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THE FUNDAMENTAL GROUP OF THE AUTOMORPHISM GROUP OF A NONCOMMUTATIVE TORUS**

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Abstract

Assume that each completely irrational noncommutative torus is realized as an inductive limit of circle algebras, and that for a completely irrational noncommutative torus A_ω of rank m there are a completely irrational noncommutative torus A_ρ of rank m and a positive integer d such that $\mathrm{tr}(A_\omega) = \frac{1}{d} \cdot \mathrm{tr}(A_\rho)$. It is proved that the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres A_ω has a group structure, denoted by $\pi_1^s(\mathrm{Aut}(A_\omega))$, which is isomorphic to $\mathbb{Z}$ if $\exists d > 1$ and $\{0\}$ if $\nexists d > 1$.

Let B_{cd} be a cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ of which no non-trivial matrix algebra can be factored out. The spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is defined by twisting $C^*(\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $B_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$. It is shown that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes C^*(\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.

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§1. Introduction

Given a locally compact abelian group G and a multiplier ρ on G, one can associate to them the twisted group C^*-algebra $C^*(G, \rho)$, which is the universal object for unitary ρ-representations of G. $C^*(\mathbb{Z}^m, \rho)$ is said to be a noncommutative torus of rank m and denoted by A_ρ. The multiplier ρ determines a subgroup S_ρ of G, called its symmetry group, and the multiplier ρ is called totally skew if the symmetry group S_ρ is trivial. And A_ρ is called completely irrational if ρ is totally skew (see [1, 13, 18]). It was shown in [1] that if G is a locally compact abelian group and ρ is a totally skew multiplier on G, then $C^*(G, \rho)$ is a simple C^*-algebra.

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The noncommutative torus A_ρ is the universal object for unitary ρ-representations of $\mathbb{Z}^m$, A_ρ is realized as $C^*(u_1, \cdots, u_m \mid u_i u_j = e^{2\pi i \theta_{ji}} u_j u_i)$, where u_i are unitaries and θ_{ji} are real numbers for $1 \le i, j \le m$. Boca[3] showed that the set of $\frac{m(m-1)}{2}$-tuples $(\theta_{12}, \theta_{13}, \cdots, \theta_{(m-1)m})$ of coefficients appearing in completely irrational noncommutative tori of rank m which are isomorphic to inductive limits of circle algebras has Lebesgue measure 1 in $[0,1)^{\frac{m(m-1)}{2}}$ for each m, where the term " circle algebra" denotes a C^*-algebra which is a finite direct sum of algebras of the form $C(\mathbb{T}^1) \otimes M_q(\mathbb{C})$.

We will assume that each completely irrational noncommutative torus appearing in this paper is an inductive limit of circle algebras.

The noncommutative torus A_ρ of rank m is obtained by an iteration of $m-1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbb{T}^1)$ (see [10]). When A_ρ is not simple, by a change of basis, A_ρ is obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$. Since the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ is a factor of the fibre of A_ρ, A_ρ can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on $A_{\frac{l}{d}}$, where the actions of $\mathbb{Z}$ on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial[4].

The definition of spherical noncommutative torus was introduced in [4, Definition 1.1]. Let B_{cd} be a cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^2$ of $S^2 \times \mathbb{T}^2$ is realized as $A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$. The crossed product $B_{cd} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_m} \mathbb{Z}$ is said to be a spherical noncommutative torus of rank m, and denoted by $\mathbb{S}_\rho^{cd}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_{cd}(\mathbb{C})$ of B_{cd} and $C(S^2)$ are trivial and $C(\mathbb{T}^2) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_m} \mathbb{Z}$ is a completely irrational noncommutative torus A_ρ of rank m. The fibre of $\mathbb{S}_\rho^d$, $c=1$, is isomorphic to the C^*-algebra $P_\rho^d := A_{\frac{l}{d}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_m} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial. Hence the fibre of $\mathbb{S}_\rho^{cd}$ is isomorphic to $P_\rho^d \otimes M_c(\mathbb{C})$.

The set $[M, BPU(cd)]$ of homotopy classes of continuous maps of a compact Hausdorff space M into the classifying space $BPU(cd)$ of the Lie group $PU(cd)$ is in bijective correspondence with the set of equivalence classes of principal $PU(cd)$-bundles over M, which is in bijective correspondence with the set of cd-homogeneous C^*-algebras over M. Thus each cd-homogeneous C^*-algebra over M is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η with base space M, fibre $M_{cd}(\mathbb{C})$, and structure group $\mathrm{Aut}(M_{cd}(\mathbb{C})) \cong PU(cd)$ (see [9, 16] for details). So each cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ is realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $S^2 \times \mathbb{T}^2$ with fibres $M_{cd}(\mathbb{C})$. Hence the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^2 with fibres $P_\rho^d \otimes M_c(\mathbb{C})$.

We are going to show that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes C^*(\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p, that $\mathcal{O}_{2u} \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_{2u} \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if cd and $2u-1$ are relatively prime, and that $\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_\infty \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$, where $\mathcal{O}_u$ and $\mathcal{O}_\infty$ denote the Cuntz algebra and the generalized Cuntz algebra, respectively.

The automorphism group of a completely irrational noncommutative torus was investigated in [12]. It is shown that for some completely irrational noncommutative torus A_ω, the group $\overline{\mathrm{Inn}}(A_\omega)$ (resp. $\overline{\mathrm{Inn}}(M_c(A_\omega))$) of approximate inner automorphisms of A_ω (resp. $M_c(A_\omega)$) is not arcwise connected.

§2. Spherical Noncommutative Tori

It was shown in [16, Proposition 2.10] that every d-homogeneous C^*-algebra over S^2 is isomorphic to one of the following C^*-subalgebras $B_{\frac{l}{d}}$, $l \in \mathbb{Z}_d$, of $C(e_+^2 \amalg e_-^2,\ M_d(\mathbb{C}))$, given as follows: if $f \in B_{\frac{l}{d}}$, then the following condition is satisfied

$$f_+(z) = U(z)^l f_-(z) U(z)^{-l}$$

for all $z \in S^1$, where e_+^2 (resp. e_-^2) denotes the 2-dimensional northern (resp. southern) hemisphere, and

$$U(z) = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & z \end{pmatrix}.$$

Since

$$[S^2, BPU(d)] \cong [S^1, PU(d)] \cong [\mathbb{T}^2, BPU(d)] \cong \mathbb{Z}_d,$$

one can construct d-homogeneous C^*-algebras over $\mathbb{T}^2$. It was shown in [5, Proposition 1] that every d-homogeneous C^*-algebra over $\mathbb{T}^2$ is isomorphic to one of the following C^*-subalgebras $A_{\frac{l}{d}}$, $l \in \mathbb{Z}_d$, of $C(\mathbb{T}^1 \times [0,1],\ M_d(\mathbb{C}))$, given as follows: if $f \in A_{\frac{l}{d}}$, then the following condition is satisfied

$$f(z,1) = U(z)^l f(z,0) U(z)^{-l}$$

for all $z \in \mathbb{T}^1$, where $U(z)$ is the unitary given above.

Note that $A_{\frac{l}{d}}$ is called a rational rotation algebra when $(d,l) = 1$.

Proposition 2.1.[4, Proposition 2.3] *Let B_{cd} be a cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^2$ of $S^2 \times \mathbb{T}^2$ is realized as $A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$, $(d,l) = 1$. Then B_{cd} is isomorphic to one of $B_{\frac{k}{c}} \otimes A_{\frac{l}{d}}$, $kd \in \mathbb{Z}_{cd}$, or one of the following C^*-subalgebras $D_{cd,k}$, $k \in \mathbb{Z}$, of $C((e_+^2 \amalg e_-^2) \times \mathbb{T}^1 \times [0,1], M_{cd}(\mathbb{C}))$, consisting of those functions f that satisfy*

$$(f|_{e_+^2 \amalg e_-^2})_+(z) = U(z)^k (f|_{e_+^2 \amalg e_-^2})_-(z) U(z)^{-k},$$
$$(f|_{\mathbb{T}^1 \times [0,1]})(w,1) = U(w)^{cl} (f|_{\mathbb{T}^1 \times [0,1]})(w,0) U(w)^{-cl}$$

for all $(z,w) \in S^1 \times \mathbb{T}^1$, where $U(z), U(w) \in PU(cd)$ are the unitaries given above.

Lemma 2.1. *Let $B_{\frac{l}{d}}$ be a d-homogeneous C^*-algebra over S^2 of which no non-trivial matrix algebra can be factored out. Then $[1_{B_{\frac{l}{d}}}] \in K_0(B_{\frac{l}{d}}) \cong \mathbb{Z}^2$ is primitive.*

Proof. It was shown in [4, Lemma 3.1] that $B_{\frac{l}{d}}$ is stably isomorphic to $C(S^2) \otimes M_d(\mathbb{C})$. So $K_0(B_{\frac{l}{d}}) \cong K_0(C(S^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since

$$[S^2, BPU(d)] \cong [S^1, PU(d)] \cong [\mathbb{T}^2, BPU(d)] \cong \mathbb{Z}_d,$$

$B_{\frac{l}{d}}$ corresponds to $A_{\frac{l}{d}}$ with respect to the conditions on sections over the boundaries S^1 of $e_+^2 \amalg e_-^2$ and $S^1 \times [0,1]$. The proof of the Elliott theorem given in [10, Theorem 2.2] implies that for a rational rotation algebra $A_{\frac{l}{d}}$ the canonical embedding of $C(\mathbb{T}^1)$ into

$A_{\frac{l}{d}}$ induces an isomorphism of $K_0(C(\mathbb{T}^2))$ into $K_0(A_{\frac{l}{d}})$ such that the primitive element $[1_{C(\mathbb{T}^2)}] \in K_0(C(\mathbb{T}^2))$ corresponds to the primitive element $[1_{A_{\frac{l}{d}}}] \in K_0(A_{\frac{l}{d}})$. The canonical embedding of $C(\mathbb{T}^1)$ into $A_{\frac{l}{d}}$ which induces the isomorphism of $K_0(C(\mathbb{T}^2))$ into $K_0(A_{\frac{l}{d}})$ corresponds to the embedding ϕ of $C(S^1)$ into $B_{\frac{l}{d}}$. The canonical embedding ϕ of $C(S^1)$ into $B_{\frac{l}{d}}$ induces an isomorphism μ of $K_0(C(S^2))$ into $K_0(B_{\frac{l}{d}})$, where $S^1 = \partial e^2_{\pm}$. The unit $1_{C(S^1)}$ corresponds to the unit $1_{C(S^2)}$ under the canonical embedding ψ of $C(S^1)$ into $C(S^2)$. $[1_{C(S^1)}] \in K_0(C(S^1)) \cong \mathbb{Z}$ corresponds to $[1_{C(S^2)}] \in K_0(C(S^2)) \cong \mathbb{Z}^2$, primitive in $K_0(C(S^2))$ (see [15]).In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^1)) & \xrightarrow{\psi_*} & K_0(C(S^2)) \\ {\scriptstyle (\mathrm{identity})_*}\big\downarrow & & \big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^1)) & \xrightarrow{\phi_*} & K_0(B_{\frac{l}{d}}), \end{array}$$

$\mu([1_{C(S^2)}]) = \phi_* \circ (\mathrm{identity})_* \circ \psi_*^{-1}([1_{C(S^2)}]) = [1_{B_{\frac{l}{d}}}]$. So $[1_{B_{\frac{l}{d}}}]$ is the image of the primitive element $[1_{C(S^2)}] \in K_0(C(S^2))$ under the isomorphism μ. Hence $[1_{B_{\frac{l}{d}}}] \in K_0(B_{\frac{l}{d}}) \cong \mathbb{Z}^2$ is primitive.

The proof given in Lemma 2.1 implies that the canonical embedding of $C(S^1)$ into $B_{\frac{l}{d}}$ induces an isomorphism of $K_0(C(S^2))$ into $K_0(B_{\frac{l}{d}})$ such that the class $[1_{C(S^2)}]$ of the unit $1_{C(S^2)}$ corresponds to the class $[1_{B_{\frac{l}{d}}}]$ of the unit $1_{B_{\frac{l}{d}}}$.

Proposition 2.2. *Let B_{cd} be a cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ defined above. Assume that no non-trivial matrix algebra can be factored out of B_{cd}. Then $K_0(B_{cd}) \cong K_1(B_{cd}) \cong \mathbb{Z}^4$, and $[1_{B_{cd}}] \in K_0(B_{cd})$ is primitive.*

Proof. It was shown in [4, Lemma 3.1] that B_{cd} is stably isomorphic to $C(S^2 \times \mathbb{T}^2) \otimes M_{cd}(\mathbb{C})$. So $K_0(B_{cd}) \cong K_0(C(S^2 \times \mathbb{T}^2)) \cong \mathbb{Z}^4$ and $K_1(B_{cd}) \cong K_1(C(S^2 \times \mathbb{T}^2)) \cong \mathbb{Z}^4$.

First, assume that B_{cd} is isomorphic to $B_{\frac{k}{c}} \otimes A_{\frac{l}{d}}$. By Lemma 2.1, $[1_{B_{\frac{k}{c}}}] \in K_0(B_{\frac{k}{c}}) \cong \mathbb{Z}^2$ is primitive. And by the Elliott theorem[10, Theoerm 2.2], $[1_{A_{\frac{l}{d}}}] \in K_0(A_{\frac{l}{d}}) \cong \mathbb{Z}^2$ is primitive. Since $[1_{B_{\frac{k}{c}} \otimes A_{\frac{l}{d}}}] \in K_0(B_{\frac{k}{c}} \otimes A_{\frac{l}{d}})$ is the image of $[1_{B_{\frac{k}{c}}}] \otimes [1_{A_{\frac{l}{d}}}] \in K_0(B_{\frac{k}{c}}) \otimes K_0(A_{\frac{l}{d}})$, which is primitive, under the isomorphism of $K_0(B_{\frac{k}{c}}) \otimes K_0(A_{\frac{l}{d}}) \oplus K_1(B_{\frac{k}{c}}) \otimes K_1(A_{\frac{l}{d}})$ into $K_0(B_{\frac{k}{c}} \otimes A_{\frac{l}{d}})$, $[1_{B_{\frac{k}{c}} \otimes A_{\frac{l}{d}}}]$ is primitive.

Next, assume that B_{cd} is isomorphic to $D_{cd,k}$. By the same reasoning as the proof given in Lemma 2.1, the canonical embedding ϕ of $C(S^1 \times \mathbb{T}^1)$ into B_{cd} induces an isomorphism μ of $K_0(C(S^2 \times \mathbb{T}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}^3$ into $K_0(B_{cd}) \cong \mathbb{Z} \oplus \mathbb{Z}^3$. The unit $1_{C(S^1 \times \mathbb{T}^1)}$ corresponds to the unit $1_{C(S^2 \times \mathbb{T}^2)}$ under the canonical embedding ψ of $C(S^1 \times \mathbb{T}^1)$ into $C(S^2 \times \mathbb{T}^2)$. $[1_{C(S^1 \times \mathbb{T}^1)}] \in K_0(C(S^1 \times \mathbb{T}^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ corresponds to $[1_{C(S^2 \times \mathbb{T}^2)}] \in K_0(C(S^2 \times \mathbb{T}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}^3$, primitive in $K_0(C(S^2 \times \mathbb{T}^2))$ (see [15]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^1 \times \mathbb{T}^1)) & \xrightarrow{\psi_*} & K_0(C(S^2 \times \mathbb{T}^2)) \\ {\scriptstyle (\mathrm{identity})_*}\big\downarrow & & \big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^1 \times \mathbb{T}^1)) & \xrightarrow{\phi_*} & K_0(B_{cd}), \end{array}$$

$\mu([1_{C(S^2 \times \mathbb{T}^2)}]) = \phi_* \circ (\mathrm{identity})_* \circ \psi_*^{-1}([1_{C(S^2 \times \mathbb{T}^2)}]) = [1_{B_{cd}}]$. So $[1_{B_{cd}}]$ is the image of the primitive element $[1_{C(S^2 \times \mathbb{T}^2)}] \in K_0(C(S^2 \times \mathbb{T}^2))$ under the isomorphism μ. Hence

$[1_{B_{cd}}] \in K_0(B_{cd})$ is primitive.

Therefore, $K_0(B_{cd}) \cong K_1(B_{cd}) \cong \mathbb{Z}^4$, and $[1_{B_{cd}}] \in K_0(B_{cd})$ is primitive.

We are going to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Theorem 2.1. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus of rank m. Assume that no non-trivial matrix algebra can be factored out of B_{cd}. Then $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^m}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.*

Proof. It was shown in [4, Theorem 3.4] that $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. By the Elliott theorem[10, Theorem 2.2],

$$\begin{aligned} K_0(\mathbb{S}_\rho^{cd}) &\cong K_0(C(S^2) \otimes A_\rho) \\ &\cong K_0(C(S^2)) \otimes K_0(A_\rho) \oplus K_1(C(S^2)) \otimes K_1(A_\rho) \\ &\cong \mathbb{Z}^2 \otimes \mathbb{Z}^{2^{m-1}} \oplus \{0\} \otimes \mathbb{Z}^{2^{m-1}} \cong \mathbb{Z}^{2^m}. \end{aligned}$$

Similarly, one can obtain that $K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^m}$. So it is enough to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. The proof is by induction on m. Assume that $m = 2$. We have obtained the result in Proposition 2.2.

So assume that the result is true for all spherical noncommutative tori of rank $m = i-1$. Write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(B_{cd}, u_3, \ldots, u_i)$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product by an action α of $\mathbb{Z}$ on $\mathbb{S}_{i-1}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(u_1^d, u_2^d, \cdots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i\theta_{ji}} u_j, j \neq 1, 2$, and sending u_j^d to $u_i u_j^d u_i^{-1} = e^{2\pi i d\theta_{ji}} u_j^d, j = 1, 2$), and which acts trivially on $C(S^2) \otimes M_{cd}(\mathbb{C})$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $\alpha_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \to K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \to \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{i-1} = 2^i$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^m}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Corollary 2.1. *Let p be a positive integer. Assume that no non-trivial matrix algebra can be factored out of B_{cd}. Then $\mathbb{S}_\rho^{cd} \otimes M_p(\mathbb{C})$ is not isomorphic to $A \otimes M_{sp}(\mathbb{C})$ for a C^*-algebra A if s is greater than 1. In particular, no non-trivial matrix algebra can be factored out of $\mathbb{S}_\rho^{cd}$, P_ρ^d, or A_ρ.*

Proof. Assume that $\mathbb{S}_\rho^{cd} \otimes M_p(\mathbb{C})$ is isomorphic to $A \otimes M_{sp}(\mathbb{C})$. Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes I_p$ corresponds to the unit $1_A \otimes I_{sp}$. So

$$[1_{\mathbb{S}_\rho^{cd}} \otimes I_p] = [1_A \otimes I_{sp}].$$

Thus there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $p[1_{\mathbb{S}_\rho^{cd}}] = (sp)[e]$. But $K_0(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^m}$ is torsion-free, so $[1_{\mathbb{S}_\rho^{cd}}] = s[e]$. This contradicts Theorem 2.1.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_p(\mathbb{C})$ is not isomorphic to $A \otimes M_{sp}(\mathbb{C})$.

We have obtained that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. This result is very useful for investigating the bundle structure of the tensor products of spherical noncommutative tori with UHF-algebras and Cuntz algebras.

§3. Tensor Products of Spherical Noncommutative Tori with UHF-Algebras and Cuntz Algebras

In this section, assume no non-trivial matrix algebra can be factored out of B_{cd}. Using the fact that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive, we are going to show that the tensor product of the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ with a UHF-algebra M_{p^∞} of type p^∞ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.

Theorem 3.1. *$\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of cd is a subset of the set of prime factors of p. To show that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^\infty}$. But there exist the C^*-algebra homomorphisms which are the canonical inclusions

$$\mathbb{S}_\rho^{cd} \otimes M_{(cd)^i}(\mathbb{C}) \hookrightarrow C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^i}(\mathbb{C}),$$
$$C(S^2) \otimes A_\rho \otimes M_{(cd)^i}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{(cd)^i}(\mathbb{C}) :$$
$$\mathbb{S}_\rho^{cd} \hookrightarrow C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{cd}(\mathbb{C}) \hookrightarrow C(S^2) \otimes A_\rho \otimes M_{(cd)^2}(\mathbb{C}) \hookrightarrow \cdots .$$

The inductive limit of the odd terms

$$\cdots \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^s}(\mathbb{C}) \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^{s+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$, and the inductive limit of the even terms

$$\cdots \to C(S^2) \otimes A_\rho \otimes M_{(cd)^s}(\mathbb{C}) \to C(S^2) \otimes A_\rho \otimes M_{(cd)^{s+1}}(\mathbb{C}) \to \cdots$$

is $C(S^2) \otimes A_\rho \otimes M_{(cd)^\infty}$. Thus by the Elliott theorem[11, Theorem 2.1], $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{(cd)^\infty}$.

Conversely, assume that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ corresponds to the unit $1_{C(S^2)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}$. So

$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{C(S^2)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}],$$
$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{S}_\rho^{cd}}] \otimes [1_{M_{p^\infty}}],$$
$$[1_{C(S^2)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] = cd([1_{C(S^2)\otimes A_\rho}] \otimes [1_{M_{p^\infty}}]).$$

Under the assumption that the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ corresponds to the unit $1_{C(S^2)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}$, if there is a prime factor q of cd such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for a projection e_∞ in M_{p^∞}. So there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $[1_{\mathbb{S}_\rho^{cd}}] = q[e]$. This contradicts Theorem 2.1. Thus the set of prime factors of cd is a subset of the set of prime factors of p.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.

In particular, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ has the trivial bundle structure if the set of prime factors of cd is a subset of the set of prime factors of p.

Let us study the tensor products of spherical noncommutative tori with (even) Cuntz algebras.

The Cuntz algebra $\mathcal{O}_u, 2 \le u < \infty$, is the universal C^*-algebra generated by u isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \cdots + s_u s_u^* = 1$. Cuntz[7,8] proved that $\mathcal{O}_u$ is simple and the K-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

Proposition 3.1. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ for a positive integer cd greater than 1. Let u be a positive integer such that cd and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_u \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.*

Proof. Let p be a prime such that $p \mid cd$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_u \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}$ corresponds to the unit $1_{\mathcal{O}_u \otimes C(S^2) \otimes A_\rho} \otimes I_{cd}$. So $[1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}] = [1_{\mathcal{O}_u \otimes C(S^2) \otimes A_\rho} \otimes I_{cd}] = cd[1_{\mathcal{O}_u \otimes C(S^2) \otimes A_\rho}]$. Hence there is a projection e in $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ such that $[1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}] = cd[e]$. But $[1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}] = [1_{\mathcal{O}_u}] \otimes [1_{\mathbb{S}_\rho^{cd}}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u) \cong \mathbb{Z}/(u-1)\mathbb{Z}$ (see [8]). But $p \mid u-1$. $[1_{\mathcal{O}_u}] \ne p[e_*]$ for a projection e_* in $\mathcal{O}_u$. So $[1_{\mathbb{S}_\rho^{cd}}] = p[e']$ for a projection e' in $\mathbb{S}_\rho^{cd}$. This contradicts Theorem 2.1. Hence cd and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_u \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if cd and $u-1$ are not relatively prime.

The following result is useful for understanding the bundle structure of $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$.

Proposition 3.2.[19, Theorem 7.2] *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces α.*

Corollary 3.1.

(1) *Let p be an odd integer such that p and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{(2u-1)p+1} \otimes M_{p^\infty}$. That is, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$.*

(2) *$\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty}$.*

Theorem 3.2. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus with fibres $P_\rho^d \otimes M_c(\mathbb{C})$. Then $\mathcal{O}_{2u} \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_{2u} \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if cd and $2u-1$ are relatively prime.*

Proof. Assume that cd and $2u-1$ are relatively prime. Let $cd = p2^v$ for some odd integer p. Then p and $2u-1$ are relatively prime. Then by Corollary 3.1 $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2u)^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(cd)^\infty}$. Thus by Theorem 3.1 $\mathcal{O}_{2u} \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes \mathbb{S}_\rho^{cd}$, which in turn is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Thus $\mathcal{O}_{2u} \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_{2u} \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

The converse was proved in Proposition 3.1.

Therefore, $\mathcal{O}_{2u} \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_{2u} \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if cd and $2u-1$ are relatively prime.

Cuntz[8] computed the K-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$.

He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

Proposition 3.2. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus with fibres $P_\rho^d \otimes M_c(\mathbb{C})$. Then $\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_\infty \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$.*

Proof. Suppose $\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_\infty \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. The unit $1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}$ corresponds to the unit $1_{\mathcal{O}_\infty \otimes C(S^2) \otimes A_\rho} \otimes I_{cd}$. By the same trick as in the proof of Proposition 3.1, one can show that $[1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}] = cd[e]$ for a projection $e \in \mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$. $[1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}] = [1_{\mathcal{O}_\infty}] \otimes [1_{\mathbb{S}_\rho^{cd}}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [8]). So $[1_{\mathbb{S}_\rho^{cd}}] = cd[e']$ for a projection $e' \in \mathbb{S}_\rho^{cd}$. This contradicts Theorem 2.1.

Therefore, $\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_\infty \otimes C(S^2) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

§4. Completely Irrational Noncommutative Tori

It was proved in [2, Theorem 1.5] that every completely irrational noncommutative torus has real rank 0, where the “ real rank 0” means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. Combining [4, Theorem 3.4] and [6, Corollary 3.3] yields that the simple C^*-algebra P_ρ^d has real rank 0 since the completely irrational noncommutative torus A_ρ has real rank 0. The Lin and Rørdam theorem[17, Proposition 3] says that the simple C^*-algebra P_ρ^d is an inductive limit of circle algebras, since $P_\rho^d \otimes \mathcal{K}(\mathcal{H}) \cong A_\rho \otimes \mathcal{K}(\mathcal{H})$ is an inductive limit of circle algebras[17, Proposition 2]. Combining the Elliott classification theorem [11, Theorem 7.1] and [14, Theorem 1.3] yields that the completely irrational noncommutative torus A_ω and the simple C^*-algebra P_ρ^d induced from A_ρ are isomorphic if the ranges of the traces equal.

Lemma 4.1. $\mathrm{tr}(K_0(P_\rho^d)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$.

Proof. P_ρ^d has a matrix representation induced from the matrix representation of the rational rotation subalgebra $A_{\frac{l}{d}}$. The diagonal entries of the matrix representation are in A_ρ, and so the range of the trace of $K_0(P_\rho^d)$ is $\mathbb{Z} + \frac{1}{d}(\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \cdots + \mathbb{Z}\gamma)$, where $\mathrm{tr}(K_0(A_\rho)) = \mathbb{Z} + \mathbb{Z}d + \mathbb{Z}\alpha + \mathbb{Z}\beta + \cdots + \mathbb{Z}\gamma$, and hence $\mathrm{tr}(K_0(P_\rho^d)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$.

Theorem 4.1. *Let A_ω be a completely irrational noncommutative torus of rank m with $\mathrm{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$ for a completely irrational noncommutative torus A_ρ of rank m. Then A_ω is isomorphic to P_ρ^d.*

Proof. One can assume that

$$\begin{aligned}\mathrm{tr}(K_0(A_\omega)) &= \mathbb{Z} + \mathbb{Z}\frac{l}{d} + \mathbb{Z}\alpha + \cdots + \mathbb{Z}\beta = \mathbb{Z}\frac{1}{d} + \mathbb{Z}\alpha + \cdots + \mathbb{Z}\beta \\ &= \frac{1}{d}(\mathbb{Z} + \mathbb{Z}d + \mathbb{Z}d\alpha + \cdots + \mathbb{Z}d\beta) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho)).\end{aligned}$$

One can divide the proof into three cases according to where the rational number $\frac{l}{d}$ comes from.

Case 1. The case that $\frac{l}{d}$ is one of θ_{ij} or $\frac{l}{d}$ can be obtained by a change of basis for $\mathbb{Z}^m$ to be $\frac{l}{d} = \theta_{ij}$. By a change of basis for $\mathbb{Z}^{r+m}$, one can easily obtain that there is a completely irrational noncommutative torus $A_{\rho'}$ with $\theta_{ij} = \frac{l}{d}$ for some i, j, and $\mathrm{tr}(K_0(A_\rho)) = \mathrm{tr}(K_0(A_{\rho'}))$.

Case 2. The case that $\frac{l}{d}$ is one of the products of θ_{ij}. By replacing a suitable product of real numbers appearing in the product giving $\frac{l}{d}$ with $\frac{l}{d}$, one can replace A_ρ with $A_{\rho'}$ up to isomorphism (by the Elliott classification theorem and the Ji and Xia result) with $\theta_{ij} = \frac{l}{d}$ for some i, j and $\mathrm{tr}(K_0(A_\rho)) = \mathrm{tr}(K_0(A_{\rho'}))$.

Case 3. The case that $\frac{l}{d}$ is a $\mathbb{Z}$-linear combination of the products of θ_{ij}. Combining the two results given above yields that A_ρ can be replaced with $A_{\rho'}$ with $\theta_{ij} = \frac{l}{d}$ for some i, j, and $\mathrm{tr}(K_0(A_\rho)) = \mathrm{tr}(K_0(A_{\rho'}))$. So one can assume that $A_{\rho'} = A_\rho$. Hence

$$\mathrm{tr}(K_0(P_\rho^d)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho)) = \mathrm{tr}(K_0(A_\omega)).$$

By the Elliott classification theorem[11, Theorem 7.1] and the Ji and Xia's result[14, Theorem 1.3], P_ρ^d is isomorphic to A_ω.

§5. Sectional C^*-Algebras with Fibres a Completely Irrational Noncommutative Torus

We are going to show that the set of all spherical noncommutative tori with fibres $A_\omega \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres $A_\omega \otimes M_c(\mathbb{C})$ for a completely irrational noncommutative torus A_ω.

Let A_ω be a noncommutative torus of rank m with $\widehat{S_\omega} \cong \mathbb{T}^1$ and fibres $M_d(\mathbb{C}) \otimes A_\rho$ for a completely irrational noncommutative torus A_ρ (see [18]). By the definition of A_ω, $C(\mathbb{T}^1)$ and A_ρ split. Since $[\mathbb{T}^1, BPU(d)] \cong \{0\}$, $C(\mathbb{T}^1)$ and $M_d(\mathbb{C})$ split. And $M_d(\mathbb{C})$ and A_ρ also split. But by Corollary 2.1, A_ω has a non-trivial bundle structure if $d > 1$. This implies that a C^*-subalgebra of A_ρ plays a role as a base space in the bundle structure. In fact, A_ω can be obtained by an iteration of $m - 2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$, and the non-triviality of the bundle structure is given by a non-trivial element of $[\mathbb{T}^2, BPU(d)] \cong [\mathbb{T}^1, PU(d)] \cong \mathbb{Z}_d$, which represents $A_{\frac{l}{d}}$ canonically embedded into A_ω.

Let d be the biggest integer among the possible integers satisfying the condition

$$\mathrm{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho)),$$

i.e., $A_\omega \cong P_\rho^d$. We want to show that each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres P_ρ^1 has the trivial bundle structure.

Lemma 5.1. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^1 with fibres P_ρ^1 has the trivial bundle structure.*

Proof. Let $P_\rho^1 = \varinjlim \Big(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}) \Big)$. The C^*-algebra $\Gamma(\eta)$ is isomorphic to an inductive limit of direct sums of $p_{i(j)}$-homogeneous C^*-algebras over $S^1 \times \mathbb{T}^1$, and each $C(S^1 \times \mathbb{T}^1)$ is canonically embedded into $\Gamma(\eta)$. So there could be a canonical homomorphism of $C(S^1) \otimes M_d(\mathbb{C})$ into the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^1 with fibres P_ρ^1 such that the non-triviality can be given by a d-homogeneous C^*-algebra over $S^1 \times \mathbb{T}^1$. Then $M_d(\mathbb{C})$ must be factored out of the circle algebra in each inductive step, and so the range of the trace of P_ρ^1 would be the form $\frac{1}{d}\mathrm{tr}(K_0(A))$ for a simple unital C^*-algebra A, which is impossible by the assumption for the range of the trace. We have two cases; one of them is the case that a C^*-subalgebra of P_ρ^1 plays a role as a base space in the bundle structure, and the other is not.

For the first case, when a C^*-subalgebra of P^1_ρ plays a role as a base space in the bundle structure and P^1_ρ is realized as a tensor product of non-trivial completely irrational noncommutative tori, the torsion-free groups in $P^1_\rho = A_\rho$ giving simple noncommutative tori which are given by twisting the torsion-free groups by totally skew multipliers must split, so all factors of P^1_ρ must split. The relation among factors of P^1_ρ is different from the relation between fibres $M_d(\mathbb{C})$ and base A_ρ in the fibres of the non-simple noncommutative torus A_ω given above, and so one can assume that all factors of P^1_ρ play roles as a base space in the bundle structure. Hence P^1_ρ plays a role as a base space in the bundle structure, and so $\Gamma(\eta)$ is isomorphic to $C(S^1) \otimes P^1_\rho$.

For the other case, since $P^1_\rho = \varinjlim \Big(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}) \Big)$, there is a non-trivial matrix algebra $M_p(\mathbb{C})$ which is embedded into P^1_ρ. Since $[S^1, BPU(p)] \cong \{0\}$, $C(S^1)$ and $M_p(\mathbb{C})$ split, i.e., any p-homogeneous C^*-algebra over S^1 has the trivial bundle structure. By the same reasoning as above, $M_p(\mathbb{C})$ cannot be factored out of the circle algebras in all inductive steps. But $\Gamma(\eta)$ has a locally trivial bundle structure. Hence $C(S^1)$ and $(M_p(\mathbb{C}) \hookrightarrow)$ P^1_ρ must split, and so $\Gamma(\eta)$ has the trivial bundle structure.

Therefore, each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^1 with fibres P^1_ρ has the trivial bundle structure.

Compare the C^*-algebra $\Gamma(\eta)$ for the second case with any C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^3 with fibres $\mathcal{K}(\mathcal{H})$.

Proposition 5.1. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^2 with fibres P^1_ρ has the trivial bundle structure.*

Proof. Let $P^1_\rho = \varinjlim \Big(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}) \Big)$. But there is a map of degree 1 from S^2 to $S^1 \times \mathbb{T}^1$. So each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres P^1_ρ is induced from the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $S^1 \times \mathbb{T}^1$ with fibres P^1_ρ. Consider the crossed product by the action α_θ of $\mathbb{Z}$ on $\Gamma(\zeta)$ for a suitable irrational number θ such that the range of the trace of $P^d_\rho \otimes A_\theta$ is not $\frac{1}{s}\times$ the range of the trace of any simple noncommutative torus of rank $m+1$ for any positive integer s greater than 1, where the action α_θ on $C(S^1) \otimes P^1_\rho$ is trivial and $C(\mathbb{T}^1) \times_{\alpha_\theta} \mathbb{Z}$ is the irrational rotation algebra A_θ. Then $\Gamma(\zeta) \times_{\alpha_\theta} \mathbb{Z}$ is obviously realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^1 with fibres $P^1_\rho \otimes A_\theta$, since by Lemma 5.1 P^1_ρ and $C(\mathbb{T}^1)$ split and so do P^1_ρ and A_θ. By Lemma 5.1, $\Gamma(\zeta) \times_{\alpha_\theta} \mathbb{Z}$ has the trivial bundle structure. Hence each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres P^1_ρ has the trivial bundle structure.

Each cd-homogeneous C^*-algebra over $S^2 \times \mathbb{T}^2$ is realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $S^2 \times \mathbb{T}^2$ with fibres $M_{cd}(\mathbb{C})$, and hence $\mathbb{S}^{cd}_\rho$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres $P^d_\rho \otimes M_c(\mathbb{C})$.

Theorem 5.1. *The set of spherical noncommutative tori with fibres $P^d_\rho \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres $P^d_\rho \otimes M_c(\mathbb{C})$.*

Proof. If $cd = 1$, we have obtained the result in Proposition 5.1. So assume that $cd > 1$. Then one can assume that there is the matrix algebra $M_{cd}(\mathbb{C})$ which is factored out of each inductive step. But $M_d(\mathbb{C})$ is not factored out of P_ρ^d, and so P_ρ^d is isomorphic to $A_{\frac{1}{d}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_m} \mathbb{Z}$. By Proposition 5.1, each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres $C^*(d\mathbb{Z} \times d\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_m} \mathbb{Z}$ has the trivial bundle structure. Hence each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is given by twisting $C^*(\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $B_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by the totally skew multiplier ρ on $\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$.

Therefore, the set of spherical noncommutative tori with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres $P_\rho^d \otimes M_c(\mathbb{C})$.

The set of spherical noncommutative tori with fibres the simple C^*-algebra P_ρ^d has a group structure.

Definition 5.1. *The group of spherical noncommutative tori with fibres the simple C^*-algebra P_ρ^d is said to be the fundamental group of the automorphism group* $\mathrm{Aut}(P_\rho^d)$*, and denoted by* $\pi_1^s(\mathrm{Aut}(P_\rho^d))$.

Theorem 5.2. $\pi_1^s(\mathrm{Aut}(P_\rho^d)) \cong \mathbb{Z}$ *if* $d > 1$, $\cong \{0\}$ *if* $d = 1$.

Proof. By Theorem 5.1, the set of spherical noncommutative tori with fibres P_ρ^d is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres P_ρ^d. So $\pi_1^s(\mathrm{Aut}(P_\rho^d))$ is in bijective correspondence with $\{D_{d,k} \mid k \in \mathbb{Z}\}$ given in the statement of Proposition 2.1. Hence $\pi_1^s(\mathrm{Aut}(P_\rho^d)) \cong \mathbb{Z}$ if $d > 1$, $\cong \{0\}$ if $d = 1$.

Elliott and Rørdam[12] proved that $\overline{\mathrm{Inn}}(A_\omega)$ is connected.

Question.[12] *Is* $\overline{\mathrm{Inn}}(A_\omega)$ *arcwise connected ?*

It is well-known (see [21, Lemma 1.8, Lemma 3.1, Lemma 3.2]) that for a locally trivial C^*-algebra bundle over S^2 with fibres A_ω there is a principal $\mathrm{Aut}(A_\omega)$-bundle over S^1, and that the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over S^2 with fibres A_ω is in bijective correspondence with the set of principal $\mathrm{Aut}(A_\omega)$-bundles over S^1. The set of principal $\mathrm{Aut}(A_\omega)$-bundles over S^1 is in bijective correspondence with (the set of principal $\overline{\mathrm{Inn}}(A_\omega)$-bundles over S^1) $\times\, GL(2^{m-1}, \mathbb{Z})$, since $GL(2^{m-1}, \mathbb{Z}) \cong \mathrm{Aut}(K_1(A_\omega))$ is discrete.

If $\mathrm{Aut}(A_\omega)$ is arcwise connected, then $\pi_1(\mathrm{Aut}(A_\omega))$ is in bijective correspondence with the set of principal $\mathrm{Aut}(A_\omega)$-bundles over S^1 by [20, Corollary 18.6]. This implies that if $\overline{\mathrm{Inn}}(A_\omega)$ is arcwise connected, then $\pi_1(\overline{\mathrm{Inn}}(A_\omega))$ is in bijective correspondence with the set of principal $\overline{\mathrm{Inn}}(A_\omega)$-bundles over S^1

Theorem 5.3. *Let A_ω be a completely irrational noncommutative torus which is isomorphic to P_ρ^d for an integer d greater than* 1*. Then* $\overline{\mathrm{Inn}}(A_\omega)$ *is not arcwise connected.*

Proof. Let P_ρ^d be a simple C^*-algebra which is given by twisting $C(\mathbb{T}^2) \otimes C^*(\mathbb{Z}^{m-2})$ in $A_{\frac{l}{d}} \otimes C^*(\mathbb{Z}^{m-2})$ by the multiplier ρ on $\widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$, where $\mathbb{T}^2$ is the primitive ideal space of $A_{\frac{l}{d}}$, and $C^*(\widehat{\mathbb{T}^2}, \text{res of } \rho) \cong C^*(\widehat{\mathbb{T}^2})$. Assume that $\overline{\mathrm{Inn}}(P_\rho^d)$ is arcwise connected. Then $\pi_1(\overline{\mathrm{Inn}}(P_\rho^d))$ is in bijective correspondence with the set of principal $\overline{\mathrm{Inn}}(P_\rho^d)$-bundles over S^1.

The elements of $\pi_1^s(\mathrm{Aut}(P_\rho^d))$ correspond to the C^*-algebras of locally trivial C^*-algebra bundles over S^2 with fibres P_ρ^d. There are principal $\mathrm{Aut}(P_\rho^d)$-bundles over S^2 corresponding to the elements. Actually, the principal $PU(d)$-bundles over S^2 correspond to the elements if $\overline{\mathrm{Inn}}(P_\rho^d)$ is arcwise connected. But $\pi_1(PU(d)) = \mathbb{Z}/d\mathbb{Z}$ is in bijective correspondence with the set of principal $PU(d)$-bundles over S^2. This contradicts the fact that $\pi_1^s(\mathrm{Aut}(P_\rho^d)) \cong \mathbb{Z}$ if $d > 1$. So $\overline{\mathrm{Inn}}(P_\rho^d)$ is not arcwise connected if $d > 1$.

Therefore, $\overline{\mathrm{Inn}}(A_\omega)$ is not arcwise connected if A_ω is isomorphic to P_ρ^d for an integer d greater than 1.

For the classifying space $B(\mathrm{Aut}(M_c(\mathbb{C}) \otimes P_\rho^d))$ of the automorphism group $\mathrm{Aut}(M_c(\mathbb{C}) \otimes P_\rho^d)$,

$$\begin{aligned}\pi_2(B(\mathrm{Aut}(M_c(P_\rho^d)))) &= [S^2, B(\mathrm{Aut}(M_c(P_\rho^d)))] = [S^1, \mathrm{Aut}(M_c(P_\rho^d))]\\ &= \pi_1(\mathrm{Aut}(M_c(P_\rho^d))),\end{aligned}$$

which is a non-trivial group if $cd > 1$. But $\pi_1^s(\mathrm{Aut}(M_c(P_\rho^d)))$ has not a group structure if $c > 1$ and $d > 1$. Assume that $\overline{\mathrm{Inn}}(M_c(P_\rho^d))$ is arcwise connected. Then $\pi_1(\overline{\mathrm{Inn}}(M_c(P_\rho^d))) \cong \pi_1^s(\mathrm{Aut}(M_c(P_\rho^d)))$, which is a contradiction. So $\overline{\mathrm{Inn}}(M_c(P_\rho^d))$ is not arcwise connected.

We have obtained that $\overline{\mathrm{Inn}}(A_\omega \otimes M_c(\mathbb{C}))$ is not arcwise connected if A_ω is isomorphic to P_ρ^d for an integer d greater than 1.

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