Chin. Ann. of Math.
21B: 3(2000),389-402.

INITIAL-BOUNDARY VALUE PROBLEM FOR THE UNSATURATED LANDAU-LIFSHITZ SYSTEM***

DING Shijin* GUO Boling**

Abstract

The existence, partial regularity and uniqueness of weak solution to the initial boundary value problem for the unsaturated Landau-Lifschitz systems are given.

Keywords Unsaturated Landau-Lifschitz systems, Ferromagnetic spin chain, Initial boundary problem
1991 MR Subject Classification 35D10, 35Q80
Chinese Library Classification O175.2 **Document Code** A
Article ID 0252-9599(2000)02-0389-14

§1. Introduction

Let $\Omega \subset R^n$ $(n=1,2)$ be a bounded smooth domain. Consider the following nonhomogeneous initial-boundary value problem for the unsaturated Landau-Lifshitz systems of ferromagnetic spin chain with Gilbert damping constant $\alpha_1 > 0$,

$$u_t = -\alpha_1 u \times (u \times \triangle u) + \alpha_2 u \times \triangle u, \quad \text{in } \ \Omega \times R_+, \tag{1.1}$$

$$u|_{\partial\Omega \times R_+} = \psi(x), \qquad u|_{\Omega \times \{t=0\}} = u_0(x), \tag{1.2}$$

where α_2 is the exchange constant, $u = (u^1, u^2, u^3)$, $u_0(x)$ is smooth and satisfies the unsaturated condition, i.e., $|u_0(x)| \not\equiv$ constant, and $\int_\Omega |\nabla u_0|^2 < +\infty$, $u_0(x)|_{\partial\Omega} = \psi(x)$. Denote $\varphi(x) = |u_0(x)|$. We assume $0 < m = \min\limits_{\overline{\Omega}} \varphi < M = \max\limits_{\overline{\Omega}} \varphi$. Throughout this paper we assume $\triangle\varphi \leq 0$.

System (1.1) describes the motion of unsaturated ferromagnetic spin chain without applied magnetic field. The magnitude of the spin is finite, i.e., $|u| = \varphi(x)$. (1.1) is implied by the conservation of energy and magnitude of u, and is a version which gives rise to a continuous wave theory.

For the saturated problem, i.e., $|u_0| \equiv$ constant (cf. $|u_0| = 1$), a lot of works contributed to the study of solutions to the Landau-Lifshitz systems of 1-dimensional motion spin chain have been made by mathematicians (see [10, 11]). For the initial problem and the homogeneous boundary problem, we refer to [3, 9, 10–13].

Manuscript received May 18, 1998. Revised October 14, 1999.

*Department of Mathematics, South China Normal University, Guangzhou 510630, China.

**Center for Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China.

***Project supported by the National Natural Science Foundation of China (No. 19971030).

In 1993, Guo and Hong[9] established the global existence and partial regularity theorems concerning the weak solutions of (1.1) ($\varphi \equiv 1$) on a 2-dimensional Riemannian manifold (without boundary) and revealed the links between the solutions and the harmonic maps. They found that the solutions have the same regularity as that of the harmonic map heat flow of [4]. Recently, Chen, Ding and Guo[3] proved that any weak solution with finite energy is partial regular and is unique.

The first work concerning the nonhomogeneous boundary value problem for the saturated Landau-Lifshitz systems was done recently by Guo and Ding[8]. In that paper, the authors dealt with this problem by some detailed analyses on the asymptotic behavior of a penalty problem to conclude that there exists unique weak solution, which is "almost smooth" (see [3]) when $\dim\Omega = 2$ and is globally smooth when the initial data is "energy small" or $\dim\Omega = 1$. The idea comes from the corresponding one in studying the asymptotics of the Ginzburg-Landau functionals due to Bethuel-Brezis-Hélein[1,2]. The uniqueness was also obtained .

In physics there are a lot of unsaturated problems. On the other hand, the unsaturated initial condition is more natural than the saturated one. For these reasons we are motivated to discuss Problem (1.1)–(1.2)

Our main results are Theorem 4.1, Theorem 4.2 and Theorem 5.1.

§2. Preliminaries

In this section, we give two equations equivalent to Equation (1.1) and construct a penalty problem related to these equations. And then we discuss the solution of the penalty problem.

Similarly to [9], we have the following two lemmas

Lemma 2.1. *Let u be a solution of* (1.1)–(1.2) *in the classical sense. We have*

$$|u(x,t)| = \varphi(x), \qquad \forall\ (x,t) \in \overline{\Omega} \times R_+. \tag{2.1}$$

Lemma 2.2 *In the classical sense, u is a solution of* (1.1)–(1.2) *if and only if u is a solution of one of the following equations subject to Condition* (1.2):

$$u_t = \alpha_1(\varphi^2 \triangle u + u|\nabla u|^2) - \frac{\alpha_1}{2} u \triangle \varphi^2 + \alpha_2 u \times \triangle u \tag{2.2}$$

or equivalently

$$\frac{\alpha_1 \varphi^2}{{\alpha_1}^2 \varphi^2 + {\alpha_2}^2} u_t - \frac{\alpha_2}{{\alpha_1}^2 \varphi^2 + {\alpha_2}^2} u \times u_t = \varphi^2 \triangle u + u|\nabla u|^2 - \frac{1}{2} u \triangle \varphi^2. \tag{2.3}$$

By Lemma 2.2, it is natural to consider the following penalty problem

$$\frac{\alpha_1 \varphi^2}{{\alpha_1}^2 \varphi^2 + {\alpha_2}^2} u_t - \frac{\alpha_2}{{\alpha_1}^2 \varphi^2 + {\alpha_2}^2} u \times u_t = \varphi^2 \triangle u + \frac{1}{\varepsilon^2} u(\varphi^2 - |u|^2) \tag{2.4}$$

accompanied by Condition (1.2).

In the sequel, we denote by ν and τ the unit outer normal vector and the unit tangential vector to $\partial\Omega$ such that (ν, τ) is directed. We also use the following notations: $\Omega(t) = \Omega \times \{t\}$, $\Omega_t = \Omega \times (0,t)$, $B_r(x) = B(x,r)$.

Similarly to [8], we have the following three lemmas

Lemma 2.3. *For any given* $\varepsilon > 0$*, Problem* (2.4)–(1.2) *admits a global smooth solution* u_ε *satisfying*

$$|u_\varepsilon| \le \varphi, \qquad on \ \ \overline{\Omega} \times [0,T], \tag{2.5}$$

and there are constants $C_1 > 0$ *independent of* ε *and* $C_2 > 0$ *such that*

$$\int_0^T \int_{\partial\Omega} \Big|\frac{\partial u_\varepsilon}{\partial \nu}\Big|^2 \le C_1, \tag{2.6}$$

$$\|u_\varepsilon\|_{W_2^{2,1}(\Omega_T)} \le C_2. \tag{2.7}$$

Lemma 2.4. *For any given* $T > 0$*, there exists a global weak solution* u *of* (2.3)–(1.2) *in* V*, where*

$$V = \Big\{u \Big| \, |u(x,t)| = \varphi(x) \ a.e. \ (x,t) \in \Omega \times [0,T], \ u \ \ is \ measurable \ and$$
$$\int_0^T \int_\Omega |u_t|^2 dxdt + \operatorname*{esssup}_{0 \le t \le T} \int_\Omega |\nabla u(\cdot,t)|^2 dx < \infty\Big\},$$

and the following identity holds for any weak solution u :

$$\int_0^T \int_\Omega \frac{\alpha_1}{\alpha_1^2\varphi^2 + \alpha_2^2}|u_t|^2 + \frac{1}{2}\int_{\Omega(T)} |\nabla u|^2 = \frac{1}{2}\int_\Omega |\nabla u_0|^2. \tag{2.8}$$

Lemma 2.5. *Let* u_ε *be the solution of* (2.4)–(1.2)*. Then we have a subsequence, denoted by* u_{ε_n}*, and a weak solution* u *of* (2.3)–(1.2)*, such that*

$$u_{\varepsilon_n t} \to u_t \ \ strongly \ in \ \ L^2(0,T;L^2(\Omega)), \tag{2.9}$$

$$\nabla u_{\varepsilon_n} \to \nabla u \ \ strongly \ in \ \ L^2(\Omega), \ \ \forall t > 0. \tag{2.10}$$

§3. Estimates Uniformly in ε

In this section we shall give some uniform estimates for the solutions of (2.2)–(1.2).

First, it follows from [8] that the following lemma holds.

Lemma 3.1. *For any given* $T > 0$*, there is a constant* $C > 0$ *independent of* ε *and* T *such that for the solution,* u_ε*, of* (2.4)–(1.2) *obtained in Lemma* 2.4*, we have*

$$\sup_{t \in [0,T]} \|\nabla u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C\varepsilon^{-1}. \tag{3.1}$$

Also from the argument in [8], one has

Lemma 3.2. *There are constants* $\lambda_0 > 0$*,* $\mu_0 > 0$ *independent of* ε *and* t *such that if*

$$\frac{1}{\varepsilon^2}\int_{\Omega \cap B_{2l}} \varphi^2(1 - |u_\varepsilon/\varphi|^2)^2 \le \mu_0 \tag{3.2}$$

for $l/\varepsilon \ge \lambda_0$*,* $0 < l \le 1$*, then*

$$|u_\varepsilon| \ge \frac{m}{2}, \ \ \forall x \in \Omega \cap B_l, \tag{3.3}$$

where B_l *is any sphere in* R^2 *with radius* l*.*

Then we can prove

Lemma 3.3.[2,Lemma IV.2] *There exists a positive integer N independent of ε and t such that $CardJ \leq N$ and*

$$|u_\varepsilon| \geq \frac{m}{2} \quad on \;\; \Omega\backslash \bigcup_{j\in J} B(x_j, \lambda_0\varepsilon). \tag{3.4}$$

Moreover, we can choose (see [2, Section IV.2]) J' : $J' \subset J$ and $\lambda \geq \lambda_0$ such that

$$\begin{cases} |x_i - x_j| \geq 8\lambda\varepsilon, & i \neq j, \quad i, j \in J', \\ \bigcup\limits_{j\in J} B(x_j, \lambda_0\varepsilon) \subset \bigcup\limits_{j\in J'} B(x_j, \lambda\varepsilon), & \\ |u_\varepsilon| \geq \frac{m}{2} & \text{on } \Omega\backslash \bigcup\limits_{j\in J'} B(x_j, \lambda\varepsilon). \end{cases} \tag{3.5}$$

In the following of this section, we want to derive some estimates uniformly in ε for the solution of (2.4)–(1.2). We call $B(x_j, \lambda\varepsilon), j \in J'$ bad disks.

Lemma 3.4. *Let $x_0 \in \overline{\Omega}$, $P_r = B_r(x_0) \times [t_0, t_0 + r^2]$. If $|u_\varepsilon| \geq \alpha_0 > 0$ on P_r, then there exists a constant $C > 0$ independent of ε such that $\int_{P_r} |D^2 u_\varepsilon|^2 \leq C$.*

Proof. The proof can be done by modifying the proof of Lemma 2.4 in [8].

Lemma 3.5. *Let $|u_\varepsilon| \geq \alpha_0 > 0$ on $Q_{r,s} = B_r(x_0) \times [t_0 - s, t_0 + s]$. Then for any $q > 2$, there is a constant $C_q > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{W_q^{2,1}(Q_{r/2,s/2})} \leq C_q. \tag{3.6}$$

Proof. First of all, we have from Lemma 3.4 that $\|\nabla u_\varepsilon\|_{L^q(Q_{r,s})} \leq C_q$. Moreover we have for $\Psi = \frac{1}{\varepsilon^2}(1 - |u_\varepsilon/\varphi|^2)$,

$$\frac{\alpha_1\varepsilon^2}{\alpha_1^2\varphi^2 + \alpha_2^2}\Psi_t - \varepsilon^2\varphi^2\triangle\Psi + 2{\alpha_0}^2\Psi \leq 2|\nabla u_\varepsilon|^2, \quad \text{ in } \; Q_{r,s}. \tag{3.7}$$

Take cut-off function $\xi(x) \in C_0^\infty(B_r(x_0))$, $\xi \equiv 1$ in $B_{r/2}(x_0)$, $\eta(t) \in C_0^\infty([t_0 - s, t_0 + s])$, $\eta \equiv 1$ in $[t_0 - s/2, t_0 + s/2]$, $|\nabla\xi| \leq C/r$, $|\eta_t| \leq C/s$, $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1$. Multiply (3.7) by $\xi^2(x)\eta^2(t)\Psi^{q-1}$ and integrate it over $Q_{r,s}$ to give

$$\begin{aligned} &\frac{1}{2}\varepsilon^2(q-1)\int_{Q_{r,s}} \xi^2\eta^2\varphi^2\Psi^{q-2}|\nabla\Psi|^2 + 2{\alpha_0}^2\int_{Q_{r,s}} \xi^2\eta^2\Psi^q \\ &\leq \sigma\int_{Q_{r,s}} \xi^2\eta^2\Psi^q + C_\sigma\int_{Q_{r,s}} \xi^2\eta^2|\nabla u_\varepsilon|^{2q} \\ &\quad + \int_{Q_{r,s}} \frac{2\alpha_1\varepsilon^2}{q(\alpha_1^2\varphi^2 + \alpha_2^2)}\xi^2\eta|\eta_t|\Psi^q + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}} \eta^2|\nabla(\xi\varphi)|^2\Psi^q. \end{aligned}$$

Set $\sigma = {\alpha_0}^2$ in above inequality. We have

$$\begin{aligned} \alpha_0^2\int_{Q_{r,s}} \xi^2\eta^2\Psi^q &\leq C\int_{Q_{r,s}} \xi^2\eta^2|\nabla u_\varepsilon|^{2q} + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}} \eta^2|\nabla(\xi\varphi)|^2\Psi^q \\ &\quad + 2\int_{Q_{r,s}} \frac{\alpha_1\varepsilon^2}{q(\alpha_1^2\varphi^2 + \alpha_2^2)}\xi^2\eta|\eta_t|\Psi^q. \end{aligned}$$

Hence

$${\alpha_0}^2\int_{Q_{r,s}} \xi^2\eta^2\Psi^q \leq C_q + C\varepsilon^2\int_{Q_{r,s}\backslash Q_{r/2,s/2}} \Big(\frac{1}{r^2}\Psi^q + \frac{1}{s}\Psi^q\Big).$$

Fixing r, s and taking ε small enough such that $\frac{C\varepsilon^2}{r^2} \leq \frac{1}{4}{\alpha_0}^2$, $\frac{C\varepsilon^2}{s} \leq \frac{1}{4}{\alpha_0}^2$, we obtain ${\alpha_0}^2\int_{Q_{r,s}} \xi^2\eta^2\Psi^q \leq C_q + \frac{{\alpha_0}^2}{2}\int_{Q_{r,s}\backslash Q_{r/2,s/2}} \Psi^q$.

It follows from hole-filling method that

$$\int_{Q_{r/2,s/2}} \Psi^q \le C_q, \qquad \forall q > 2, \tag{3.8}$$

and it is concluded from (3.8) and L^q theory of parabolic systems that (3.6) holds.

Corollary 3.1. *Under the assumption of Lemma* 3.5, *we have for any* $\gamma \in (0,1)$,

$$\|\nabla u_\varepsilon\|_{L^\infty(Q_{r,s})} \le C, \tag{3.9}$$

$$\|u_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(Q_{r,s})} \le C, \tag{3.10}$$

with C *independent of* ε.

§4. Smooth Solution and "Almost Smooth" Solution

In this section, we shall prove the following claims. For small initial data, there exists a global smooth solution for Problem (1.1)–(1.2). For general initial data with finite energy, the weak solution of (2.3)–(1.2) obtained in Section 2 is "almost smooth", i.e., it is smooth away from a set consisting of at most finitely many points and thus also solves (1.1)-(1.2).

Theorem 4.1. *Let* μ_0 *be determined in Lemma* 3.2 *and* $\int_\Omega |\nabla u_0|^2 \le \mu_0/2$. *Then for any given* $T > 0$, *Problem* (1.1)–(1.2) *has a smooth solution.*

Proof. It suffices to prove this theorem for Problem (2.3)–(1.2).

For any $t > 0$ it follows from (2.11) that

$$\sup_{0\le t\le T} \frac{1}{\varepsilon^2}\int_\Omega \varphi^2(1-|u_\varepsilon/\varphi|^2)^2 \le 2\int_\Omega |\nabla u_0|^2. \tag{4.1}$$

From this inequality we infer from Lemma 3.2 that $|u_\varepsilon| \ge \frac{m}{2}$, on $\Omega_T = \Omega\times[0,T]$. From Section 3, this implies for any $\gamma \in (0,1)$, $\|u_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(\Omega_T)} \le C$ with C independent of ε. Then Problem (2.3)–(1.2) admits a solution in $C^{1+\gamma,(1+\gamma)/2}(\overline{\Omega}_T)$. The conclusion of Theorem 4.1 follows from Schauder's method.

In the following, let $\int_\Omega |\nabla u_0|^2 < \infty$. We have

Theorem 4.2. *Let* u *be the weak solution of* (2.3)–(1.2) *obtained in Lemma* 2.4. *Then,* u *is smooth away from a set* A *where* A *is a set in* $\overline{\Omega}\times[0,\infty)$ *consisting of at most finitely many points. Moreover,* u *also solves* (1.1)–(1.2).

According to the proof of Theorem 4.1, it suffices to give $C^{1+\gamma,(1+\gamma)/2}$ estimates uniformly in ε for u_ε on the compact subset of $(\overline{\Omega}\times[0,\infty)\backslash A)$.

Lemma 4.1. *There exists* $\tilde{T}_1 > 0$ *independent of* ε *such that*

$$|u_\varepsilon| \ge \frac{m}{2}, \quad \textit{on } \ \overline{\Omega}\times[0,\tilde{T}_1]. \tag{4.2}$$

Proof. For any $x_0 \in \overline{\Omega}$, let ξ be the standard cut-off function on $B_{2R}(x_0)$ such that $0 \le \xi \le 1$, $\xi \equiv 1$ on $B_R(x_0)$. Test (2.4) by $\xi^2\varphi^{-2}u_{\varepsilon t}$ to give for any $\beta > 0$,

$$\int_0^t\int_\Omega \frac{\alpha_1}{\alpha_1^2\varphi^2+\alpha_2^2}\xi^2|u_{\varepsilon t}|^2 + \sup_{0\le\tau\le t}\Big[\frac12\int_{\Omega(\tau)}\xi^2|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega(\tau)}\xi^2\varphi^2(1-|u_\varepsilon/\varphi|^2)^2\Big]$$
$$\le \frac12\int_\Omega \xi^2|\nabla u_0|^2 + \beta\int_0^t\int_\Omega \xi^2|u_{\varepsilon t}|^2 + C_\beta\int_0^t\int_\Omega |\nabla\xi|^2|\nabla u_\varepsilon|^2.$$

Suitably taking β in above inequality, we have

$$\begin{aligned}&\sup_{0\le\tau\le t}\Big[\frac{1}{2}\int_{\Omega(\tau)}\xi^2|\nabla u_\varepsilon|^2+\frac{1}{4\varepsilon^2}\int_{\Omega(\tau)}\xi^2\varphi^2(1-|u_\varepsilon/\varphi|^2)^2\Big]\\&\le\frac{1}{2}\int_\Omega\xi^2|\nabla u_0|^2+C\int_0^t\int_\Omega|\nabla\xi|^2|\nabla u_\varepsilon|^2\le\frac{1}{2}\int_\Omega\xi^2|\nabla u_0|^2+\frac{4Ct}{R^2}\int_\Omega|\nabla u_0|^2.\end{aligned}\tag{4.3}$$

Fixing $R=R_0>0$, $t=\tilde{T}_1>0$ in (4.3) such that

$$\frac{1}{2}\int_{B_{2R_0}}|\nabla u_0|^2\le\mu_0/4,\qquad\frac{4C\tilde{T}_1}{R_0}\int_\Omega|\nabla u_0|^2\le\mu_0/4,$$

we deduce

$$\sup_{0\le t\le\tilde{T}_1}\frac{1}{\varepsilon^2}\int_{B_{R_0}(x_0)}\varphi^2(1-|u_\varepsilon/\varphi|^2)^2\le\mu_0/2.$$

It follows from this and Lemma 3.2 that $|u_\varepsilon|\ge\frac{m}{2}$, on $B_{R_0}(x_0)\times[0,\tilde{T}_1]$. This implies the desired result.

Now we define $T_1\ge\tilde{T}_1$ by

$$T_1=\inf\{\ T\mid T>0,\ \text{ there is } x_0\in\Omega \text{ such that} \liminf_{\varepsilon\to 0}|u_\varepsilon(x_0,T)|=0\ \}.\tag{4.4}$$

From the definition of T_1 we know that there is no bad disk on $\Omega(t)$ if $0\le t<T_1$ and for any $0\le T<T_1$ there holds $\|u_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{\Omega}\times[0,T])}\le C$.

Denote the bad disks on $\Omega(T_1)$ by $\{B(x_i^\varepsilon,\lambda\varepsilon)\times\{T_1\}\}$, $i=1,\cdots,\tilde{N}_1$, where $\tilde{N}_1\le N$, N is determined by Lemma 3.3. Passing to a subsequence, we assume $x_i^{\varepsilon_n}\to a_j^1$, $j=1,\cdots,N_1$, $N_1\le\tilde{N}_1$, $a_l^1\ne a_k^1$ $(l\ne k)$.

At this time, on any compact subset of $\overline{\Omega}\times[0,T_1]\backslash\bigcup\limits_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})$, we have $|u_{\varepsilon_n}|\ge m/2$ if n is large enough. Therefore the conclusion of Corollary 3.1 holds on such compact subset.

Now we work starting from $t=T_1$. We first prove

Lemma 4.2. *For the function Ψ defined in* (3.7) *we have*

$$\Psi\in L^\infty_{\rm loc}\Big(\overline{\Omega}\times[0,T_1]\Big\backslash\bigcup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\})\Big).\tag{4.5}$$

Proof. The interior estimates and the estimates near the boundary are done in the following one step. Denote

$$K=B_{2r}(x_0)\times[0,T_1]\subset\Omega\times[0,T_1]\backslash\bigcup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\}),\qquad x_0\in\Omega,$$

$$\tilde{K}=(B_{2r}(x_0)\cap\Omega)\times[0,T_1]\backslash\bigcup_{j=1}^{N_1}(\{a_j^1\}\times\{T_1\}),\qquad x_0\in\partial\Omega.$$

Again denote by ξ the standard cut-off function of $B_{2r}(x_0)$. Then we get

$$\begin{aligned}&\frac{\alpha_1{\varepsilon_n}^2}{\alpha_1^2\varphi^2+\alpha_2^2}\frac{\partial}{\partial t}(\xi\Psi)-{\varepsilon_n}^2\varphi^2\triangle(\xi\Psi)+\frac{m^2}{2}\xi\Psi\\&\le 2\xi|\nabla u_{\varepsilon_n}|^2-2{\varepsilon_n}^2\varphi^2\nabla\xi\cdot\nabla\Psi-{\varepsilon_n}^2\varphi^2\Psi\triangle\xi.\end{aligned}\tag{4.6}$$

It follows from above that on the compact subsets K and $\tilde{K}$, the right-hand side of (4.6) is bounded uniformly in n. Then Lemma 4.2 follows from the maximum principle (see also the proof of Step A.5 of [1]).

Lemma 4.3. *There exists $\tilde{T}_2 > T_1$ independent of ε_n such that on any compact subset M of $\overline{\Omega} \times [T_1, \tilde{T}_2] \setminus \bigcup_{j=1}^{N_1} (\{a_j^1\} \times [T_1, \tilde{T}_2])$,*

$$|u_{\varepsilon_n}| \geq \frac{m}{2}, \qquad on \quad M. \tag{4.7}$$

Proof. For any $x_0 \in \overline{\Omega} \setminus \bigcup_{j=1}^{N_1} \{a_j^1\}$, take $R > 0$ so small that $B_{2R}(x_0)$ doesn't contain a_j^1 $(1 \leq j \leq N_1)$. Let $\xi(x)$ be the cut-off function of $B_{2R}(x_0)$ and define

$$E_\xi(u) = \frac{1}{2} \int_\Omega \xi^2 |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega \xi^2 \varphi^2 (1 - |u/\varphi|^2)^2.$$

It follows from simple computations that for $t > T_1$,

$$\begin{aligned} E_\xi(u_{\varepsilon_n}(x,t)) &\leq E_\xi(u_{\varepsilon_n}(x,T_1)) + C \int_{T_1}^t \int_\Omega |\nabla \xi|^2 |\nabla u_{\varepsilon_n}|^2 \\ &\leq \frac{1}{2} \int_{B_{2R}(x_0) \times \{T_1\}} \xi^2 |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4{\varepsilon_n}^2} \int_{B_{2R}(x_0) \times \{T_1\}} \xi^2 \varphi^2 \Big(1 - \Big|\frac{u_{\varepsilon_n}}{\varphi}\Big|^2\Big)^2 \\ &\quad + \frac{C}{R^2} \int_{T_1}^t \int_{B_{2R}(x_0)} |\nabla u_{\varepsilon_n}|^2. \end{aligned}$$

Hence we have from this inequality, the following fact

$$\int_0^t \int_\Omega \frac{\alpha_1}{\alpha_1^2 \varphi^2 + \alpha_2^2} |u_{\varepsilon t}|^2 + \int_{\Omega(t)} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega(t)} \varphi^{-2} (|u_\varepsilon|^2 - |\varphi|^2)^2 = \frac{1}{2} \int_\Omega |\nabla u_0|^2$$

and Lemma 2.5 and Lemma 4.2 that $E_\xi(u_{\varepsilon_n}(x,t)) \leq CR^2 + o(1) + \frac{C(t-T_1)}{R^2}$.

Now the desired conclusion follows from Lemma 3.2 if one fixes $R = R_0$, $t = \tilde{T}_2 > T_1$ such that $C{R_0}^2 + o(1) + \frac{C(\tilde{T}_2 - T_1)}{{R_0}^2} \leq \frac{\mu_0}{16}$.

As before, we define $T_2 > T_1$ by

$$T_2 = \inf \Big\{ T \,\Big|\, T > T_1, \text{ there is } x_0 \in \Omega \setminus \bigcup_{j=1}^{N_1} \{a_j^1\} \text{ such that } \liminf_{\varepsilon \to 0} |u_\varepsilon(x_0, T)| = 0 \Big\}. \tag{4.8}$$

Denote the bad disks on $\Omega(T_2)$ by $B(x_k^\varepsilon, \lambda\varepsilon)$, $k = 1, \cdots, \tilde{N}_2$, $\tilde{N}_2 \leq N$. Passing to a further subsequence, still denoted by u_{ε_n}, we assume $x_k^{\varepsilon_n} \to a_l^2$, $l = 1, \cdots, N_2 \leq \tilde{N}_2$ with a_l^2 different from each other. On the compact subset of $\overline{\Omega} \times [T_1, T_2] \setminus \Big(\bigcup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \cup \bigcup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\} \Big)$, repeating above proof, we obtain

Lemma 4.4. *For any $\gamma \in (0,1)$ and any compact subset M of*

$$\overline{\Omega} \times [T_1, T_2] \setminus \Big(\bigcup_{j=1}^{N_1} \{a_j^1\} \times [T_1, T_2] \cup \bigcup_{l=1}^{N_2} \{a_l^2\} \times \{T_2\} \Big),$$

we have for some constant $C > 0$ independent of n that $\|u_{\varepsilon_n}\|_{C^{1+\gamma,(1+\gamma)/2}(M)} \leq C$.

Summing up, we have proved

Proposition 4.1. *There exist $T_1 < T_2 < \cdots < T_L$ and $a_j^i \in \overline{\Omega}$, $j = 1, \cdots, N_i$, $i = 1, \cdots, L$, $N_i \leq N$, such that, $\forall \gamma \in (0,1)$, we have for some sequence $\{u_{\varepsilon_n}\}$ $u_{\varepsilon_n} \to$*

u, in $C^{1+\gamma,(1+\gamma)/2}_{\rm loc}(\overline{\Omega}\times[o,\infty)\backslash B)$, *where* $B=\bigcup\limits_{i=1}^{L}\bigcup\limits_{j=1}^{N_i}(\{a^i_j\}\times[T_i,\infty))$, *$u$ is a solution of* (2.3)–(1.2).

In what follows, we claim that L in Proposition 4.1 has a uniform bound.

Lemma 4.5. *Let u be the solution obtained in Lemma* 2.4 *and $T_1,\cdots,T_L$ be determined in Proposition* 4.1. *Then*

$$\int_{\Omega(T_i)}|\nabla u|^2\le\int_\Omega|\nabla u_0|^2-(N_1+\cdots+N_i)\frac{\mu_0}{2}\ ,\qquad i=1,\cdots,L. \tag{4.9}$$

Proof. We first prove this inequality for $i=1$. Let

$$E(u,\Omega)=\frac12\int_\Omega|\nabla u|^2+\frac{1}{4\varepsilon^2}\int_\Omega\varphi^2(1-|u/\varphi|^2)^2.$$

We have for any $\delta>0$ and n large enough

$$\begin{aligned}&\frac12\int_{\Omega(T_1)\backslash\bigcup\limits_{j=1}^{N_1}B(a^1_j,\delta)}|\nabla u_{\varepsilon_n}|^2+\frac{1}{4{\varepsilon_n}^2}\int_{\Omega(T_1)\backslash\bigcup\limits_{i=1}^{N_1}\bigcup\limits_{j\in\Lambda^1_i}B(x^{\varepsilon_n}_j,2\lambda\varepsilon_n)}\varphi^2(1-|u_{\varepsilon_n}/\varphi|^2)^2\\&+\frac{1}{4{\varepsilon_n}^2}\sum_{i=1}^{N_1}\sum_{j\in\Lambda^1_i}\int_{B(x^{\varepsilon_n}_j,2\lambda\varepsilon_n)}\varphi^2(1-|u_{\varepsilon_n}/\varphi|^2)^2\\&\le E(u_{\varepsilon_n},\Omega(T_1))\le\frac12\int_\Omega|\nabla u_0|^2,\end{aligned}$$

where $\Lambda^1_i=\{j,x^{\varepsilon_n}_j\to a^1_i,n\to\infty\}$.

From the definition of bad disk, we have $\frac{1}{{\varepsilon_n}^2}\int_{B(x^{\varepsilon_n}_j,\lambda\varepsilon_n)}\varphi^2(1-|u_{\varepsilon_n}/\varphi|^2)^2\ge\mu_0$. This implies

$$\int_{\Omega(T_1)\backslash\bigcup\limits_{j=1}^{N_1}B(a^1_j,\delta)}|\nabla u_{\varepsilon_n}|^2\le\int_\Omega|\nabla u_0|^2-N_1\frac{\mu_0}{2}.$$

Sending $\varepsilon_n\to0$, we get

$$\int_{\Omega(T_1)\backslash\bigcup\limits_{j=1}^{N_1}B(a^1_j,\delta)}|\nabla u|^2\le\int_\Omega|\nabla u_0|^2-N_1\frac{\mu_0}{2}.$$

Thus it follows by sending $\delta\to0$ in above inequality that

$$\int_{\Omega(T_1)}|\nabla u|^2\le\int_\Omega|\nabla u_0|^2-N_1\frac{\mu_0}{2},$$

i.e., (4.9) holds for $i=1$.

Next, suppose (4.9) is true for $i=k$. Then one can prove that it is also true for $i=k+1$ since $E(u(x,t),\Omega)$ is nonincreasing in t. Lemma 4.5 follows.

Now we are in a position to prove Theorem 4.2.

In fact, from Lemma 4.5 we know, every singular point results in a deduction from the energy of u_0 by at least the quantity $\mu_0/2$. However, since $\int_\Omega|\nabla u_0|^2<\infty$, we conclude that the total number of singular points must be less than or equal to $N_0=[\frac{2}{\mu_0}\int_\Omega|\nabla u_0|^2]$. Consequently, in Proposition 4.1 the real singular set consists of at most finitely many points but other than lines which is denoted by B in Proposition 4.1.

On any compact subset K of $\overline{\Omega}\times[0,\infty)\backslash A$, $|u_{\varepsilon_n}|\ge m/2$ when $n\ge N_K$ where N_k is determined by K and independent of n. Therefore, on such a K we have a uniform $C^{1+\gamma,(1+\gamma)/2}$

estimate for u_{ε_n} and get a solution $u \in C^{1+\gamma,(1+\gamma)/2}(K)$ for (2.3)-(1.2). Theorem 4.2 follows from this and the Schauder's method.

Remark. Let $\Omega \in R^1$. Then there exists at least one smooth solution to the Problem (1.1)–(1.2).

§5. The Uniqueness

On the uniqueness, we have

Theorem 5.1. *Let $\Omega \subset R^n$ $(n = 1, 2)$ be a bounded smooth domain and u, v are two weak solutions of* (2.3) *satisfying the energy inequality*

$$\int_0^t \int_\Omega \frac{\alpha_1}{\alpha_1^2\varphi^2 + \alpha_2^2}|u_t|^2 + \frac{1}{2}\int_\Omega |\nabla u(\cdot, t)|^2 \le \frac{1}{2}\int_\Omega |\nabla u_0|^2, \quad \forall t > 0, \tag{5.1}$$

with the same initial-boundary condition (1.2). *Then $u = v$ in Ω_T for any $T \in (0, \infty)$.*

The proof of this theorem depends on the following lemma.

Lemma 5.1. *Suppose that $\int_\Omega |\nabla u_0|^2 < \infty$ and u is a weak solution of* (2.3)–(1.2) *satisfying* (5.1). *Then there is $t_1 > 0$ depending on u_0 such that*

$$\nabla u \in L^2((0, t_1), W^{1,4/3}(\Omega)). \tag{5.2}$$

Moreover, if $u \in L^2((0, t_1), W^{2,4/3}(\Omega))$ is a solution of (2.3)–(1.2), *then*

$$\nabla u \in L^4((0, t_1), W^{1,4/3}(\Omega)). \tag{5.3}$$

In order to prove this lemma, we list some known lemmas without proving.

Lemma 5.2 (Hodge Decomposition).[6] *Let W be in $\Lambda^l L^p(\Omega)$. Then, there exists an $(l-1)$-form A and an $(l+1)$-form B, such that*

$$W = dA + \delta B + h, \quad \delta A = dB = 0, \quad \|A\|_{W^{1,p}} + \|B\|_{W^{1,p}} \le C(p, \Omega)\|W\|_{L^p}, \tag{5.4}$$

where the differential forms A and B belong to $W^{1,p}(\Omega)$, h is a smooth harmonic form, d and δ are the differential and co-differential operators.

Lemma 5.3.[7] *If the matrix $H = (h_{ij}(x))_{1\le i,j\le n} \in W^{1,1}(B)$ and $\max\limits_{1\le i,j\le n} \|h_{i,j}\| < \pi^{1/2}/4$, where B is a unit ball in R^2 with center at origin, then there exists a vector $b^l = (b_1^l, \cdots, b_n^l) \in L^\infty(B)$ such that*

$$\frac{db_k^l}{d\bar{z}} = \sum_{j=1}^n h_{ij} b_j^l, \qquad \|b^l - e^l\|_{L^\infty} \le \frac{1}{3}, \tag{5.5}$$

where $e^l = (e_1^l, \cdots, e_n^l) = \sum\limits_{j=1}^n m_{lj} b^j$ and $e_i^l = \delta_{lj}$.

Lemma 5.4. *Assume that Ω is a bounded open set in R^n and u and v are in $W^{1,2}(\Omega)$. Also assume that $u|_{\partial\Omega}$ and $v|_{\partial\Omega}$ are in $W^{1,2}(\partial\Omega)$. Then $u_{x_i}v_{x_j} - u_{x_j}v_{x_i}$ belongs to the Hardy space $\mathcal{H}^1(\Omega)$ and*

$$\|u_{x_i}v_{x_j} - u_{x_j}v_{x_i}\|_{\mathcal{H}^1(\Omega)} \le C(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}), \tag{5.6}$$

where C is a constant depending on Ω.

Proof. It follows from the condition that u and v can be extended to a larger domain Ω' such that $\Omega \subset\subset \Omega'$ and u, $v \in W^{1,2}(\Omega')$. Then the conclusion follows from [5].

Proof of (5.2). From (5.1) we find that as $t \to 0$,

$$\int_\Omega |\nabla u(\cdot, t)|^2 \le \int_\Omega |\nabla u_0|^2. \tag{5.7}$$

Due to the boundedness of $\|\nabla u(\cdot,t)\|_{L^2(\Omega)}$, we may choose a subsequence $t_\nu \to 0$ as $\nu \to \infty$, such that

$$\nabla u(\cdot, t_\nu) \to \omega \quad \text{weakly in} \quad L^2(\Omega). \tag{5.8}$$

We have $\omega = \nabla u_0$ since for $t > 0$,

$$\|u(\cdot,t) - u(\cdot,0)\|_{L^2(\Omega)} \le \int_\Omega \Big| \int_0^t \frac{\partial u}{\partial \tau}(x,\tau) d\tau \Big|^2 \le Ct,$$

and therefore, as $t \to 0$,

$$u(\cdot,t) \to u_0 \quad \text{strongly in} \quad L^2(\Omega); \qquad \nabla u(\cdot,t) \to \omega \quad \text{weakly in} \quad L^2(\Omega).$$

Moreover, from the latest relations we not only know that any weakly convergent subsequence of $\nabla u(\cdot,t)$ must converge weakly in $L^2(\Omega)$ to ∇u_0, but also conclude that $\nabla u(\cdot,t)$ (not only a subsequence) weakly converges to ∇u_0 as t tends to 0. Further, one can prove

$$\nabla u(\cdot,t) \to \nabla u_0 \quad \text{strongly in} \quad L^2(\Omega). \tag{5.9}$$

In fact it follows from (5.7) that

$$\begin{aligned}\int_\Omega |\nabla u(x,t) - \nabla u_0(x)|^2 &= \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u_0|^2 - 2\int_\Omega \langle \nabla u, \nabla u_0 \rangle \\ &\le 2\int_\Omega |\nabla u_0|^2 - 2\int_\Omega \langle \nabla u, \nabla u_0\rangle \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

So for $\varepsilon > 0$ (ε will be determined later in getting (5.25)) there is $t_1 = t_1(\varepsilon) > 0$ such that

$$\int_\Omega |\nabla u(\cdot,t) - \nabla u_0|^2 < \frac{\varepsilon}{2}, \qquad \forall t \in (0,t_1). \tag{5.10}$$

In the following, we consider u is defined on $(\Omega \cap B) \times R_+$ where B is a unit ball in R^2 with center at $\partial\Omega$. We want to prove

Claim.

$$\nabla u \in L^2((0,t_1); W^{1,4/3}(\Omega \cap B)). \tag{5.11}$$

Proof of Claim. For any $z_0 = (x_0,t_0) \in (\Omega \cap B) \times (0,t_1)$, denote $I_r = (t_0 - r^2, t_0)$, $\Omega_r = \Omega \cap B_r(x_0)$ and $P_r = \{(x,t) \mid x \in \Omega_r,\ t \in I_r\}$. Since $\int_\Omega |\nabla u_0|^2 < \infty$, for given $\varepsilon > 0$ in (5.10), there is $r_0 = r_0(\varepsilon, z_0) > 0$, such that $P_{r_0} \subset (\Omega \cap B) \times (0,t_1)$ and $\int_{B_{r_0}} |\nabla u_0|^2 < \varepsilon/2$ which combined with (5.10) gives

$$\int_{B_{r_0}} |\nabla u(\cdot,t)|^2 \le \int_\Omega |\nabla u(\cdot,t) - \nabla u_0|^2 + \int_{B_{r_0}} |\nabla u_0|^2 \le \varepsilon, \qquad \forall t \in (0,t_1). \tag{5.12}$$

Let

$$\begin{aligned} f_k &= \frac{1}{4}\Big[\frac{\alpha_1}{\alpha_1^2\varphi^2 + \alpha_2^2} u_t^k - \frac{\alpha_2 \varphi^{-2}}{\alpha_1^2 \varphi^2 + \alpha_2^2}(u \times u_t)^k + \frac{1}{2}\frac{u^k}{\varphi^2}\triangle \varphi^2\Big], \\ H_{kj} &= u^k \frac{\partial u^j}{\partial \bar z} - u_j \frac{\partial u^k}{\partial \bar z}. \end{aligned}$$

From Equation (2.3) and the fact $|u| = \varphi$, one may have

$$\begin{aligned} \frac{\partial^2 u^k}{\partial \bar z \partial z} &= \frac{1}{4}\triangle u^k = f_k - \frac{1}{\varphi^2}\Big\langle \frac{\partial u}{\partial \bar z}, \frac{\partial u}{\partial z}\Big\rangle u^k \\ &= F_k - \frac{1}{\varphi^2}\Big\langle \Big(u^k \frac{\partial u}{\partial \bar z} - u\frac{\partial u^k}{\partial \bar z}\Big), \frac{\partial u}{\partial z}\Big\rangle = F_k - \frac{1}{\varphi^2}\sum_{j=1}^3 H_{kj}\frac{\partial u^j}{\partial z}, \end{aligned} \tag{5.13}$$

where $F_k = f_k - \frac{1}{2\varphi^2}\langle \frac{\partial u^k}{\partial \bar{z}}, \frac{\partial \varphi^2}{\partial z}\rangle$.

Consider the following 1-form

$$w_{kj} = u^k du^j - u^j du^k \in L^\infty((0,t_1), L^2(\Omega_{r_0})), \tag{5.14}$$

$$\tilde{w}_{kj} = u^k d(\xi u^j) - u^j d(\xi u^k) \in L^\infty((0,t_1), L^2(\Omega_{r_0})), \tag{5.15}$$

where ξ is a cut-off function such that $\xi = 1$ on $B_{\frac{2r_0}{3}}(x_0)$, $\xi = 0$ on $R^2 \backslash B_{r_0}(x_0)$ and $|\nabla \xi| \le 4/r_0$.

Obviously $w_{kj} = \tilde{w}_{kj}$, on $\Omega_{\frac{2r_0}{3}} \times (0,t_1)$.

By Lemma 5.2, there exist $A_{kj}(\cdot,t) \in W^{1,2}$, $B_{kj}(\cdot,t) \in \Lambda^2 W^{1,2}$ and smooth harmonic form h_{kj} such that $\delta A_{kj} = dB_{kj} = 0$,

$$\tilde{w}_{kj} = dA_{kj} + \delta B_{kj} + h_{kj} \quad \text{for a.e. } t \in (0,t_1), \tag{5.16}$$

$$\|A_{kj}\|_{W^{1,2}} + \|B_{kj}\|_{W^{1,2}} \le C\|\tilde{w}_{kj}\|_{L^2} \le C\|\nabla u\|_{L^2} \le C.$$

From (5.16), (5.13) and (2.3)

$$\triangle A_{kj} = \delta \tilde{w}_{kj} = \delta w_{kj} = u^k \triangle u^j - u^j \triangle u^k = 4u^k f_j - 4u^j f_k, \quad \text{on } P_{\frac{r_0}{2}}.$$

So $\triangle A_{kj} \in L^2(P_{\frac{r_0}{2}})$, and then

$$dA_{kj} \in L^2(I_{\frac{r_0}{2}}, W^{1,2}(\Omega_{\frac{r_0}{2}})) \subset L^2(I_{\frac{r_0}{2}}, L^4(\Omega_{\frac{r_0}{2}})). \tag{5.17}$$

On the other hand, it follows from (5.16) and (5.14) that

$$\triangle B_{kj} = d\tilde{w}_{kj} = dw_{kj} = du^k \wedge du^j, \quad \text{on} \quad \Omega_{\frac{7r_0}{12}} \times (0,t_1),$$

where "$\wedge$" is the Wedge product in R^2. Since $u|_{\partial\Omega} = v|_{\partial\Omega} = \psi$ and ψ is smooth, by Lemma 5.4, for a.e. $t \in (0,t_1)$, we have that $\triangle B_{kj} \in \mathcal{H}^1(\Omega_{\frac{7r_0}{12}})$, and for any $0 < r < \frac{7r_0}{12}$,

$$\|\triangle B_{kj}(t)\|_{\mathcal{H}^1(\Omega_r)} \le C(r)\|du(t)\|^2_{L^2\left(\Omega_{\frac{7r_0}{12}}\right)}. \tag{5.18}$$

This implies that $B_{kj} \in L^\infty((0,t_1), W^{2,1}(\Omega_{\frac{7r_0}{12}}))$.

In particular

$$\delta B_{kj} \in L^\infty(I_{\frac{r_0}{2}}, W^{1,1}(\Omega_{\frac{7r_0}{12}})). \tag{5.19}$$

Using the definition of H_{kj} and w_{kj}, one may rewrite H_{kj} as

$$H_{kj} = \frac{1}{2}(w_{kj}(\partial x) + i w_{kj}(\partial y)),$$

where $w_{kj}(\partial x) = u^k \frac{\partial u^j}{\partial x} - u^j \frac{\partial u^k}{\partial x}$, $\quad w_{kj}(\partial y) = u^k \frac{\partial u^j}{\partial y} - u^j \frac{\partial u^k}{\partial y}$.

Applying the Hodge decomposition (5.16) and noting that $\tilde{w}_{kj} = w_{kj}$ on $P_{\frac{r_0}{2}}$, one may further rewrite H_{kj} as

$$H_{kj} - h_{kj} = H^1_{kj} + H^2_{kj}, \tag{5.20}$$

where $H^1_{kj} = \frac{1}{2}(dA_{kj}(\partial x) + i dA_{kj}(\partial y))$, $\quad H^2_{kj} = \frac{1}{2}(\delta B_{kj}(\partial x) + i\delta B_{kj}(\partial y))$.

(5.17) and (5.19) give that

$$H^1_{kj} \in L^2(I_{\frac{r_0}{2}}, L^4(\Omega_{\frac{r_0}{2}})), \qquad H^2_{kj} \in L^\infty(I_{\frac{r_0}{2}}, W^{1,1}(\Omega_{\frac{r_0}{2}})). \tag{5.21}$$

Moreover, from (5.12) and (5.18) we know

$$\|H^2_{kj}\|_{L^\infty\left(I_{\frac{r_0}{2}}, W^{1,1}\left(\Omega_{\frac{r_0}{2}}\right)\right)} \le C\varepsilon. \tag{5.22}$$

Using (5.20), we rewrite (5.13) as

$$\frac{\partial^2 u^k}{\partial \bar{z}\partial z} = F_k - \frac{1}{\varphi^2}\Big(\sum_{j=1}^{3} H^1_{kj}\frac{\partial u^j}{\partial z} + \sum_{j=1}^{3} H^2_{kj}\frac{\partial u^j}{\partial z} + \sum_{j=1}^{3} h_{kj}\frac{\partial u^j}{\partial z}\Big). \tag{5.23}$$

For the last term we have

$$\sup_{\Omega_{\frac{r_0}{2}}} |h_{kj}| \le C\|h_{kj}\|_{L^2\left(\Omega_{\frac{r_0}{2}}\right)} \le C[\|w_{kj}\|_{L^2} + \|dA_{kj}\|_{L^2} + \|\delta B_{kj}\|_{L^2}] \le C,$$

$$\Big\|h_{kj}\frac{\partial u^k}{\partial z}\Big\|_{L^{4/3}\left(\Omega_{\frac{r_0}{2}}\right)} \le C\Big(\sup_{\Omega_{\frac{r_0}{2}}}|h_{kj}|\Big)\|\nabla u\|_{L^2} \le C.$$

Hence, it is easy to see from the definition of F_k and (5.21) that

$$G_k := F_k - \frac{1}{\varphi^2}\Big(\sum_{j=1}^{3} H^1_{kj}\frac{\partial u^j}{\partial z} + \sum_{j=1}^{3} h_{kj}\frac{\partial u^j}{\partial z}\Big) \in L^2(I_{\frac{r_0}{2}}, L^{4/3}(\Omega_{\frac{r_0}{2}})). \tag{5.24}$$

On the other hand, from (5.22), if ε is small enough, by Lemma 5.3 it is possible to construct solutions $b^l \in L^\infty(\Omega_{r_0/2})$, $l = 1, 2, 3$ to the system

$$\frac{\partial b^l}{\partial \bar{z}} = -\frac{1}{\varphi^2}\sum_{j=1}^{3} H^2_{kj} b^l_j \quad \text{for each fixed} \quad t \in I_{\frac{r_0}{2}}, \tag{5.25}$$

and to have an invertible matrix $M = (m_{ij})_{1\le i,j\le 3} \in L^\infty(\Omega_{r_0/2})$, such that (5.5) holds. Then from (5.5) we get

$$\frac{\partial u^k}{\partial z} = \sum_{j=1}^{3}\delta_{kj}\frac{\partial u^j}{\partial z} = \sum_{j=1}^{3} e^k_j\frac{\partial u^j}{\partial z} = \sum_{j=1}^{3}\sum_{l=1}^{3} m_{kj} b^l_j \frac{\partial u^j}{\partial z}. \tag{5.26}$$

Using (5.23) and (5.25) we compute

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\Big(\sum_{k=1}^{3} b^l_k\frac{\partial u^k}{\partial z}\Big) &= \sum_{k=1}^{3}\Big[\Big(\frac{\partial b^l_k}{\partial \bar{z}}\Big)\frac{\partial u^k}{\partial z} + b^l_k\frac{\partial^2 u^k}{\partial \bar{z}\partial z}\Big] \\
&= -\frac{1}{\varphi^2}\sum_{k=1}^{3}\sum_{j=1}^{3} H^2_{kj} b^l_j\frac{\partial u^k}{\partial z} + \sum_{k=1}^{3} b^l_k G_k - \frac{1}{\varphi^2}\sum_{k=1}^{3}\sum_{j=1}^{3} H^2_{kj} b^l_j\frac{\partial u^j}{\partial z} \\
&= \sum_{k=1}^{3} b^l_k G_k,
\end{aligned} \tag{5.27}$$

where we have used the fact $H_{kj} = -H_{jk}$. By (5.24) and the fact $b^l \in L^\infty\left(\Omega_{\frac{r_0}{2}}\right)$ for all $t \in I_{\frac{r_0}{2}}$, (5.27) leads to the result

$$\sum_{k=1}^{3} b^l_k\frac{\partial u^k}{\partial z} \in L^2\Big(I_{\frac{r_0}{2}}, W^{1,4/3}\big(\Omega_{\frac{r_0}{2}}\big)\Big). \tag{5.28}$$

(5.28) combined with (5.26) yields $\frac{\partial u^k}{\partial z} \in L^2\Big(I_{\frac{r_0}{2}}, W^{1,4/3}\big(\Omega_{\frac{r_0}{2}}\big)\Big)$, $k = 1, 2, 3$. Then the Claim follows from standard covering argument. (5.2) is proved.

Next, we prove (5.3). The following lemma is regarding to the following problem

$$\partial_t u^i - G_1(x,t)\Delta u = G_2(x,t)\Delta u + g(x,t), \quad x \in \Omega, \quad t > 0, \tag{5.29}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.30}$$

$$u|_{\partial\Omega} = u_0|_{\partial\Omega}, \tag{5.31}$$

where $G_i(x,t)$ $(i=1,2)$ are matrices and $g(x,t)$ is a vector.

Lemma 5.5. *Let Ω be a two-dimensional bounded smooth domain. Suppose that in* (5.29)–(5.31)

(1) $G_1(x,t)\Delta u$ *is strongly elliptic.*

(2) $G_1 \in C^\infty(\Omega\times(0,T))$, $G_2 \in L^\infty(\Omega\times(0,T))$, $g\in L^4(0,T;\ L^{4/3}(\Omega))$ *and* $u_0\in H^1(\Omega)$. *Then, there exists a constant $\varepsilon_1>0$ depending only on Ω and T, such that if*

$$\|G_2\|_{L^\infty(\Omega\times(0,T))} \le \varepsilon_1, \tag{5.32}$$

then the problem (5.29)–(5.31) *has a unique solution in* $L^s(0,T;W^{2,4/3}\ (\Omega))$, *for any* $s\in[2,4]$.

The proof of this lemma can be found in [3].

Now we use Lemma 5.5 to prove (5.3).

Let $u\in L^2(0,T;W^{2,4/3})$ be a weak solution of (2.2)-(1.2). From (2.2), one has

$$\partial_t u - G(u)\Delta u = \alpha_1\varphi^2|\nabla u|^2u - \frac{\alpha_1}{2}u\Delta\varphi^2, \tag{5.33}$$

with $|u|=\varphi$, where

$$G(u) = \begin{pmatrix} \alpha_1\varphi^2 & -\alpha_2u^3 & \alpha_2u^2\\ \alpha_2u^3 & \alpha_1\varphi^2 & -\alpha_2u^1\\ -\alpha_2u^2 & \alpha_2u^1 & \alpha_1\varphi^2\end{pmatrix}.$$

Since $|u|=\varphi$, for any $\varepsilon>0$ we can decompose $G(u)$ into the form

$$G(u) = G_1(u)+G_2(u), \tag{5.34}$$

where $G_1(u)\in C^\infty(\Omega\times(0,T))$, $G_2(u)\in L^\infty(\Omega\times(0,T))$ with

$$\|G_1(u)\|_{L^\infty(\Omega\times(0,T))} \le C\|u\|_{L^\infty(\Omega\times(0,T))}, \tag{5.35}$$

$$\|G_2(u)\|_{L^\infty(\Omega\times(0,T))} \le \epsilon. \tag{5.36}$$

Inserting (5.34) into (5.33), one has

$$\partial_t u - G_1(u)\Delta u = G_2(u)\Delta u + g(x,t), \tag{5.37}$$

where $g(x,t)=\alpha_1\varphi^2|\nabla u|^2u - \frac{\alpha_1}{2}u\Delta\varphi^2$; then, (5.37) takes the same form as (5.29). Moreover,

$$\|g\|_{L^4(0,T;L^{4/3})} \le C\|\nabla u\|_{L^\infty([0,T],L^2)}\|\nabla u\|_{L^2([0,T],L^4)} + C.$$

Hence, we can see that if ϵ is sufficiently small, then the assumptions in Lemma 5.5 and (5.32) are satisfied. Therefore, by Lemma 5.5, the problem (5.37) with the condition (1.2) admits a unique solution $v\in L^4(0,T;W^{2,4/3})\subset L^2(0,T;W^{2,4/3})$ and the solution in the space $L^2(0,T;W^{2,4/3})$ is also unique. On the other hand, it is obvious that if $u\in L^2(0,T;W^{2,4/3})$ is a weak solution of (2.3)-(1.2), then u is also a solution of (5.37) and (1.2) in $L^2(0,T;W^{2,4/3})$. Therefore, $u=v\in L^4(0,T;W^{2,4/3})$. This proves (5.3).

Proof of Theorem 5.1. We may assume that $\nabla u\in L^4([0,T),W^{1,4/3}(\Omega))$ and v is the "almost smooth" solution obtained in Theorem 4.1. We are going to show that $u=v$ on $\Omega\times[0,T]$. Let $w=u-v$. It follows from (2.2) that w solves

$$w_t - \alpha_1\varphi^2\triangle w = \alpha_2(u\times\triangle w + w\times\triangle u) + \alpha_1[u(|\nabla u|^2-|\nabla v|^2)+|\nabla v|^2w] - \frac{\alpha_1}{2}w\triangle\varphi^2 \tag{5.38}$$

with the condition $w(x,0)=0$, $w(x,t)|_{\partial\Omega\times[0,T]}=0$. Let $|\nabla U|^2=|\nabla u|^2+|\nabla v|^2$. Testing (5.38) by w and integrating it by parts, we obtain for almost every $t\in[0,T]$ that

$$\frac{d}{dt}\|w\|_{L^2}^2 + \int_\Omega|\nabla w|^2 \le C\int_\Omega |w|^2|\nabla U|^2. \tag{5.39}$$

Using Sobolev imbedding inequality

$$\|w\|_{L^4}^4 \le 4\|w\|_{L^2}^2 \|dw\|_{L^2}^2,$$

we have

$$\begin{aligned}\int_M |w|^2|\nabla U|^2 &\le C\Big(\int_M |\nabla U|^4\Big)^{1/2}\Big(\int_M |w|^2\Big)^{1/2}\Big(\int_M |\nabla w|^2\Big)^{1/2}\\ &\le \frac{1}{4}\int_M |\nabla w|^2 + C\int_M |\nabla U|^4 \int_M |w|^2.\end{aligned}$$

This combined with (5.39) implies

$$f'(t) + g(t) \le Cp(t)f(t),$$

where $f(t) = \int_M |w|^2$, $g(t) = \int_M |\nabla w|^2$, $p(t) = \int_M |\nabla U|^4$. Since $u, v \in L^4(0, T; W^{2,4/3})$, $p(t) \in L^1(0, T)$. We have from the Gronwall inequality that $f(t) = 0$ for a.e. $t \in [0, T]$, and hence $w = 0$ a.e. on $M \times [0, T]$.

Now u meets the identity

$$\int_0^T \int_\Omega \frac{\alpha_1}{\alpha_1^2\varphi^2 + \alpha_2^2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{1}{2}\int_\Omega |\nabla u(\cdot, T)|^2 = \frac{1}{2}\int_\Omega |\nabla u_0|^2. \tag{5.40}$$

Subtracting (5.40) from (5.1), we have

$$\int_T^t \int_\Omega \frac{\alpha_1}{\alpha_1^2\varphi^2 + \alpha_2^2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{1}{2}\int_\Omega |\nabla u(\cdot, t)|^2 \le \frac{1}{2}\int_\Omega |\nabla u(\cdot, T)|^2.$$

Therefore, we can consider u as a solution of (2.3) on $\Omega \times [T, \infty)$ with initial data $u(x, T)$ satisfying the assumption in Theorem 5.1. Then we can repeat above argument again. The conclusion of Theorem 5.1 can be obtained by iterating this argument.

References

[1] Bethuel, F., Brezis, H. & Hélein, F., Asymptotics for the minimization of a Ginzburg-Landau functional [J], *Calc. Var. PDE,* **1**(1993), 123–148.

[2] Bethuel, F., Brezis, H. & Hélein, F., Ginzburg-Landau Vortices [M], Birkhäuser, 1994.

[3] Chen, Y., Ding, S. & Guo, B. L., Partial regularity for the two dimensional Landau-Lifshitz equations [J], *Acta Math. Sinica, New Series,* **14:**3(1998), 423–432.

[4] Chen, Y. & Struwe, M., Existence and partial regularity results for the heat flow of harmonic maps [J], *Math. Z.,* **201**(1989), 83–103.

[5] Chen, Y. & Wang, C., Partial regularity for weak flows into Riemannian homogeneous space [J], *Comm. P. D. E.,* **21:**(5,6)(1996), 735–761.

[6] Freire, A., Uniqueness for the harmonic map flow in two dimensions [J], *Calc. Var.,* **3**(1995), 95–105.

[7] Gilkey, P., Invariance theory, the heat equation and the Atiyah-Singer index theorem [M], Publish of Perish, Inc., 1984.

[8] Guo, B. L. & Ding, S., Initial-boundary value problem of Landau-Lifshitz system of ferromagnetic spin chain with Gilbert damping, I, II [J], *Progress Nat. Sci.,* **8:**1(1998), 11–23; **8:**2(1998), 147–151.

[9] Guo, B. L. & Hong, M. C., The Landau-Lifshitz equations of the ferromagnetic spin chain and harmonic maps [J], *Calc. Var. PDE.,* **1**(1993), 311–334.

[10] Zhou, Y. L. & Guo, B. L., Existence of weak solution for boundary problems of systems of ferromagnetic chain [J], *Science in China,* **A27**(1984), 799–811.

[11] Zhou, Y. L., Guo, B. L. & Tan, S., Existence and uniqueness of smooth solution of system of ferromagnetic chain [J], *Science in China,* **A34**(1991), 257–266.

[12] Zhou, Y. L. & Guo, B. L., Weak solution systems of ferromagnetic chain with several variables [J], *Science in China,* **A30**(1987), 1251–1266.

[13] Zhou, Y. L., Sun, H. S. & Guo, B. L., Multidimensional system of ferromagnetic chain type [J], *Science in China,* **A36**(1993), 1422–1434.