Chin. Ann. of Math.
21B: 3(2000),381-388.

THE GENERALIZED SMASH PRODUCT AND COPRODUCT

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Abstract

The author discusses the braiding structures of the generalized smash product bialgebra and the cobraiding structures of the generalized smash coproduct bialgebra. It is pointed out that doublecrossed product determined by a cocycle is the generalized smash product and that doublecocrossed coproduct determined by a weak R-matrix is the generalized smash coproduct.

Keywords Generalized smash product, Generalized coproduct, Braiding structure, Cobraiding structure
1991 MR Subject Classification 16W30
Chinese Library Classification O153.3 **Document Code** A
Article ID 0252-9599(2000)03-0381-08

§1. Preliminaries

Throughout the paper all spaces are over a fixed ground field K. If C is a coalgebra, then we always denote the comultiplication and counit by $\triangle$ and ε respectively. Set $\triangle(c) = \sum c_1 \otimes c_2$ for $c \in C$. If M is a right (left) C -comodule, then we use ρ for the structure map of M, and set $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$ $(\rho(m) = \sum m_{(-1)} \otimes m_{(0)})$ for $m \in M$. Let H be a bialgebra. Denote left H-module category by ${}_HM$ and left H-comodule category by ${}^HM$. If M and N are in ${}_HM$, then $M \otimes N$ is also a left H-module by pull-back along $\triangle$, i.e.,

$$h \cdot (m \otimes n) = \sum h_1 m \otimes h_2 n \quad \text{for } h \in H, \quad m \in M, \quad n \in N.$$

Dually if M and N are in ${}^HM$, then $M \otimes N$ is also a left H -comodule by pull-out along the multiplication of H, i.e.,

$$\rho(m \otimes n) = \sum m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \quad \text{for } m \in M, \quad n \in N.$$

For any left H-module algebra A, we have Smash product $A\#H$, which is an associative algebra with the identity $1\#1$. Also $j_A : A \longmapsto A\#H$ ($j_A(a) = a\#1$) and $i_H : H \longmapsto A\#H$ $(i_H(h) = 1\#h)$ are algebra embeddings. Similarly, for any left H-comodule coalgebra C, we also have Smash coproduct $C \diamond H$ with coproduct

$$\triangle(c \diamond h) = \sum (c_1 \diamond c_{2_{(-1)}} h_1) \otimes (c_{2_{(0)}} \diamond h_2).$$

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Also $\pi_C : C \diamond H \longmapsto C$ $(\pi_C(c \diamond h) = c\varepsilon(h))$ and $\pi_H : C \diamond H \longmapsto H$ $(\pi_C(c \diamond h) = \varepsilon(c)h)$ are coalgebra surjective maps.

Let A and H be bialgebras. A bilinear form $\tau : A \otimes H \longmapsto K$ is called a skew pairing[5] if the following equalities hold:

(Sp1) $\tau(ab,\ h) = \sum \tau(a,\ h_1)\tau(b,\ h_2)$,

(Sp2) $\tau(a,\ hg) = \sum \tau(a_1,\ g)\tau(a_2,\ h)$,

(Sp3) $\tau(1,\ h) = \varepsilon(h)$,

(Sp4) $\tau(a,\ 1) = \varepsilon(a)$,

where $a, b \in A,\ h, g \in H$. If τ is invertible (in $(A \otimes H)^*$), then (Sp3) and (Sp4) follow from (Sp1) and (Sp2). Also if τ is invertible, then τ^{-1} also satisfies the following equalities:

(Sp1′) $\tau^{-1}(ab,\ h) = \sum \tau^{-1}(a,\ h_2)\tau^{-1}(b,\ h_1)$,

(Sp2′) $\tau^{-1}(a,\ hg) = \sum \tau^{-1}(a_1,\ h)\tau^{-1}(a_2,\ g)$,

(Sp3′) $\tau^{-1}(1,\ h) = \varepsilon(h)$,

(Sp4′) $\tau^{-1}(a,\ 1) = \varepsilon(a)$,

where $a, b \in A,\ h, g \in H$. If A has an antipode s (H has a pode $\overline{s}$), then τ is invertible with $\tau^{-1}(a, h) = \tau(s(a), h)$ $(\tau^{-1}(a, h) = \tau(a, \overline{s}(h)))$. Clearly (H, τ) is a coquasitriangular bialgebra if τ is a skew pairing of (H, H) such that $gh = \sum \tau(h_1, g_1)h_2g_2\tau^{-1}(h_3, g_3)$. Let τ be an invertible skew pairing of (A, H). Then we have a bialgebra $A \bowtie_\tau H$ with the comultiplication of the tensor coproduct and with the multiplication $(a \bowtie h)(b \bowtie g) = \sum \tau(b_1, h_1)ab_2 \bowtie h_2g\tau^{-1}(b_3, h_3)$, where $a, b \in A,\ h, g \in H$. All coquasitriangular structures of $A \bowtie_\tau H$ are given in [3].

Let A and H be bialgebras. In [2], an invertible element $R = \sum R' \otimes R''$ in $A \otimes H$ is called a weak R-matrix of (A, H) if the following equalities hold:

(WR1) $(\triangle \otimes 1)R = \sum R' \otimes r' \otimes R''r''$,

(WR2) $(1 \otimes \triangle)R = \sum R'r' \otimes r'' \otimes R''$,

where $r = \sum r' \otimes r'' = R$. If R is a weak R-matrix of (A, H), clearly $(\varepsilon \otimes 1)R = 1_H$ and $(1 \otimes \varepsilon)R = 1_A$. R^{-1} has also similar properties. In fact, the notation of weak R-matrix is dual notation of skew pairing. Clearly (H, R) is a quasitriangular bialgebra if R is a weak R-matrix of (H, H) such that $\triangle^{\mathrm{cop}}(h) = R\triangle(h)R^{-1}$ for any $h \in H$. For any weak R-matrix of (A, H), a doublecrossed coproduct $A \bowtie^R H$ is defined as follows: $A \bowtie^R H = A \otimes H$ as algebras, and the coproduct is given by

$$\triangle(a \bowtie h) = \sum a_1 \bowtie R''h_1(R^{-1})'' \otimes R'a_2(R^{-1})' \bowtie h_2,$$

where $a \in A,\ h \in H$. Then $A \bowtie^R H$ is a bialgebra. In [2], it has been proved that a doublecrossed coproduct of $A^\phi \bowtie^\psi H$ is quasitriangular if and only if both A and H are quasitriangular and there exists a weak R-matrix R of (A, H) such that $A^\phi \bowtie^\psi H = A \bowtie^R H$ as bialgebras. Furthermore, all quasitriangular structures of $A \bowtie^R H$ are given.

§2. Generalized Smash Product and Coproduct

In this section, necessary and sufficient conditions that a generalized smash product admits a bialgebra structure with tensor coproduct are given. The coquasitriangular structure of the generalized smash product is discussed. Some important examples of the generalized smash product is investigated. Similarly the generalized smash coproduct is considered.

Theorem 2.1. *Let H be a bialgebra.*

(1) *Let A be a left H-module algebra and X a left H-comodule algebra. Define the generalized smash product $A\#X$ as follows: $A\#X = A\otimes X$ as vector spaces, and the multiplication is defined by*

$$(a\#x)(b\#y) = \sum a(x_{(-1)}\cdot b)\#x_{(0)}y,$$

where $a, b \in A,\ x, y \in X$. Then $A\#X$ is an associative algebra with the identity $1\#1$.

(2) *Furthermore, if both A and X are bialgebas, then $A\#X$ is a bialgebra with the comultiplication of tensor coproduct if and only if the map $f : a\#x \longmapsto \sum x_{(-1)}\cdot a \ \#\ x_{(0)}$ from $A\#X$ to $A\#X$ is a coalgebra map. Also $i_A : A \longmapsto A\#X$ $(a \longmapsto a\#x)$ and $i_X : X \longmapsto A\#X$ $(x \longmapsto 1\#x)$ are bialgebra injections. If A and X are Hopf algebras, then $A\#X$ is also a Hopf algebra with the antipode $s(a\#x) = \sum(1\#s(x))(s(a)\#1)$.*

Proof. (1) For $a, b, c \in A,\ x, y, z \in X$,

$$\begin{aligned}((a\#x)(b\#y))(c\#z) &= \Big(\sum a(x_{(-1)}\cdot b)\#x_{(0)}y\Big)(c\#z)\\ &= \sum a(x_{(-2)}\cdot b)(x_{(-1)}y_{(-1)}\cdot c)\#x_{(0)}y_{(0)}z\\ &= \sum a(x_{(-1)}\cdot(b(y_{(-1)}\cdot c)))\#x_{(0)}y_{(0)}z\\ &= (a\#x)\Big(\sum b(y_{(-1)}\cdot c)\#y_{(0)}z\Big)\\ &= (a\#x)((b\#y)(c\#z)),\end{aligned}$$

so $A\#X$ is associative. Clearly $1\#1$ is the identity.

(2)

$$\begin{aligned}\triangle((a\#x)(b\#y)) &= \triangle\Big(\sum a(x_{(-1)}\cdot b)\#x_{(0)}y\Big)\\ &= \sum(a_1(x_{(-1)}\cdot b)_1\#x_{(0)_1}y_1)\otimes\sum a_2(x_{(-1)}\cdot b)_2\#x_{(0)_2}y_2,\\ \triangle(a\#x)\triangle(b\#y) &= \Big(\sum a_1\#x_1\otimes a_2\#x_2\Big)\Big(\sum b_1\#y_1\otimes b_2\#y_2\Big)\\ &= \sum(a_1\#x_1)(b_1\#y_1)\otimes(a_2\#x_2)(b_2\#y_2)\\ &= \sum a_1(x_{1_{(-1)}}\cdot b_1)\#x_{1_{(0)}}y_1\otimes a_2(x_{2_{(-1)}}\cdot b_2)\#x_{2_{(0)}}y_2.\end{aligned}$$

So $\triangle((a\#x)(b\#y)) = \triangle(a\#x)\triangle(b\#y)$ if and only if

$$\sum(x_{(-1)}\cdot b)_1\#x_{(0)_1}\otimes(x_{(-1)}\cdot b)_2\#x_{(0)_2} = \sum(x_{1_{(-1)}}\cdot b_1)\#x_{1_{(0)}}\otimes(x_{2_{(-1)}}\cdot b_2)\#x_{2_{(0)}}.$$

Similarly $\varepsilon((a\#x)(b\#y)) = \varepsilon(a\#x)\varepsilon(b\#y)$ if and only if $\varepsilon(x)\varepsilon(b) = \varepsilon(x_{(-1)}\cdot b)\varepsilon(x_{(0)})$. The other statements can easily be got.

Remark 2.1. In [4] and [7], the concept of the generalized smash product has appeared. For simplicity, we give the following definition.

Definition 2.1. *Let H be a bialgebra. $A\#X$ is called the generalized smash product associated to H if X is a left H comodule algebra and A is a left module algebra. In addition, $A\#X$ is called the generalized smash product bialgebra associated to H if $A\#X$ is a bialgebra with the comultiplication of tensor coproduct, i.e.,* (2) *in Theorem* 2.1 *holds.*

Proposition 2.1. *Let A and H be bialgebras, τ an invertible skew pairing of $A\otimes H$. Let $A\bowtie_\tau H$ be a doublecrossed product which is a bialgebra with tensor coproduct and the product defined by $(a\bowtie h)(b\bowtie g) = \sum\tau(b_1, h_1)ab_2\bowtie h_2g\tau^{-1}(b_3, h_3)$. If H is a Hopf algebra with invertible antipode s, then $A\bowtie_\tau H$ is the generalized smash product.*

Proof. For any $h, g \in H$ and $a \in A$, define

$$\rho(h) = \sum (s^{-1}(h_3) \otimes h_1) \otimes h_2 \in (H^{\mathrm{op}} \otimes H) \otimes H,$$
$$(h \otimes g) \cdot a = \sum \tau(a_1, g) a_2 \tau(a_3, h).$$

One can directly verify that H is a left $H^{\mathrm{op}} \otimes H$ comodule algebra and A is a left $H^{op} \otimes H$ module algebra. So one can from the generalized smash product $A\#H$ according to Theorem 2.1. Furthermore,

$$\begin{aligned}(a\#h)(b\#h) &= \sum a(h_{-1} \cdot b)\#h_{(0)}g \\ &= \sum a((s^{-1}(h_3) \otimes h_1) \cdot b)\#h_2 g \\ &= \sum a\tau(b_1, h_1) b_2 \tau(b_3, s^{-1}(h_3))\#h_2 g \\ &= \sum \tau(b_1, h_1) a b_2 \# h_2 g \tau^{-1}(b_3, h_3). \\ &= (a \bowtie h)(b \bowtie g).\end{aligned}$$

Therefore, $A \bowtie_\tau H$ is the same as $A\#H$ as bialgebras.

Proposition 2.2. *Let H be a finite dimension Hopf algebra. Then the Drinfeld double $D(H)$ is the generalized smash product.*

Proof. For any $h, g, k \in H$ and $h^* \in H^*$, define

$$\rho(h) = \sum (s^{-1}(h_3) \otimes h_1) \otimes h_2 \in (H^{\mathrm{op}} \otimes H) \otimes H,$$
$$\langle (h \otimes g) \cdot h^*, k \rangle = \langle h^*, hkg \rangle.$$

One can directly verify that H is a left $H^{\mathrm{op}} \otimes H$ comodule algebra and $(H^*)^{\mathrm{cop}}$ is a left $H^{\mathrm{op}} \otimes H$ module algebra. So one can from the generalized smash product $H^{*\mathrm{cop}}\#H$ according to Theorem 2.1. Furthermore,

$$\begin{aligned}(h^*\#h)(g^*\#g) &= \sum h^*((s^{-1}(h_3) \otimes h_1) \cdot g^*)\#h_2 g \\ &= \sum h^* \langle g^*, s^{-1}(h_3) \otimes h_1 \rangle \# h_2 g \\ &= (h^* \bowtie h)(g^* \bowtie g).\end{aligned}$$

Hence $D(H)$ is actually the generalized smash product.

Remark 2.2. Let A be a bialgebra and H a Hopf algebra. Let A be an H bimodule algebra, i.e., A is an H bimodule, and A is a left H module algebra and a right H module algebra. In [8], the authors constructed an algebra $A \star H$ which is equal to $A \otimes H$ as vector space and which has the mutiplication

$$(a \otimes h)(b \otimes g) = \sum a(h_1 \rightharpoonup b \leftharpoonup s(h_3)) \otimes h_2 l,$$

where $a, b \in A$ and $h, g \in H$, s is the antipode of H. If one defines

$$\rho(h) = \sum (s(h_3) \otimes h_1) \otimes h_2 \in (H^{\mathrm{op}} \otimes H) \otimes H,$$
$$(h \otimes g) \rightharpoonup a = g \rightharpoonup a \leftharpoonup h,$$

where $a \in A$ and $h \in H$, then H is a left $H^{\mathrm{op}} \otimes H$ comodule algebra and A is a left $H^{\mathrm{op}} \otimes H$ module algebra. Clearly the generalized smash product $A\#H$ is the same as $A \star H$.

Lemma 2.1. *Let H be a Hopf algebra, $A\#X$ the generalized smash product bialgebra associated to H. Let B be a bialgebra. If $\alpha : A \longmapsto B$ and $\beta : X \longmapsto B$ are bialgebra maps,* *then there exists a unique bialgebra map $F : A\#X \longmapsto B$ such that $Fi_A = \alpha$ and $Fi_X = \beta$ if and only if $\sum \alpha(x_{(-1)} \cdot a)\beta(x_{(0)}) = \beta(x)\alpha(a)$.*

Proof. Define $F : A\#X \longmapsto B$ by $F(a\#x) = \alpha(a)\beta(b)$, where $a \in A, x \in X$. It is very easy to check that F is a bialgebra map if and only if

$$\sum \alpha(x_{(-1)} \cdot a)\beta(x_{(0)}) = \beta(x)\alpha(a).$$

Theorem 2.2. *Let $A\#X$ be the generalized smash product bialgebra associated to H. If $(A\#X, \sigma)$ is a coquasitriangular bialgebra, then A and X are coquasitriangular bialgebras, and there exists an invertible skew pairing τ of (A, X) such that $A\#X$ is the same as $A \bowtie_\tau X$ as bialgebras.*

Proof. Since A and X are subbialgebras of $A\#X$, A and X are coquasitriangular bialgebras. For $a, b \in A$ and $x, y \in X$, by definition we have

$$\sum \sigma(a_1\#x_1, b_1\#y_1)(a_2\#x_2)(b_2\#y_2) = (b_1\#y_1)(a_1\#x_1)\sigma(a_2\#x_2, b_2\#y_2).$$

Letting $b = 1$ and $x = 1$, we have

$$\sum \sigma(a_1\#1, 1\#y_1)(a_2\#y_2) = (1\#y_1)(a_1\#1)\sigma(a_2\#1, 1\#y_2).$$

So

$$(1\#y)(a\#1) = \sum \sigma(a_1\#1, 1\#y_1)(a_2\#y_2)\sigma^{-1}(a_3\#1, 1\#y_3).$$

Define $\tau(a, x) = \sigma(a\#1, 1\#x)$ for $a \in A$ and $x \in X$. Clearly τ is an invertible skew pairing of (A, X) and

$$(a\#x)(b\#y) = \sum \tau(b_1, x_1)ab_2\#x_2y\tau^{-1}(b_3, x_3).$$

Remark 2.3. In [3], the author has proved the following fact: If (A, ζ) and (X, η) are coquasitriangular bialgebras, then $A \bowtie_\tau X$ is also a coquasitriangular bialgebra with braiding

$$\sigma(a \bowtie x, b \bowtie y) = \sum \tau(a_1, y_1)\zeta(a_2, b_1)\eta(x_1, y_2)\tau^{-1}(b_2, x_2).$$

Also all braidings are given.

Now we turn to the discussion on the generalized smash coproduct. Let H be a bialgebra, C a left H comodule coalgebra, X a left module coalgebra. If we set $C \diamond X = C \otimes X$ as vector spaces and define

$$\triangle(c \diamond x) = \sum (c_1 \diamond c_{2_{(-1)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond x_2), \quad \varepsilon(c \diamond x) = \varepsilon(c)\varepsilon(x),$$

where $c \in C, x \in X$, then $(C \diamond X, \triangle, \varepsilon)$ is a coalgebra. In fact,

$$\begin{aligned}
(1 \otimes \triangle) \triangle (c \diamond x) &= (1 \otimes \triangle) \sum (c_1 \diamond c_{2_{(-1)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond x_2) \\
&= \sum (c_1 \diamond c_{2_{(-1)}} c_{3_{(-2)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond c_{3_{(-1)}} \cdot x_2) \otimes (c_{3_{(0)}} \diamond x_3) \\
&= \sum (c_1 \diamond c_{2_{(-1)}} \cdot (c_{3_{(-2)}} \cdot x_1)_1 \otimes (c_{2_{(0)}} \diamond (c_{3_{(-2)}} \cdot x_1)_2 \otimes (c_{3_{(0)}} \diamond x_3) \\
&= \sum \triangle(c_1 \diamond c_{2_{(-1)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond x_2) \\
&= (\triangle \otimes 1) \triangle (c \diamond x).
\end{aligned}$$

Therefore $(C \diamond X, \triangle, \varepsilon)$ is a coalgebra. We call $C \diamond X$ the generalized smash coproduct associated to H. The following theorem gives necessary and sufficient conditions for $C \diamond X$ to be a bialgebra.

Theorem 2.3. *Let H be a bialgebra, $C\diamond X$ the generalized smash coproduct associated to H. Assume that both C and X are bialgebras. Then the following are equivalent*:

(1) *$C\diamond X$ is a bialgebra whose multiplication is tensor product.*

(2) *The following equalities hold*:

(i) $\sum(ab)_{(-1)}\cdot xy\otimes(ab)_{(0)}=\sum(a_{(-1)}\cdot x)(b_{(-1)}\cdot y)\otimes a_{(0)}b_{(0)}$,

(ii) $\sum 1_{(-1)}\cdot 1_X\otimes 1_{(0)}=1_X\otimes 1_C$,

where $\rho(1_C)=\sum 1_{(-1)}\otimes 1_{(0)},\ a,b\in C,\ x,y\in X$.

(3) *The following equalities hold*:

(i) $\sum a_{(-1)}\cdot xy\otimes a_{(0)}=\sum(a_{(-1)}\cdot x)(1_{(-1)}\cdot y)\otimes a_{(0)}1_{(0)}$,

(ii) $\sum(ab)_{(-1)}\cdot x\otimes(ab)_{(0)}=\sum(a_{(-1)}\cdot x)(b_{(-1)}\cdot 1)\otimes a_{(0)}b_{(0)}$,

(iii) $\sum 1_{(-1)}\cdot 1_X\otimes 1_{(0)}=1_X\otimes 1_C$,

where $\rho(1_C)=\sum 1_{(-1)}\otimes 1_{(0)},\ a,b\in C,\ x,y\in X$.

(4) *The following equalities hold*:

(i) $\sum a_{(-1)}\cdot xy\otimes a_{(0)}=\sum(1_{(-1)}\cdot x)(a_{(-1)}\cdot y)\otimes 1_{(0)}a_{(0)}$,

(ii) $\sum(ab)_{(-1)}\cdot x\otimes(ab)_{(0)}=\sum(a_{(-1)}\cdot 1)(b_{(-1)}\cdot x)\otimes a_{(0)}b_{(0)}$,

(iii) $\sum 1_{(-1)}\cdot 1_X\otimes 1_{(0)}=1_X\otimes 1_C$,

where $\rho(1_C)=\sum 1_{(-1)}\otimes 1_{(0)},\ a,b\in C,\ x,y\in X$.

Proof. (1) $\Longrightarrow$ (2) Since $1_C\diamond 1_X$ is the identity of $C\diamond X$,

$$\triangle(1_C\diamond 1_X)=(1_C\diamond 1_X)\otimes(1_C\diamond 1_X)=\sum(1_C\diamond 1_{(-1)}\cdot 1_X)\otimes(1_{(0)}\diamond 1_X).$$

Hence $\sum 1_{(-1)}\cdot 1_X\otimes 1_{(0)}=1_X\otimes 1_C$, where $\rho(1_C)=\sum 1_{(-1)}\otimes 1_{(0)}$. Since

$$\begin{aligned}\triangle((a\#x)(b\#y))&=\triangle(ab\#xy)\\&=\sum(a_1b_1\diamond(a_2b_2)_{(-1)}\cdot x_1y_1)\otimes((a_2b_2)_{(0)}\diamond x_2y_2)\\&=\triangle(a\#x)\triangle(b\#y)\\&=\sum(a_1\diamond a_{2_{(-1)}}\cdot x_1)\otimes(a_{2_{(0)}}\diamond x_2)(b_1\diamond b_{2_{(-1)}}\cdot y_1)\otimes(b_{2_{(0)}}\diamond y_2)\\&=\sum(a_1b_1\diamond(a_{2_{(-1)}}\cdot x_1)(b_{2(-1)}\cdot y_1)\otimes((a_{2_{(0)}}b_{2_{(0)}}\diamond x_2y_2),\end{aligned}$$

we have

$$\begin{aligned}&\sum(a_1b_1\diamond(a_2b_2)_{(-1)}\cdot x_1y_1)\otimes((a_2b_2)_{(0)}\diamond x_2y_2)\\=&\sum(a_1b_1\diamond(a_{2_{(-1)}}\cdot x_1)(b_{2(-1)}\cdot y_1)\otimes((a_{2_{(0)}}b_{2_{(0)}}\diamond x_2y_2).\end{aligned}$$

Applying $\varepsilon\otimes 1\otimes 1\otimes\varepsilon$, we have

$$\sum(ab)_{(-1)}\cdot xy\otimes(ab)_{(0)}=\sum(a_{(-1)}\cdot x)(b_{(-1)}\cdot y)\otimes a_{(0)}b_{(0)}.$$

So (2) holds.

(2) $\Longrightarrow$ (1) We can reverse the procedure above.

(2) $\Longrightarrow$ (3) Clearly.

(3) $\Longrightarrow$ (2)

$$\begin{aligned}\sum(ab)_{(-1)}\cdot xy\otimes(ab)_{(0)}&=\sum(a_{(-1)}\cdot xy)(b_{(-1)}\cdot 1_X)\otimes a_{(0)}b_{(0)}\\&=\sum(a_{(-1)}\cdot x)(1_{(-1)}\cdot y)(b_{(-1)}\cdot 1_X)\otimes a_{(0)}1_{(0)}b_{(0)}\\&=\sum(a_{(-1)}\cdot x)(b_{(-1)}\cdot y)\otimes a_{(0)}b_{(0)}.\end{aligned}$$

(2) $\Longleftrightarrow$ (4) Similarly.

We call $C \diamond X$ the generalized smash coproduct bialgebra if $C \diamond X$ is a bialgebra whose multiplication is tensor product, i.e., one of the equivalent conditions holds. Let $C \diamond X$ be the generalized smash coproduct bialgebra. Define

$$\Pi_C : C \diamond X \longmapsto C \quad \text{by} \quad \Pi_C(c \diamond x) = c\varepsilon(x)$$

and

$$\Pi_X : C \diamond X \longmapsto X \quad \text{by} \quad \Pi_C(c \diamond x) = \varepsilon(c)x.$$

Clearly Π_C and Π_X are bialgebra surjections.

Proposition 2.3. *Let A and H be bialgebras, $R = \sum R^{'} \otimes R''$ a weak R-Matrix of (A, H). Then doublecrossed coproduct $A \bowtie^R H$ is the generalized smash coproduct.*

Proof. Define

$$\rho(a) = \sum (R^{-1})'' \otimes R'' \otimes R' a (R^{-1})',$$

where $a \in A$, R is the weak R-matrix of (A, H), and R^{-1} is the inverse of R. We exclaim that A is a left $H^{op} \otimes H$ comodule coalgebra. For $a \in A$,

$$\begin{aligned}
\sum a_{(-1)} \otimes \triangle(a_{(0)}) &= \sum (R^{-1})'' \otimes R'' \otimes \triangle(R' a (R^{-1})') \\
&= \sum (R^{-1})'' \otimes R'' \otimes \triangle(R') \triangle (a) \triangle ((R^{-1})') \\
&= \sum (r^{-1})''(R^{-1})'' \otimes R''r'' \otimes (R' \otimes r')(a_1 \otimes a_2)((R^{-1})' \otimes (r^{-1})') \\
&= \sum (r^{-1})''(R^{-1})'' \otimes R''r'' \otimes (R' a_1 (R^{-1})' \otimes r' a_2 (r^{-1})') \\
&= \sum (R^{-1})'' \cdot (r^{-1})'' \otimes R''r'' \otimes (R' a_1 (R^{-1})' \otimes r' a_2 (r^{-1})') \\
&= \sum a_{1_{(-1)}} a_{2_{(-1)}} \otimes a_{1_{(0)}} \otimes a_{1_{(0)}},
\end{aligned}$$

where

$$R = \sum R^{'} \otimes R'' = r = \sum r^{'} \otimes r'', \quad r^{-1} = \sum (r^{-1})^{'} \otimes (r^{-1})''.$$

By the definition of weak R-matrix, we have

$$\sum a_{(-1)} \varepsilon(a_{(0)}) = \sum (R^{-1})'' \otimes R'' \otimes \varepsilon(R' a (R^{-1})') = \varepsilon(a) 1 \otimes 1.$$

So A is a left $H^{\rm op} \otimes H$ comodule coalgebra. For any $h \otimes g \in H^{\rm op} \otimes H,\ k \in H$, define the action as follows

$$(h \otimes g) \cdot k = gkh.$$

Thus H is a left $H^{\rm op} \otimes H$ module coalgebra. Whence we can form the generalized smash coproduct $A \diamond H$. Moreover

$$\begin{aligned}
\triangle(a \diamond h) &= \sum (a_1 \diamond a_{2_{(-1)}} \cdot h_1) \otimes (a_{2_{(0)}} \diamond h_2) \\
&= \sum (a_1 \diamond ((R^{-1})'' \otimes R'') \cdot h_1) \otimes (R' a_2 (R^{-1})' \diamond h_2) \\
&= \sum (a_1 \diamond (R^{-1})'' h_1 R'') \otimes (R' a_2 (R^{-1})' \diamond h_2) \\
&= \triangle(a \bowtie h).
\end{aligned}$$

Therefore, $A \bowtie^R H = A \diamond H$ as bialgebras.

Let H be a bialgebra, $C \diamond X$ a generalized smash coproduct bialgebra associated to H. If C and X are Hopf algebras, then it is easy to check that $C \diamond X$ is also a Hopf algebra with antipode

$$s(c \diamond x) = \sum (1 \diamond s(c_{(-1)} \cdot x)(s(c_{(0)}) \diamond 1),$$

where $c \in cC, x \in X$.

Theorem 2.4. *Let H be a bialgebra, $C \diamond X$ a generalized smash coproduct bialgebra associated to H. If $(C \diamond X, R)$ is a quasitriangular bialgebra, then C and X are quasitriangular bialgebras, and there exists a weak R-matrix of (C, X) such that $C \diamond X = C \bowtie^R X$.*

Proof. Since $\Pi_C : C \diamond X \longmapsto C$ $(\Pi_C(c \diamond x) = c\varepsilon(x))$ and $\Pi_X : C \diamond X \longmapsto X$ $(\Pi_C(c \diamond x) = \varepsilon(c)x)$ are surjections, $(C, (\Pi_C \otimes \Pi_C)R)$ and $(X, (\Pi_X \otimes \Pi_X)R)$ are quasitriangular bialgebras. By the definition,

$$\triangle^{\mathrm{cop}}(c \diamond x) = R \triangle (c \diamond x) R^{-1},$$

this is

$$\sum (c_{2_{(0)}} \diamond x_2) \otimes (c_1 \diamond c_{2_{(-1)}} \cdot x_1) = R\Big(\sum (c_1 \diamond c_{2_{(-1)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond x_2)\Big) R^{-1}.$$

Let $U = (1 \otimes \varepsilon \otimes \varepsilon \otimes 1)R$. Clearly U is a weak R-matrix of (C, X). Applying $1 \otimes \varepsilon \otimes \varepsilon \otimes 1$ on the two sides of the equality above, we have

$$\sum c_{(0)} \otimes c_{(-1)} \cdot x = U(c \otimes x) U^{-1}.$$

So $\sum c_{(-1)} \cdot x \otimes c_{(0)} = T(U)(x \otimes c) T(U^{-1})$, where T is the usual twist map, i.e., $T(c \otimes x) = x \otimes c$. Thus

$$\begin{aligned} \triangle(c \diamond x) &= \sum (c_1 \diamond c_{2_{(-1)}} \cdot x_1) \otimes (c_{2_{(0)}} \diamond x_2) \\ &= \sum c_1 \diamond T(U)(x_1 \otimes c_2) T(U^{-1}) \diamond x_2 \\ &= \triangle(c \bowtie x). \end{aligned}$$

References

[1] Radford, D. E., The structure of Hopf algebra with a projection [J], *J. Alg.,* **92**(1985), 322–347.
[2] Chen, H. X., Quasitriangular structures of doublecrossed coproducts [J], *J. Algebra,* **204**(1998), 504–531.
[3] Chen, H. X., Braiding structures of doublecrossed coproducts [J], *Tsukuba J. Math.,* **22:**3(1998), 589–609.
[4] Chen, H. X., Cocycle deformations for double crossproducts [J], *Chin. Ann. of Math.,* **20A:**1(1999), 39–46(in Chinese).
[5] Takeuchi, M., Ext_{ad}(spR, μ^A)≅ $\widehat{B_r}(A/k)$ [J], *J. Alg.,* **67**(1980), 436–475.
[6] Doi, Y. & Takeuchi, M., Multiplication alteration by two-cocycles [J], *Comm. Alg.,* **22:**14(1994), 5715–5732.
[7] Kassel, C., Quantum groups [M], Graduate Texts in Math., 155, Springer-Verlag, New YorK-Berlin, 1995.
[8] Caenepeel, S., Militaru, G. & Zhu, S., Doi-Hopf modules, Yetter-Drinfeld modules and Frobenius type properties [J], *Trans. Amer. Soc.,* **349**(1997), 4311–4342.
[9] Caenepeel, S., Militaru, G. & Zhu, S., A Maschke type theorem for Doi-Hopf modules and applications [J], *J. Algebra,* **187**(1997), 388–412.
[10] Wang, S. H. & Li, J. Q., On the twisted smash products for bimodule algebras and the Drinfeld double [J], *Comm. Alg.,* **26:**8(1998), 2435–2244.