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GRAPH-DIRECTED STRUCTURES OF SELF-SIMILAR SETS WITH OVERLAPS***

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Abstract

Some kinds of the self-similar sets with overlapping structures are studied by introducing the graph-directed constructions satisfying the open set condition that coincide with these sets. In this way, the dimensions and the measures are obtained.

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§1. Introduction

The self-similar sets (SSS) is one of the most important fractal classes, but the most properties such as dimensions, measures $\cdots$ have been established upon the open set condition (OSC). It is a difficult problem to determine the structure and only a few results are known when this condition is absent. On the other hand, for the graph-directed sets (GDS), a generalization of SSS, if the OSC is satisfied, then analogous properties of the self-similar sets will hold still. The main purpose of this paper is to obtain graph-directed set with OSC starting from some kinds of self-similar set with overlapping satisfying certain arithmatical properties. In this way, we will determine completely the structure of this kind SSS with overlapping.

1.1. Self-Similar Set with Overlaps

Let $\{S_j\}_{j=1}^m$ be a family of contracting similarities on $\mathbb{R}^d$, which we call iterated function system (IFS).

By [4], there exists a unique compact set K, such that

$$K = \bigcup_{j=1}^{m} S_j(K).$$

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The set K is called the self-similar set (SSS) of the IFS, and it is called also invariant set or attractor of the IFS.

If there is an open set V with

$$\bigcup_{j=1}^{m} S_j(V) \subseteq V,$$

where the left side is a disjoint union, then we say the IFS satisfies the open set condition (OSC). Under OSC, the dimensions and measures of the SSS are determined completely (see [4]).

A closed set D satisfying

$$\bigcup_{j=1}^{m} S_j(D) \subseteq D \tag{1.1}$$

is called a basic domain of the IFS. If the left side of (1.1) is a non-overlapping union, that is, their interiors are disjoint, then the IFS satisfies OSC with open set D°, the interior of D. Otherwise we will say that the self-similar set generated by the IFS has overlaps. The structure of the self-similar set with overlaps is an interesting but difficult subject in fractal geometry (see for example [5,6,8,12])).

Example 1.1. Let $\{S_j\}_{j=1}^3$ be an IFS on $\mathbb{R}$ defined by

$$S_1(x) = \frac{x}{3}, \quad S_2(x) = \frac{x}{3} + \lambda, \quad S_3(x) = \frac{x}{3} + \frac{2}{3},$$

where $\lambda \in [0, 1/3]$.

We denote by C_λ the SSS of the IFS and we call it the λ-Cantor set with parameter λ (see [12]).

When $\lambda = 0$, C_λ is the classical middle-third Cantor set; when $\lambda = 1/3$, C_λ is the interval $[0, 1]$. For $\lambda \in (0, \frac{1}{3})$, C_λ is a SSS with overlaps. This setting has been studied in details in [6] and [12].

It is known that for $\lambda = 2/5$, C_λ satisfies OSC (see [12]). This means that a SSS with overlaps may satisfy OSC still.

Example 1.2. Suppose $1/3 < \beta < 2/5$ is a positive number. Let $\{S_j\}_{j=1}^3$ be an IFS on $\mathbb{R}$ defined by

$$S_1(x) = \beta x, \ S_2(x) = \beta(x + 1), \ S_3(x) = \beta(x + 3)$$

and let K_β denote the SSS of IFS. Then

$$K_\beta = \Big\{ \sum_{n\geq 1} a_n \beta^n : \quad \text{for each} \ \ n, \ a_n \in \{0, 1, 3\} \Big\}.$$

The set K_β does not satisfy OSC, it is introduced by Keane (see [5]). The problem to determine the structure is called the 0,1,3 problem (for the details, see [5,11,14] and [7]).

1.2. Graph-Directed Sets (GDS)

The graph-directed set is a natural generalization of self-similar set (see [2, 9, 1,10, 3]). In this paper we adopt the terminologies in [3].

Let $\mathcal{V}$ be a finite set with q elements that we call the vertices and we label $\{1, 2, \cdots, q\}$, and let $\mathcal{E}$ be a set of directed edges with each edge starting and ending at a vertex so that

$(\mathcal{V}, \mathcal{E})$ is a directed graph. A pair of vertices may be joined by several edges and we also allow edges starting and ending at the same vertex. We write $\mathcal{E}_{i,j}$ for the set of edges from vertex i to vertex j, and $\mathcal{E}^k_{i,j}$ for the set of sequences of k edges $(e_1, e_2, \cdots, e_k)$ which form a directed path from vertex i to j. For each edge $e \in \mathcal{E}$, let $F_e : \mathbb{R} \to \mathbb{R}$ be a contracting similarity of ratio r_e with $0 < r_e < 1$. Then (see [3] for example), there is a unique family of non-empty compact sets $K_1, \cdots, K_q$ such that

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} F_e(K_j). \tag{1.2}$$

The sets $K_1, \cdots, K_q$ are called the GDS determined by $(\mathcal{V}, \mathcal{E}, F_e)$.

We assume that the union in (1.2) are disjoint for all i; this separation condition may be relaxed to an open set condition. The dimension of graph-directed set is determined in terms of associated $q \times q$ matrices $M^{(s)}$ with (i, j)-th entry given by

$$M^{(s)}_{i,j} = \sum_{e \in \mathcal{E}_{i,j}} r_e^s.$$

The matrix $M^{(s)}$ is called the Perron-Frobenius matrix.

Theorem A.[1,10] *Let the GDS $\{K_i\}_{1 \le i \le q}$ satisfy OSC.*

Let $\rho(M^{(s)})$ be the largest eigenvalue of $M^{(s)}$ and α be the unique non-negative real number such that $\rho(M^{(\alpha)}) = 1$. Then

$$\dim_H K_i = \alpha, \quad 1 \le i \le q.$$

Theorem B.[3] *Let $\{K_i\}_{1 \le i \le q}$ be graph-directed sets* (*even without OSC and assuming the maps are conformal*), *then*

$$\dim_H K_i = \dim_P K_i = \dim_B K_i, \quad 1 \le i \le q.$$

That is, all graph-directed sets are regular sets.

1.3. Main Results

As we mentioned above, it is difficult to study the structure and properties of a self-similar set with overlapping. On the other hand, that of graph-directed sets with OSC are clear by Theorems A and B. The main purpose of this paper is to obtain the graph-directed set with OSC starting from some kinds of self-similar set with overlapping.

An algebraic integer β is called a Pisot number if $|\beta| > 1$ and all of its conjugates are inside the unit circle.

Suppose $\beta = \rho^{-1}$ is a Pisot number. Put

$$\mathbb{Z}[\rho] = \Big\{ \sum_{i=0}^{n} a_i \rho^i : a_i \in \mathbb{Z},\ n \in \mathbb{N} \Big\}.$$

Let $\{S_j\}_{j=1}^m$ be an IFS on $\mathbb{R}$ with

$$S_j(x) = \rho x + d_j, \tag{1.3}$$

where $d_j \in \mathbb{Z}[\rho]$. Examples 1.1 and 1.2 are included in this setting.

In general, the SSS defined by (1.3) has overlaps, but we have the following theorem that is the main result of this paper.

Main Theorem. *Let the IFS be defined as* (1.1) *and let K be the SSS generated by the IFS. Then there exists a GDS with OSC, denoted by E, such that $K = E$.*

§2. System of First Labels

In this section, associated with a given Pisot number, we introduce a sequence of the cylinders which constitute partitions of the interval $[0,1]$. Then we introduce the first label on the cylinders, which is the first step to construct the desired GDS.

2.1. A Property of Pisot Number

From now on, we assume that $\beta = \rho^{-1}$ is a Pisot number. The following known result about Pisot number will play an important role in our studies.

Proposition 2.1.[13] *Suppose that β is a Pisot number. Let M, N be two positive constants and let*

$$\Omega_{M,N} = \Big\{ \sum_{i=0}^{k} a_i \beta^i : |a_i| < M,\ a_i \in \mathbb{N},\ k \in \mathbb{N} \Big\} \cap [0, N].$$

Then $\Omega_{M,N}$ is a finite set.

2.2. Number System in Base $\boldsymbol{\beta}$

Let $\beta > 1$ and x be a positive real number. Then the expansion of the form $x = \sum\limits_{i=N_0}^{\infty} a_i \beta^{-i}$ is said to be a greedy expansion if

$$\Big| x - \sum_{N_0 \le i \le N} a_i \beta^{-i} \Big| < \beta^{-N}$$

holds for every N, where a_i are nonnegative integers with $0 \le a_i < \beta$.

When $0 \le x < 1$, the greedy expansion is a natural generalization of binary or decimal expansion, so we call the greedy expansion the number system in base β.

2.3. First Label on the Cylinders

2.3.1. Partitions of the unit interval [0,1] associated with $\boldsymbol{\rho}$

Let $k \in \mathbb{N}$. The partition of order k associated with ρ is defined by

$$I_k = \Big\{ \sum_{i=1}^{k} a_i \rho^i : 0 \le a_i < \rho^{-1}, a_i \in \mathbb{N} \Big\} \cup \{1\}.$$

By the definition, we see that

(1) I_k is a finite set of $[0,1]$ that consists of all greedy expansion with length less than or equal to k;

(2) For any k, $I_{k+1} \subset I_k$;

2.3.2. Cylinders of Order $\boldsymbol{k}$ and the First Label

Now we arrange the elements of the partition of order k by an increasing order and we denote it by $\{z_0 = 0,\ z_1, \cdots, z_p = 1\}$. The intervals $[z_i, z_{i+1}]$ $(1 \le i \le p-1)$ are called the cylinders of order k (or k-cylinder) that we denote by $\mathcal{C}_{\rho,k}$. The set $\mathcal{C}_\rho = \bigcup\limits_{k\ge 1} \mathcal{C}_\rho$ is the set of all cylinders.

Let $\tau \in \mathcal{C}_{\rho,k}$ be a k-cylinder. We define the first label of τ by

$$L_{\rho,k}(\tau) = \rho^{-k} |\tau|,$$

where $|\tau|$ is the length of the interval τ. Set

$$\mathcal{L}_\rho = \{L_{\rho,k}(\tau) :\ \tau \in \mathcal{C}_{\rho,k}, k \in \mathbb{N}\}.$$

Then $\mathcal{L}_\rho$ is the set of the first labels of the cylinders.

By the definitions above, it is easy to see that:

(1) Two k-cylinders have the same first label if and only if they have the same length;

(2) Assume that $\tau \in \mathcal{C}_{\rho,k}$ and $\sigma \in \mathcal{C}_{\rho,n+k}$, then τ and σ have the same first label if and only if $|\tau| = \rho^{-n}|\sigma|$.

(3) For any $\tau \in \mathcal{C}_{\rho,k}$, $|\tau| \le \rho^k$.

Proposition 2.2. *With the notations and the definitions above, $\mathcal{L}_\rho$ is a finite set.*

Proof. Let $\tau = [z_i, z_{i+1}] \in \mathcal{C}_{\rho,k}$, let $z_i = \sum\limits_{i=1}^{k} a_i\rho^i$ and $z_{i+1} = \sum\limits_{i=1}^{k} b_i\rho^i$. Then

$$\begin{aligned} L_\rho(\tau) &= \rho^{-k}|\tau| = \rho^{-k}|z_{i+1} - z_i| \\ &= \rho^{-k}\Big|\sum_{i=1}^{k} a_i\rho^i - \sum_{i=1}^{k} b_i\rho^i\Big| \\ &= \Big|\sum_{i=1}^{k}(a_i - b_i)\beta^{k-i}\Big|. \end{aligned}$$

Noticing that $|\tau| \le \rho^k$ and $|a_i - b_i| \le 2\rho^{-1}$, we have $L_\rho(\tau) \in \Omega_{M,N}$, by taking $M = 2\rho^{-1}$, $N = 1$, where $\Omega_{M,N}$ is defined by Propsition 2.1. Therefore $\mathcal{L}_\rho \subseteq \Omega_{M,N}$ is a finite set from Proposition 2.3.

Let $\tau \in \mathcal{C}_{\rho,k}$ be a k-cylinder and let

$$O(\tau) = \{\sigma : \sigma \subset \tau \ \text{ and } \sigma \in \mathcal{C}_{\rho,k+1}\}.$$

Then τ is a nonoverlapping union of the elements ($k+1$-cylinders) of the set $O(\tau)$. The elements of $O(\tau)$ are called the offspring of τ.

Proposition 2.3. *Let $\tau \in \mathcal{C}_k$. Then*

(1) $\sharp O(\tau) = p := [\frac{L_{\rho,k}(\tau)}{\rho}] + 1$*;*

(2) *Let $\sigma_1, \cdots, \sigma_p$ be the offspring of τ arranged from left to right, then*

$$L_{\rho,k+1}(\sigma_1) = \cdots = L_{\rho,k+1}(\sigma_{p-1}) = 1, \quad L_{\rho,k+1}(\sigma_p) = \rho^{-1}L_{\rho,k}(\tau) - (p-1),$$

that is, the cardinality of $O(\tau)$ and the first labels of the elements of $O(\tau)$ are determined completely by the first label of τ.

Proof. Denote by $\bar\tau$ the left endpoint of τ. Then $\bar\tau + |\tau|$ is the right endpoint of τ.

Case 1. If $p = 1$, then by the definition of p and $L_{\rho,k}(\tau)$, $|\tau| \le \rho^{k+1}$, in this case, by the constructions of I_k and I_{k+1}, the elements of I_{k+1} falling in the interval τ are exactly $\bar\tau$ and $\bar\tau + |\tau|$. Therefore $O(\tau) = \tau$ and $L_{\rho,k+1}(\tau) = \rho^{-1}L_{\rho,k}(\tau)$.

Case 2. If $p > 1$, then $|\tau| > \rho^{k+1}$. In this case, the elements of I_{k+1} contained in the interval τ are

$$\bar\tau, \bar\tau + \rho^{k+1}, \cdots \bar\tau + (p-1)\rho^{k+1}, \bar\tau + |\tau|.$$

From this, we get immediately

$$\begin{aligned} &L_{\rho,k+1}(\sigma_1) = \cdots = L_{\rho,k+1}(\sigma_{p-1}) = \rho^{-(k+1)}|\sigma_1| = 1, \\ &L_{\rho,k+1}(\sigma_p) = \rho^{-1}L_{\rho,k}(\tau) - (p-1). \end{aligned}$$

2.4. Example

Example 2.1. Let $\rho = b^{-1}$, where $b > 2$ is an integer. The k-cylinders of $[0,1]$ consist of the intervals of length b^{-k}, so all first labels of the k-cylinders are 1.

Example 2.2. Let $\rho = \frac{\sqrt{5}-1}{2}$. The length of the k-cylinders of $[0,1]$ is either ρ^k or ρ^{k+1}. So there are two types of intervals, denoted by A and B. $L(A) = \rho$ and $L(B) = \rho^2$. The offspring of A is A and B, the offspring of B is A. More precisely, we have $A \to A, B$ and $B \to A$.

§3. Graph-Directed Constructions of Self-Similar Sets

In this section, we will introduce the second labels on the set of cylinders $\mathcal{C}_\rho$, associated with the given IFS (with overlapping). Then we construct a GDS with OSC that identifies the self-similar set generated by the IFS.

3.1. Second Labels on the Set of Cylinders $\mathcal{C}_\rho$

In this subsection, we suppose always that the compact set K is the SSS generated by the IFS (1.3). Set $\mathcal{D} = \{d_1, \cdots, d_m\}$, where d_j are defined as in (1.3).

Without loss of generality we assume that $0 = d_1 < d_2 < \cdots < d_m$. Then $J = [0, \frac{d_m}{1-\rho}]$ is a basic domain. We call the sets

$$J_{i_1\cdots i_k} = S_{i_1} \circ \cdots S_{i_k}(J),\ 1 \le i_1, \cdots, i_k \le m,$$

the k-cylinders of K and denote the set of them by $\mathcal{C}_{K,k}$. The set of all k-cylinders of K is denoted by $\mathcal{C}_K$. Let N_0 be the smallest integer such that $\rho^{-N_0} \ge \frac{d_m}{1-\rho}$ and let $I = [0, \rho^{-N_0}]$.

We have defined the first label on $\mathcal{C}_\rho$ in the last section, and we see that the labels only depend on the Pisot number $\beta = \rho^{-1}$. To establish the relations between $\mathcal{C}_{\rho,k}$ and $\mathcal{C}_{K,k}$, we need another label on $\mathcal{C}_\rho$ that relies on closely the structure of the cylinders of K.

Definition 3.1. *Suppose $\tau \in \mathcal{C}_{\rho,k}$. The neighborhood of τ with respect to $\mathcal{C}_{K,k}$ is defined as*

$$\mathcal{N}_k(\tau) = \{\delta : \delta \in \mathcal{C}_{K,k} \text{ and } \delta \cap \tau \ne \emptyset\}. \tag{3.1}$$

Notice that a k-cylinder of $\mathcal{C}_{K,k}$ is an interval of length $\rho^k|J|$, so it is completely determined by its left endpoint. For this point, we define

$$\bar{\mathcal{N}}_k(\tau) = \{\bar{\delta} : \delta \in \mathcal{C}_{K,k} \text{ and } \delta \cap \tau \ne \emptyset\}, \tag{3.2}$$

where, as above, $\bar{\delta}$ is the left endpoint of the interval δ.

Let $\tau, \sigma \in \mathcal{C}_{\rho,k}$. Then by (3.1) and (3.2)

$$\mathcal{N}_k(\tau) = \mathcal{N}_k(\sigma) \Rightarrow \bar{\mathcal{N}}_k(\tau) = \bar{\mathcal{N}}_k(\sigma).$$

Now we define the second label over $\mathcal{C}_\rho$ with respect to $\mathcal{C}_K$. Let $\tau \in \mathcal{C}_{\rho,k}$. The second label of τ is defined as

$$R_{\rho,k}(\tau) = \{\rho^{-k}(\bar{\delta} - \bar{\tau}) : \delta \in \mathcal{N}_k(\tau)\},$$
$$\mathcal{R}_\rho = \bigcup_{k\ge 0} \{R_{\rho,k}(\tau) : \tau \in \mathcal{C}_{\rho,k}\}.$$

Remark 3.1. By convention, the interval $I = [0, \rho^{-N_0}]$ is defined as a 0-cylinder, and its first label and second label are 1 and $\{0\}$ respectively.

Proposition 3.1. *Let ρ^{-1} be a Pisot number. Then $\sharp\mathcal{R}_\rho < \infty$.*

Proof. Suppose $\tau \in \mathcal{C}_{\rho,k}$ and $\delta \in \mathcal{N}_k(\tau)$. By the definitions of $\bar{\tau}$ and $\bar{\delta}$, we have

$$\bar{\tau} = \sum_{i=1-N_0}^{k-N_0} a_i \rho^i, \quad a_i \in [0, \rho^{-1}] \cap \mathbb{N},$$

$$\bar{\delta} = \sum_{i=1}^{k} b_i \rho^i, \quad b_i \in \mathcal{D}.$$

For each j, $d_j = \sum\limits_{p=0}^{\deg(d_j)} c_p \rho^p \in \mathbb{Z}(\rho)$ is a polynomials of ρ. We denote by $\deg(d_j)$ and $H(d_j)$ the degree and the height of the polynomial d_j (the height of a polynomial $\sum\limits_{1}^{m} u_j x^j$ is defined as $\max\limits_{1 \le i \le m} |u_i|$), and let

$$s = \max_{1 \le j \le m} \{\deg(d_j)\}, \quad h = \max_{1 \le j \le m} \{H(d_j)\}.$$

Let

$$\rho^{-s} := R_{\rho,k}(\tau) = \{\rho^{-k-s}(\bar{\delta} - \bar{\tau}) : \delta \in \mathcal{N}_k(\tau)\}.$$

Let $b_i = \sum\limits_{j=0}^{\deg(b_i)} c_{ij} \rho^j$, and notice that

$$\rho^{-(k+s)} \bar{\delta} = \sum_{i=1}^{k} b_i \beta^{s+k-i} = \sum_{i=1}^{k} \sum_{j=1}^{\deg(b_i)} c_{ij} \beta^{s+k-i-j} = \sum_{n=1}^{s+k} \sum_{i+j=n} c_{ij} \beta^{k+s-n}.$$

Since $0 \le j \le s$, $\sum\limits_{i+j=n} c_{ij} \le (s+1)h$, we get thus $H(\rho^{-(s+k)}\bar{\delta}) \le (s+1)h$. On the other hand, $H(\rho^{-(s+k)}\bar{\tau}) \le \rho^{-1}$, we obtain

$$H(\rho^{-(s+k)}(\bar{\tau} - \bar{\delta})) \le H(\rho^{-(s+k)}\bar{\delta}) + H(\rho^{-(s+k)}\bar{\tau}) \le (s+1)h + \rho^{-1}.$$

On the other hand, $\delta \cap \tau \ne \emptyset$, thus

$$|\bar{\tau} - \bar{\delta}| \le \max\{|\tau|, |\delta|\} \le \rho^{k-N_0},$$

so

$$\rho^{-k-s}|\bar{\tau} - \bar{\delta}| \le \rho^{-N_0-s}.$$

Let $M = (s+1)h + \rho^{-1}$ and $N = \rho^{-N_0-s}$. Then for each $k \in \mathbb{N}$,

$$R_{\rho,k}(\tau) \subset \rho^s \Omega_{M,N} (= \{\rho^s x : x \in \Omega_{M,N}\}),$$

which yields

$$\mathcal{R}_\rho \subset 2^{\rho^s \Omega_{M,N}},$$

where 2^A denotes the collection of all subsets of A as usual. By Proposition 2.1, $\sharp\Omega_{M,N} < \infty$, from which follows our conclusion.

3.2. The Graph-Directed Structure of K

Let $\delta = S_{i_1 \cdots i_n}(J) \in \mathcal{C}_K$, and let $1 \le l \le m$. Define $\delta_l := S_{i_1 \cdots i_n}(S_l(J))$. By the definition, we see that $\delta_l \subset \delta$ and $\delta_l \in \mathcal{C}_{K,n+1}$.

By the above definition, we get easily:

Lemma 3.1. *Let $\delta, \delta' \in \mathcal{C}_K$, and g be an affine map such that $g(\delta) = \delta'$. Then $g(\delta_l) = \delta'_l$.*

Let $\tau_1 \in \mathcal{C}_{\rho,k_1}$, and $\tau_2 \in \mathcal{C}_{\rho,k_2}$ be two cylinders of $\mathcal{C}_\rho$ with $L_{\rho,k_1}(\tau_1) = L_{\rho,k_2}(\tau_2)$. Then by Proposition 2.3, we have $\sharp O(\tau_1) = \sharp O(\tau_2) = p$ and $L_{\rho,k_1+1}(\sigma_i^{(1)}) = L_{\rho,k_2+1}(\sigma_i^{(2)})$, where $\{\sigma_i^{(1)}\}_{1\le i\le p}$ and $\{\sigma_i^{(2)}\}_{1\le i\le p}$ are respectively the offsprings (arranged from left to right) of τ_1 and τ_2.

Proposition 3.2. *Let $\tau_1 \in \mathcal{C}_{\rho,k_1}$ and $\tau_2 \in \mathcal{C}_{\rho,k_2}$ with $L(\tau_1) = L(\tau_2)$ and $R(\tau_1) = R(\tau_2)$. Then for $1 \le j \le p$,*

$$R_{\rho,k_1+1}(\sigma_j^{(1)}) = R_{\rho,k_2+1}(\sigma_j^{(2)}),$$

where the notations are as above.

Proof. Suppose that g is the affine mapping such that $\tau_2 = g(\tau_1)$. Then $\sigma_j^{(2)} = g(\sigma_j^{(1)}),\ 1 \le j \le p$, and it is easy to see that

$$\mathcal{N}_{k_1+1}(\sigma_j^{(1)}) = \{\delta_l : 1 \le l \le m,\ \delta \in \mathcal{N}_{k_1}(\tau_1) \text{ and } \delta_l \cap \sigma_j^{(1)} \ne \emptyset\}.$$

So by Lemma 3.1,

$$\begin{aligned}\mathcal{N}_{k_2+1}(\sigma_j^{(2)}) &= \{\delta'_l : 1 \le l \le m, \delta' \in \mathcal{N}_{k_2}(\tau_2) \ \text{ and } \ \delta'_l \cap \sigma_j^{(2)} \ne \emptyset\}\\ &= \{g(\delta_l) : 1 \le l \le m, \delta \in \mathcal{N}_{k_1}(\tau_1) \ \text{ and } \ \delta_l \cap \sigma_j^{(1)} \ne \emptyset\}\\ &= \{g(\delta_l) : \delta_l \in \mathcal{N}_{k_1+1}(\sigma_j^{(1)})\},\end{aligned}$$

which yields $R(\sigma_j^{(1)}) = R(\sigma_j^{(2)})$.

3.3. The Graph-Directed Construction of *K*

In this subsection, we are going to define a GDS with OSC which coincides with K.

(1) Definition of $\mathcal{V}$.

Let $\tau \in \mathcal{C}_{\rho,k}$. Define

$$V_k(\tau) = (L_k(\tau), R_k(\tau)),$$
$$\mathcal{V} = \{V(\tau) :\ \tau \in \mathcal{C}\} \subset \mathcal{L} \times \mathcal{R}.$$

Then $\mathcal{V}$ is a finite set because both $\mathcal{L}$ and $\mathcal{R}$ are finite sets by Propositions 2.3 and 3.2.

(2) Definition of $\mathcal{E}$.

Let $\alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2) \in \mathcal{V}$. Let $\tau \in \mathcal{C}_{\rho,k}$ such that $V_k(\tau) = \alpha$ and let $\beta(\tau) := \{\sigma \in O(\tau);\ V(\sigma) = \beta\}$. Define

$$\mathcal{E}_{\alpha,\beta} := \{(\alpha, \sigma)\}_{\sigma\in\beta(\tau)},$$
$$\mathcal{E} := \bigcup_{(\alpha,\beta)\in\mathcal{V}\times\mathcal{V}} \mathcal{E}_{\alpha,\beta}.$$

Notice that the above definition is independent of the choice of τ.

(3) Definition of F.

Keep the notations of (1) and (2). Suppose $\sigma \in \alpha(\tau)$. Define

$$F_{\alpha,\sigma}(x) := \rho x + \rho^{-k}(\bar{\sigma} - \bar{\tau}).$$

Geometrically, $F_{\alpha,\sigma}(x)$ is the contracting similarity with contraction ratio ρ which maps the interval τ onto its subinterval σ. Moreover, these similarity mappings satisfy OSC. Thus in this way, for any directed edge $e \in \mathcal{E}$, F_e is well defined.

By (1),(2) and (3), we define a graph-directed set $(\mathcal{V}, \mathcal{E}, F)$ with OSC.

We note that, to reconstruct K by the GDE above, the cylinders that do not meet K have no contribution, and these cylinders should be deleted. This fact is equivalent to that we should delete the verties of $\mathcal{V}$ of which the second label is empty. We put so

$$\mathcal{V}^* = \{(\gamma_1, \gamma_2) \in \mathcal{V} : \ \gamma_2 \neq \emptyset\}.$$

By restricting $(\mathcal{V}, \mathcal{E}, F)$ over $\mathcal{V}^*$ and F, we get a new GDS $(\mathcal{V}^*, \mathcal{E}^*, F^*)$. More precisely, the GDS is written as

$$E_i = \bigcup_{j=1}^{\sharp\mathcal{V}^*} \bigcup_{e \in \mathcal{E}^*_{i,j}} F^*_e(E_j).$$

As we pointed out, these GDS's $E_1, \cdots, E_{\sharp\mathcal{E}^*}$ satisfy OSC.

Denote by E the GDS corresponding to the vertex $\gamma = (1, \{0\})$.

Theorem 3.1. *With the above notations, we have $K = E$.*

Proof. Let $\mathcal{S}([0,1]) := \bigcup_{1\le i\le m} S_i([0,1])$ and $\mathcal{T}([0,1]) := \bigcup_{\alpha\in\mathcal{V}^*} \bigcup_{e\in\mathcal{E}^*_{\gamma,\alpha}} F^*_e([0,1])$. Denote by $\mathcal{S}^k$ and $\mathcal{T}^k$ the k-th iterations of $\mathcal{S}$ and $\mathcal{T}$ respectively. Then by [3],

$$K(k) := \mathcal{S}^k([0,1]) \xrightarrow{d_H} K,$$

$$E(k) := \mathcal{T}^k([0,1]) \xrightarrow{d_H} E,$$

where d_H denotes the Hausdorff metric.

Notice that $K(k)$ and $E(k)$ consist of the union of k-cylindes of $\mathcal{C}_{K,k}$ and $\mathcal{C}_{\rho,k}$ respectively. By a simple calculation from the definitions of $E,\ E(k),\ K,\ K(k)$, we have for any $k \in \mathbb{N}$

$$d_H(K, E) \le d_H(K, K(k)) + d_H(K(k), E(k)) + d_H(E(k), E) \le 3\rho^k.$$

Thus $d_H(K, E) = 0$ which yields $K = E$.

§4. Examples

In the last section, for a kind of IFS with overlapping, we proved that the SSS generated by the IFS is a GDS with OSC, and moreover we gave in fact the algorithm to get the GDS. In this section, we give some examples.

4.1. $\boldsymbol{\lambda}$-Cantor Sets

Suppose C_λ is the λ-Cantor set in Example 1.1. Then by Theorem 3.1, we have

Theorem 4.1. *If λ is a rational number or a Pisot number, then C_λ is a GDS with OSC.*

Let $\lambda = \frac{2}{9}$. It is easy to calculate that there are four types of cylinders of $[0,1]$, whose labels are $A = \{1, \{0\}\}, B = \{1, \{0, \frac{2}{3}\}\}, C = \{1, \{-\frac{1}{3}\}\}, D = \{1, \emptyset\}$ respectively. So $\mathcal{V} = \{A, B, C, D\}$. By an easy calculation we get

$$A \to B, C, A; \quad B \to B, C, B; \quad C \to C, A, D; \quad D \to D, D, D,$$

and by this we get the set $\mathcal{E}$. By deleting the empty element D we finally get $\mathcal{V}^* = \{A, B, C\}$ and $\mathcal{E}^*$ is

$$A \to B, C, A; \quad B \to B, C, B; \quad C \to C, A.$$

The Perron-Frobenius matrix of the graph is

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and the maximum eigenvalue of M is $\frac{3+\sqrt{5}}{2}$, so the dimension of this λ-Cantor set is

$$\dim C_{2/9} = \frac{\log \frac{3+\sqrt{5}}{2}}{\log 3}.$$

4.2. $\{0, 1, 3\}$-Problem

Suppose K_β is defined in Example 1.2. Then by Theorem 3.1, we have

Theorem 4.2. *If β^{-1} is a Pisot number, then K_β is a GDS with OSC.*

For β such that $1 = 2\beta + 2\beta^2$, using the above algorithm we obtain a 36×36 matrix M (also see [7]) and

$$\dim_H K_\beta = \frac{\log \rho(M)}{-\log \beta}.$$

Remark 4.1. It was shown (see [5]) that for β such that $1 = 2\beta + 2\beta^2 + \cdots + 2\beta^n$,

$$\dim_H K_\beta < 1.$$

This leads us to ask the following question.

Open Question. Let $1/3 < \beta < 2/5$ and β^{-1} be a Pisot number. Is the Hausdorff dimension of K_β always strictly less than 1?

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