products defined above, the $M(A)$-$M(B)$ bimodule $M_1(X)$ is an $M(A)$-$M(B)$ Hilbert bimodule with separately strictly continuous module actions and inner products, and the norms on $M_1(X)$ defined by inner products and by X^{**}/N coincide. Moreover there is an $M(A)$-$M(B)$ bimodule isomorphism Φ from $M(X)$ to $M_1(X)$ which preserves the strict topologies, i.e. both Φ and Φ^{-1} are strictly continuous, and the inner products defined above for $M_1(X)$ and defined in §1 for $M(X)$ are preserved. As a consequence, $M(X)$ is also an $M(A)$-$M(B)$ Hilbert bimodule with separately strictly continuous module actions and inner products.*

Corollory 2.1. *If ${}_AX_B$ is a Hilbert bimodule with $X^{**} = X$, then its multiplier Hilbert bimodule is ${}_{M(A)}X_{M(B)}$, i.e. $M(X) = X$.*

We divide the proof of our main theorem in bits:

Lemma 2.1. (1) *The module actions of $M(A)$ and $M(B)$ on $M_1(X)$ are separately strictly continuous, i.e. if $\omega \in M_1(X)$, $k \in M(A)$, $l \in M(B)$, and $\{\omega_\lambda\} \subseteq M_1(X)$, $\{k_\lambda\} \subseteq M(A)$, $\{l_\lambda\} \subseteq M(B)$ are the nets with strict limits ω, k, l respectively, then*

$$k\omega_\lambda \xrightarrow{s} k\omega, \quad \omega_\lambda l \xrightarrow{s} \omega l, \quad k_\lambda\omega \xrightarrow{s} k\omega, \quad \omega l_\lambda \xrightarrow{s} \omega l.$$

(2) *The $M(A)$-valued and $M(B)$-valued inner products on $M_1(X)$ are separately strictly continuous, i.e. if $\omega_\lambda \xrightarrow{s} \omega$, $\nu_\lambda \xrightarrow{s} \nu$, then*

$${}_{M(A)}\langle \omega_\lambda, \nu\rangle \xrightarrow{s} {}_{M(A)}\langle \omega, \nu\rangle, \quad {}_{M(A)}\langle \omega, \nu_\lambda\rangle \xrightarrow{s} {}_{M(A)}\langle \omega, \nu\rangle,$$
$$\langle \omega_\lambda, \nu\rangle_{M(B)} \xrightarrow{s} \langle \omega, \nu\rangle_{M(B)}, \quad \langle \omega, \nu_\lambda\rangle_{M(B)} \xrightarrow{s} \langle \omega, \nu\rangle_{M(B)}.$$

Proof. Directly checking.

Lemma 2.2. *There is an $M(A)$-$M(B)$ Banach bimodule isomorphism Φ from $M(X)$ to $M_1(X)$ which is isometric and preserves the strict topologies.*

Proof. Let $\{E_\lambda\}$ and $\{F_\lambda\}$ be the bounded approximate units for A and B respectively. For every $m = (m_A, m_B) \in M(X)$, since the unit ball of X^{**} is ω^* compact, let x^{**} be one of the cluster points of $\{m_A(E_\lambda)\}$ and the subnet $\{m_A(E_{\lambda_\mu})\}$ be ω^* convergent to x^{**}. Since for every $\psi \in X^*$, $m_A^\dagger(\psi) \in A^*$, where the map $m_A^\dagger$ from X^* to A^* is the dual of m_A as the Banach space map from A to X, we have

$$x^{**}(\psi) = \lim_\mu \psi(m_A(E_{\lambda_\mu})) = m_A^\dagger(\psi)(1).$$

So x^{**} is the unique cluster point of $\{m_A(E_\lambda)\}$. Similarly let y^{**} be the unique cluster point of $\{m_B(F_\lambda)\}$. Then

$$ax^{**} = m_A(a) = ay^{**}, \quad x^{**}b = m_B(b) = y^{**}b.$$

So $\widetilde{x^{**}} = \widetilde{y^{**}} \in M_1(X)$. Let Φ be a map from $M(X)$ to $M_1(X)$: $\Phi(m) = \widetilde{x^{**}}$. For every $k \in M(A)$, $l \in M(B)$,

$$\Phi(km) = \widetilde{ky^{**}} = k\Phi(m), \quad \Phi(ml) = \widetilde{x^{**}l} = \Phi(m)l.$$

So Φ is an $M(A)$-$M(B)$ bimodule map. Moreover for every $x \in X$, $\Phi(x) = \tilde{x} = x$ for $\lim_\lambda x_A(E_\lambda) = x$, so $\Phi|_X = id$. On the other hand, for every $\widetilde{x^{**}} \in M_1(X)$, $a\widetilde{x^{**}}, \widetilde{x^{**}}b \in X$. So similar to [1, Proposition 1.2], let Ψ be a map from $M_1(X)$ to $M(X)$: $\Psi(\widetilde{x^{**}}) = ((\widetilde{x^{**}})_A, (\widetilde{x^{**}})_B)$, where

$$(\widetilde{x^{**}})_A(a) = a\widetilde{x^{**}}, \quad (\widetilde{x^{**}})_B(b) = \widetilde{x^{**}}b.$$

Then Ψ is an $M(A)$-$M(B)$ bimodule map with $\Psi|_X = id$, and clearly $\Psi\Phi = id$. On the other hand we have $\Phi\Psi = id$, since for every $\widetilde{x^{**}} \in M_1(X)$, $a \in A$,

$$a(\widetilde{x^{**}} - \Phi\Psi(\widetilde{x^{**}})) = a\widetilde{x^{**}} - \Phi\Psi(a\widetilde{x^{**}}) = 0.$$

Therefore Φ is isomorphic. Since

$$\begin{aligned}\|m\| &= \|m_A\| = \sup_{\|a\|\le 1} \|m_A(a)\| = \sup_{\|a\|\le 1}\sup_{\|b\|\le 1} \|m_A(a)b\| \\ &= \sup_{\|a\|\le 1}\sup_{\|b\|\le 1} |ax^{**}b| = \sup_{\|a\|\le 1, \|b\|\le 1, \|\phi\|\le 1} |ax^{**}b(\phi)| \\ &= \sup_{\|a\|\le 1, \|b\|\le 1, \|\phi\|\le 1} |x^{**}(b\phi a)| \le \|\widetilde{x^{**}}\| \le \|x^{**}\| \le \|m_A\|, \\ \|\widetilde{x^{**}}\| &= \|x^{**}\| = \|m\|,\end{aligned}$$

i.e. Φ is isometric. From this the strict continuity of Φ and Ψ also follows.

Proof of Main Theorem. For every $\omega, \nu \in M_1(X)$,

$$\begin{aligned}({}_{M(A)}\langle \omega, \nu\rangle)^* a &= (a^*({}_{M(A)}\langle \omega, \nu\rangle))^* \\ &= ({}_A\langle a^*\omega, \nu\rangle)^* = {}_A\langle \nu, a^*\omega\rangle = ({}_{M(A)}\langle \nu, \omega\rangle)a,\end{aligned}$$

so $({}_{M(A)}\langle \omega, \nu\rangle)^* = {}_{M(A)}\langle \nu, \omega\rangle$. Similarly $(\langle \omega, \nu\rangle_{M(B)})^* = \langle \nu, \omega\rangle_{M(B)}$. Since

$$\begin{aligned}b^*(\langle k\omega, k\omega\rangle_{M(B)})b &= \langle k\omega b, k\omega b\rangle_B \le \|k\|^2\langle \omega b, \omega b\rangle_B = \|k\|^2 b^*\langle \omega, \omega\rangle_{M(B)}b, \\ \langle k\omega, k\omega\rangle_{M(B)} &\le \|k\|^2\langle \omega, \omega\rangle_{M(B)}.\end{aligned}$$

Similarly

$$_{M(A)}\langle \omega l, \omega l\rangle \le \|l\|^2({}_{M(A)}\langle \omega, \omega\rangle).$$

Since

$$\begin{aligned} a({}_{M(A)}\langle k\omega, \nu\rangle) &= {}_A\langle ak\omega, \nu\rangle = (ak)({}_{M(A)}\langle \omega, \nu\rangle) \\ &= a(k({}_{M(A)}\langle \omega, \nu\rangle)), \\ {}_{M(A)}\langle k\omega, \nu\rangle &= k({}_{M(A)}\langle \omega, \nu\rangle). \end{aligned}$$

Similarly

$$\langle \omega, \nu l\rangle_{M(B)} = (\langle \omega, \nu\rangle_{M(B)})l.$$

So $M_1(X)$ is an $M(A)$-$M(B)$ Hilbert bimodule.

By Lemma 2.1 and Lemma 2.2, for completing the proof it is enough to prove Φ preserves the inner products. In fact for every $a, c \in A$,

$$\begin{aligned} a({}_{M(A)}\langle \Phi(m), \Phi(n)\rangle)c &= {}_A\langle a\Phi(m), c^*\Phi(n)\rangle = {}_A\langle m_A(a), n_A(c^*)\rangle \\ &= {}_A\langle n_A^* m_A(a), c^*\rangle = {}_A\langle a({}_{M(A)}\langle m, n\rangle), c^*\rangle \\ &= a({}_{M(A)}\langle m, n\rangle)c. \end{aligned}$$

So ${}_{M(A)}\langle \Phi(m), \Phi(n)\rangle = {}_{M(A)}\langle m, n\rangle$. Similarly

$$\langle \Phi(m), \Phi(n)\rangle_{M(B)} = \langle m, n\rangle_{M(B)}.$$

Similar to [1, Definition 1.8], a Hilbert bimodule homomorphism from A-B Hilbert bimodule X to C-D Hilbert bimodule Y is a triple (ϕ, ρ, ψ), where ϕ and ψ are the C^*-homomorphisms from A to C and B to D respectively, ρ is a continuous linear map from X to Y with $\rho(axb) = \phi(a)\rho(x)\psi(b)$, and

$$\phi({}_A\langle x_1, x_2\rangle) = {}_C\langle \rho(x_1), \rho(x_2)\rangle, \quad \psi(\langle x_1, x_2\rangle_B) = \langle \rho(x_1), \rho(x_2)\rangle_D.$$

A Hilbert bimodule homomorphism (ϕ, ρ, ψ) is called surjective, if ϕ, ρ, ψ all are surjective.

By [1, Proposition 1.9], if (ϕ, ρ, ψ) is a surjective Hilbert bimodule homomorphism from ${}_AX_B$ to ${}_CY_D$, then (ϕ, ρ, ψ) has a unique Hilbert bimodule homomorphism extension $(\bar\phi, \bar\rho, \bar\psi)$ from ${}_{M(A)}M(X)_{M(B)}$ to ${}_{M(C)}M(Y)_{M(D)}$ with $\bar\phi$, $\bar\rho$, $\bar\psi$ strictly continuous. Thanks to the proof of [3, Theorem 1.1.26], we have

Tietze Extension Theorem. *If (ϕ, ρ, ψ) is a surjective Hilbert bimodule homomorphism from ${}_AX_B$ to ${}_CY_D$ with A,B,C,D σ-unital, then its extension $(\bar\phi, \bar\rho, \bar\psi)$ is also surjective.*

Proof. Since (ϕ, ρ, ψ) is surjective with A,B,C,D σ-unital, $\bar\phi$ and $\bar\psi$ are surjective (see [4, 3.12.10]), and it is enough to prove $\rho : M(X) \to M(Y)$ is surjective. Let $\omega \in M(Y)$. Since C (D) is σ-unital, there is a bounded sequence $\{y_n\} \subseteq Y$ with $y_n \xrightarrow{s} \omega$, $y_1 = y_0 = 0$ (for example, $y_n = E_n\omega$, $n > 1$, where $\{E_n\}$ is a bounded approximate unit of C). Let h, g be the strict positive elements of A and B respectively for A,B σ-unital. Without loss of generality, we assume

$$\|\phi(h)(y_n - y_{n-1})\| < 1/2^n, \quad \|(y_n - y_{n-1})\psi(y)\| < 1/2^n \ (n = 0, 1, 2, \cdots).$$

We claim that there are $x_n \in X (n = 0, 1, 2, \cdots)$ with $\rho(x_n) = y_n$, and

$$\|h(x_n - x_{n-1})\| < 1/2^n, \quad \|(x_n - x_{n-1})g\| < 1/2^n.$$

In fact, by induction we assume that there are $x_0,x_1,\cdots,x_n \in X$ satisfying the norm inequations above, and we want to choose $x_{n+1} \in \rho^{-1}(y_{n+1}) \subseteq X$ such that $x_0,x_1,\cdots,x_n,x_{n+1}$ also satisfy them. First let x be a preimage of y_{n+1} under ρ. Since $\ker\phi$ is an ideal of A, there is an approximate unit $\{u_\lambda\}$ of $\ker\phi$ which is quasi-central for A (see [4, 3.12.14]). Let $z_\lambda = x - u_\lambda(x - x_n)$. Then $\rho(z_\lambda) = \rho(x) = y_{n+1}$,

$$\begin{aligned}\|(z_\lambda - x_n)g\| &= \|(I - u_\lambda)(x - x_n)g\| \\ &= \|(I - u_\lambda)_A\langle(x - x_n)g, (x - x_n)g\rangle(I - u_\lambda)\|^{1/2} \\ &\to \|\phi(_A\langle(x - x_n)g, (x - x_n)g\rangle)\|^{1/2} \quad (\lambda \to \infty) \\ &= \|_A\langle\rho((x - x_n)g), \rho((x - x_n)g)\rangle\|^{1/2} \\ &= \|(y_{n+1} - y_n)\psi(g)\| < 1/2^n.\end{aligned}$$

So $\lim\limits_{\lambda\to\infty} \|(z_\lambda - x_n)g\| < 1/2^n$. Similarly

$$\|(I - u_\lambda)h(x - x_n)\| \to \|\phi(h)(y_{n+1} - y_n)\| < 1/2^n.$$

Since $\{u_\lambda\}$ is quasi-central for A, i.e. for every $a \in A$, $\|au_\lambda - u_\lambda a\| \to 0$,

$$\|h(z_\lambda - x_n)\| = \|h(I - u_\lambda)(x - x_n)\| \to \|\phi(h)(y_{n+1} - y_n)\| < 1/2^n.$$

So we can choose a λ_0 and let $x_{n+1} = z_{\lambda_0}$ such that

$$\|h(x_{n+1} - x_n)\| < 1/2^n, \quad \|(x_{n+1} - x_n)g\| < 1/2^n.$$

By the definition and strict positivity of h, g, x_n is strictly convergent to an element $\nu \in M(X)$. So

$$\bar\rho(\nu) = \lim_{n\to\infty} \rho(x_n) = \lim_{n\to\infty} y_n = \omega$$

by the strict continunity of $\bar\rho$, i.e. $\bar\rho$ is surjective.

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