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# HEAT FLOW FOR YANG-MILLS-HIGGS FIELDS, PART I

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**Abstract**

The Yang-Mills-Higgs field generalizes the Yang-Mills field. The authors establish the local existence and uniqueness of the weak solution to the heat flow for the Yang-Mills-Higgs field in a vector bundle over a compact Riemannian 4-manifold, and show that the weak solution is gauge-equivalent to a smooth solution and there are at most finite singularities at the maximum existing time.

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## §1. Introduction

Let $(M,g)$ be a four dimensional Riemannian manifold and let $P$ be a principal bundle with compact Lie structure group $G$ over $M$. The Yang-Mills functional is defined by

$$\mathrm{YM}(D) = \int_M |F|^2 \, dM,$$

where $F$ denotes the curvature of a connection $D$ and $dM$ is the volume form induced by the metric $g$ on $M$. All integrals on $M$ in this paper will be with the volume form $dM$, so we will not write it explicitly.

Donaldson[3] and K. Uhlenbeck[18,19] obtained many important results about Yang-Mills equations (see also [5] and [7]). In physics, Yang-Mills fields represent forces. If they interact with a second type of field – the field of a particle, one can view the Yang-Mills functional as a special case of a more general Yang-Mills-Higgs functional.

Let $V$ be a finite dimensional vector space and $G \subset GL(V)$. The Yang-Mills-Higgs functional is defined through a section $u$ of a vector bundle $\eta = P \times_G V$ in the following:

$$\mathrm{YMH}(u,D) = \frac{1}{2}\int_M |F|^2 + |Du|^2 + \frac{\lambda}{4}\left(1-|u|^2\right)^2, \tag{1.1}$$

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where $D$ is a connection or covariant directive on $\Gamma(\eta)$ compatible with the structure group $G$, $F$ is the curvature of $D$, $Du$ is the covariant derivative of $u$, and $\lambda \geq 0$ is a constant[8]. A pair $u \in \Gamma(\eta)$ and $D$ is a solution to the Yang-Mills-Higgs equations if and only if

$$D^*F = -J, \quad D^*Du = \frac{\lambda}{2}u(1-|u|^2), \tag{1.2}$$

where if $G = SO(r)$ or $SU(r)$, then Lemma 2.1 shows

$$J = J(u, D) = \frac{1}{2}\left(Du \otimes u^* - u \otimes (Du)^*\right) \in \Omega^1(Ad\,\eta).$$

Let $\mathcal{G}$ be the group of gauge transformations of $\Gamma(\eta)$. Then $S \in \mathcal{G}$ acts on a connection $D$ and section $u$ as follows

$$\tilde{D} = S^*D := S^{-1} \circ D \circ S, \quad \tilde{u} := S^{-1}u. \tag{1.3}$$

The system (1.2) is not elliptic since it is invariant under the above gauge transformation (1.3). There exist many related results about Yang-Mills-Higgs equations in four dimension (e.g., see [1] and [12]).

On the other hand, the heat flow for Yang-Mills equation in four dimension has played an important role for Yang-Mills theory. Atiyah and Bott suggested the heat flow for Yang-Mills. The first contribution was made by Donaldson[4] in the case of a holomorphic vector bundle. Generally, Yang-Mills flow in a $G$-bundle over a 4-manifold may blow up in a finite time. Struwe[16] and Schlatter[13] established global existence and uniqueness of weak solutions to the Yang-Mills flow in a principal $G$ bundle over a compact 4-manifold with $G \subset SO(r)$. For Yang-Mills flows in higher dimensions, see [2].

In this paper, we consider a heat flow for the Yang-Mills-Higgs field. A pair $(u(t), D(t))$ is said to be a solution to the Yang-Mills-Higgs flow if it satisfies

$$\frac{\partial D}{\partial t} = -D^*F - J, \quad \frac{\partial u}{\partial t} = -D^*Du + \frac{\lambda}{2}u(1-|u|^2) \tag{1.4}$$

with initial conditions

$$D(0) = D_0, \quad u(0) = u_0, \tag{1.5}$$

where $D_0$ and $u_0$ are given.

Fix a connection $D_{\text{ref}} : \Gamma(\eta) \to \Omega^1(\eta) := \Gamma(\eta \otimes \Lambda^1(M))$; then any connection $D$ can be expressed as $D = D_{\text{ref}} + A$ with $A \in \Omega^1(Ad\,\eta)$, a one form on $M$ with values in the Lie algebra $\mathfrak{g}$ of $G$ (see Section 2 for the definitions and notations). Hence if we fix $D_{\text{ref}}$, we can think $D = D_{\text{ref}} + A$ as a one form $A$.

**Definition 1.1.** *A family of* $(u(t), A(t))$ *is a weak solution to* (1.4) *on* $[0, T)$ *if*

$$A \in L^2\left([0,T); L^2\left(\Omega^1(Ad\,\eta)\right)\right), \quad u \in L^2\left([0,T); H^{1,2}\left(\Omega^0(\eta)\right)\right),$$
$$F \in L^\infty\left([0,T); L^2\left(\Omega^2(Ad\,\eta)\right)\right), \quad \partial_t u \in L^2\left([0,T), L^2\left(\Omega^0(\eta)\right)\right),$$

*such that for all* $a \in C^\infty\left([0,T); \Omega^1(Ad\,\eta)\right)$ *vanishing near* $t = 0$ *and* $t = T$, *and* $\phi \in C^\infty\left([0,T); \Omega^0(\eta)\right)$

$$\int_0^T \left\{\left(A, \frac{\partial a}{\partial t}\right) - (F, Da) - (J, a)\right\} dt = 0, \tag{1.6}$$

$$\int_0^T \left\{\left(\frac{\partial u}{\partial t}, \phi\right) + (Du, D\phi) - \frac{\lambda}{2}\left(u(1-|u|^2), \phi\right)\right\} dt = 0. \tag{1.7}$$

We say that a connection $D$ is irreducible if for all $s \in H^{1,2}\left(\Omega^0(Ad\,\eta)\right)$ there is a constant $C = C(D)$, such that

$$\|s\|_{H^{1,2}} \leq C\,\|Ds\|_{L^2}. \tag{1.8}$$

The main result of this paper is

**Theorem A.** (i) *For any $H^{1,2}$ connection $D_0$ and $H^{1,2}$ section $u \in \Gamma(\eta)$, there is a $T > 0$ and a weak solution $(D, u) = (D_{\text{ref}} + A, u)$ to* (1.4) *and* (1.5) *for $0 \le t < T$ such that*

$$A \in C^0\left([0,T); L^2\left(\Omega^1(Ad\,\eta)\right)\right) \cap H^{1,2}\left([0,T); L^2\left(\Omega^1(Ad\,\eta)\right)\right),$$
$$u \in C^0\left([0,T); H^{1,2}\left(\Omega^0(\eta)\right)\right) \cap H^{1,2}\left([0,T); L^2\left(\Omega^0(\eta)\right)\right),$$
$$F \in C^0\left([0,T); L^2\left(\Omega^2(Ad\,\eta)\right)\right).$$

*Moreover, $D$ and $u$ are gauge-equivalent to a smooth solution to* (1.4) *in the following sense*:

*There are solutions $\hat{D} = D_{\text{ref}} + \hat{A}$ and $\hat{u}$ to* (1.4) *with*

$$\hat{A} \in C^0\left([0,T); L^2\left(\Omega^1(Ad\,\eta)\right)\right), \quad \hat{u} \in C^0\left([0,T); H^{1,2}\left(\Omega^0(\eta)\right)\right),$$

*and they are smooth for $0 < t < T$. Furthermore, there is a sequence of smooth gauge transformations $\hat{S}_k$ and a sequence $t_k \searrow 0$ such that $\hat{S}_k \to \hat{S}_0$ in $H^{1,2}$, $\hat{S}_k^*(\hat{D})(t_k) \to D_0$ in $H^{1,2}$, $\hat{S}_k^{-1}\hat{u}(t_k) \to u_0$ in $H^{1,2}$, and $D = \hat{S}_0^*(\hat{D})$, $u = \hat{S}_0^{-1}\hat{u}$. Finally, $D$ and $u$ are smooth if $D_0$ and $u_0$ are smooth.*

(ii) *If $D(t)$ is irreducible in the sense of* (1.8) *for all $t$, then $D$ is unique.*

(iii) *The maximal existence time $T$, if it is finite, is characterised by*

$$T = \sup\Big\{\bar{t} > 0; \exists R > 0 : \sup_{x_0 \in M,\, 0 \le t \le \bar{t}} \Big(\int_{B_R(x_0)} |F(t)|^2 + |Du(t)|^2\Big) < \epsilon_0\Big\},$$

*where $\epsilon_0 = \epsilon_0(\eta) > 0$. At $\bar{t}_1 = T$, energy concentrates in at most finite many points $\bar{x}_1^j$, $j = 1, \cdots, l_1$, in the sense that*

$$\forall R > 0 : \limsup_{t \nearrow \bar{t}_1} \int_{B_R(x_0)} (|F(t)|^2 + |Du|^2) \ge \epsilon_0.$$

We will present the long time behavior and the blow up phenomenon in [6], the part II of this article.

## §2. Preliminaries

Let $(M, g)$ be a 4-dimensional compact Riemannian manifold and let $(P, M, \pi)$ be a principal $G$ bundle with a compact Lie group $G$. Let $V$ be an $r$ dimensional (complex or real) vector space and $G$ has a representation $\rho : G \to GL(V)$. Then $\pi : \eta := P \times_\rho V \to M$ is a rank $r$ vector bundle (real or complex) associated with $P$. Our main interset is in the cases that the structure group $G$ is either $SO(r)$ or $SU(r)$, then $V = \mathbb{R}^r$ or $\mathbb{C}^r$ and $\eta$ has a Riemannian or Hermitian structure, denoted by $\langle\cdot, \cdot\rangle$, which is invariant under $G$.

Let $T^*M$ be the cotangent bundle of $M$ and $\Lambda^p(M)$ the $p$-form bundles on $M$ such that $T^*M = \Lambda^1(M)$. We have the associated bundle $\eta \otimes \Lambda^p(M)$. Let $\Omega^p(\eta) = \Gamma(\eta \otimes \Lambda^p(M))$, in particular $\Gamma(\eta) = \Omega^0(\eta)$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $Ad : G \to GL(\mathfrak{g})$ be the adjoint representation and $ad\,\eta := P \times_{Ad} \mathfrak{g} \to M$ the associated vector bundle. Let $\Omega^p(Ad\,\eta) = \Gamma(Ad\,\eta \otimes \Lambda^p(M))$. All connections here are considered compatible with the structure group $G$. For any $x \in M$, $\pi^{-1}(x) \subset \eta$ is the fibre on $x$. The gauge group $\mathcal{G}$ of $\eta$ consists of maps $S : \eta \to \eta$ keeping fibres and satisfying, for any two $u$, $v \in \Omega^0(\eta)$, $\langle Su, Sv\rangle = \langle u, v\rangle$. Thus $\mathcal{G}$ is the set of sections of the bundle $\text{Aut}(\eta)$. Each fibre of $\text{Aut}(\eta)$ is isomorphic to $G$. Hence with the usual $\exp : \mathfrak{g} \to G$ we have $\exp : \Omega^0(Ad\,\eta) \to \mathcal{G}$.

If $S \in \mathcal{G}$, then $S(x)$ is an orthonormal or unitary matrix. A $p$-form $a \in \Omega^p(Ad\,\eta)$ has values in anti-symmetric or anti-hermitian matrix. Thus $a$ can be written as

$$a(x) = \sum A_{i_1, \cdots, i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where $A_{i_1,\cdots,i_k}$'s are entries of an anti-symmetric or anti-hermitian matrix.

If $D:\Omega^0(\eta)\to\Omega^1(\eta)$ is a Riemannian or unitary connection of $\eta$, i.e.,

$$d\langle u,v\rangle=\langle Du,\,v\rangle+\langle u,\,Dv\rangle\,,\quad \forall u,\,v\in\Omega^0(\eta),$$

then it induces $D:\Omega^{p-1}(\eta)\to\Omega^p(\eta)$, for any $p>0$, by combination of $D$ on sections of $\eta$ and the exterior differential on forms. There is another extension of $D$, $\nabla:\Omega^{p-1}(\eta)\to\Gamma(\eta\otimes\Lambda^{p-1}(M)\otimes T^*M)$, by using $D$ on the sections of $\eta$ and the Levi-Civita connection of $(M,g)$ on the forms. Let $P:\Gamma(\eta\otimes\Lambda^{p-1}(M)\otimes T^*M)\to\Gamma(\eta\otimes\Lambda^p(M))$ be the projection with $D=P\circ\nabla$. The connection $D:\Omega^0(\eta)\to\Omega^1(\eta)$ also induces operators $D:\Omega^p(Ad\,\eta)\to\Omega^{p+1}(Ad\,\eta)$, for $p\geq 0$, defined by $Da=D\circ a-(-1)^pa\circ D,\quad \forall a\in\Omega^p(Ad\,\eta)$, which means that for any $u\in\Gamma(\eta)$, $(Da)u=D(au)-(-1)^pa\wedge Du$. Similarly there is a

$$\nabla:\Omega^p(Ad\,\eta)\to\Gamma\left(\Lambda^p(M)\otimes T^*M\otimes ad\,\eta\right)\quad\text{for }\ p\geq 0,$$

such that $D=P\circ\nabla$.

Using the Killing form of $\mathfrak{g}$ we can define inner product on $\Omega^p(Ad\,\eta)$. Locally, let

$$b(x)=\sum B_{i_1,\cdots,i_p}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_p}.$$

Then the inner product is equivalent to

$$\langle a,b\rangle=-\sum \mathrm{Trace}(A_{i_1,\cdots,i_p}B^*_{j_1,\cdots,j_p})\left\langle dx^{i_1}\wedge\cdots\wedge dx^{i_p},dx^{j_1}\wedge\cdots\wedge dx^{j_p}\right\rangle,$$

where $B^*$ is the transpose or conjugate transpose of matrix depending on whether $G=SO(r)$ or $SU(r)$ and $\left\langle dx^{i_1}\wedge\cdots\wedge dx^{i_p},dx^{j_1}\wedge\cdots\wedge dx^{j_p}\right\rangle$ is the inner product induced by $g$. Then we can check that for any $\phi,\,\psi\in\Omega^p(\eta)$ or $\Omega^p(Ad\,\eta)$, $d\langle\phi,\,\psi\rangle=\langle\nabla\phi,\,\psi\rangle+\langle\phi,\,\nabla\psi\rangle$. Then we have an inner product $(a,b):=\int_M\langle a,b\rangle$ for $\Omega^p(\eta)$ or $\Omega^p(Ad\,\eta)$. Then we can define $L^2\left(\Omega^p(\eta)\right)$ or $L^2\left(\Omega^p(Ad\,\eta)\right)$ as the completion of $C^\infty$ sections under these $L^2$ norms. Similarly, we can define the Sobolev spaces $H^{l,2}\left(\Omega^p(Ad\,\eta)\right)$ and $H^{l,2}\left(\Omega^p(\eta)\right)$, $l\geq 0$, as the completions of $C^\infty$ sections under the norms $\|a\|_{H^{l,2}}:=\Big(\sum\limits_{i=0}^{l}\|\nabla^{(i)}a\|^2_{L^2}\Big)^{1/2}$.

Let $\eta^*$ be the dual bundle of $\eta$ with the induced Riemannian or Hermitian structure. We see that $ad\,\eta\subset\mathrm{End}(\eta)\cong\eta\otimes\eta^*$. Note the fibrewise inner product for $\Omega^p(Ad\,\eta)$ is also defined on $\Gamma(\eta\otimes\eta^*\otimes\Lambda^p(M))$.

For any $v\in\Omega^0(\eta)$ we define $v^*\in\Omega^0(\eta^*)$ by $v^*(u)=\langle u,v\rangle$ for any $u\in\Omega^0(\eta)$. Let $v\in\Omega^p(\eta)$ and $v=\sum v_i\otimes\omega^i$, $\omega_i\in\Lambda^p(M)$, $v_i\in\Omega^0(\eta)$. Then we define $v^*=\sum v_i^*\otimes\omega^i\in\Omega^p(\eta^*)$. Let $a\in\Omega^1(Ad\,\eta)$, $b\in\Omega^1(\eta)$ and $u\in\Omega^0(\eta)$. There is an identity

$$\langle au,\,b\rangle=\langle a,\,b\otimes u^*\rangle\,. \tag{2.1}$$

Let $*:\Lambda^p(M)\to\Lambda^{n-p}(M)$ be the star operator of $(M,g)$. Then $**=(-1)^{p(n-p)}$ and

$$\langle\alpha,\beta\rangle\,dM=\alpha\wedge*\beta,\quad\forall\alpha,\ \beta\in\Lambda^p(M),\ p\geq 0.$$

Since $n=4$, we have

$$D^*=-*\circ D\circ *:\Omega^p(\eta)\to\Omega^{p-1}(\eta)\quad(D^*:\Omega^p(Ad\,\eta)\to\Omega^{p-1}(Ad\,\eta))$$

is the dual of $D$ respectively, i.e., $(Da,b)=(a,D^*b)$ for all $a\in\Omega^{p-1}(\eta)$, $b\in\Omega^p(\eta)$ or $a\in\Omega^{p-1}(Ad\,\eta)$, $b\in\Omega^p(Ad\,\eta)$.

**Lemma 2.1.**

$$J=J(u,D)=\frac{1}{2}\left(Du\otimes u^*-u\otimes(Du)^*\right)\in\Omega^1(Ad\,\eta). \tag{2.2}$$

**Proof.** Let $(D,u)$ be a stationary point of the YMH-functional (1.1) and $a\in\Omega^1(\eta)$. An anti-symmetric or anti-hermitian matrix $a$ is perpendicular to symmetric or hermitian matrices, i.e. $\langle a, Du\otimes u^* + u\otimes (Du)^*\rangle \equiv 0$ on $M$. Thus by (2.1),

$$\frac{d}{dt}\mathrm{YMH}(u, D+ta)\Big|_{t=0} = 2\Re(D^*F + \frac{1}{2}\left(Du\otimes u^* - u\otimes (Du)^*\right), a) = 0.$$

Since $a$ is arbitrary, we have proved (2.2).

Note that if locally $Du = D_i u dx_i$, then $Du\wedge (Du)^* = D_i u\otimes (D_j u)^* dx_i\wedge dx_j$. Let $v\in \Omega^0(\eta)$, $A$, $B\in \Omega^1(\eta)$, $A = A_i dx_i$, $B = B_i dx_i$. By definition $\langle v, B\rangle = \langle v, B_i\rangle\, dx_i$, $\langle A, B\rangle = \langle A_i, B_j\rangle\, dx_i\wedge dx_j$, where the $\langle\,,\rangle$ is the inner product of $\eta$.

**Lemma 2.2.**

$$DJ = \frac{1}{2}\left(Fu\otimes u^* - u\otimes (Fu)^* - 2(Du)\wedge (Du)^*\right), \tag{2.3}$$

$$D^*J = \frac{1}{2}\left(D^*Du\otimes u^* - u\otimes (D^*Du)^*\right). \tag{2.4}$$

**Proof.** Since $J\in \Omega^1(Ad\,\eta)\subset \Omega^1(\eta\otimes\eta^*)$ and $DJ = D\circ J + J\circ D$, we obtain

$$\begin{aligned}(D(Du\otimes u^*))\,v &= D(Du\,\langle v,u\rangle) + Du\wedge\langle Dv, u\rangle \\ &= Fu\,\langle v,u\rangle - Du\wedge\langle v, Du\rangle = (Fu\otimes u^* - Du\wedge (Du)^*)\,v,\end{aligned}$$

$$(D(u\otimes (Du)^*))\,v = (Du\wedge (Du)^* + u\otimes (Fu)^*)\,v.$$

Since $v$ is arbitrary, (2.3) is true. Similarly, using $D^* = -*\circ D\circ *$ and $a\wedge (*b)^* = -(*a)\wedge b^*$ for any $a$, $b\in\Omega^1(\eta)$, we can prove that

$$\begin{aligned}D^*J &= \frac{1}{2}\left(D^*Du\otimes u^* - u\otimes (D^*Du)^*\right) + \frac{1}{2}*\left[Du\wedge (*Du)^* + (*Du)\wedge (Du)^*\right] \\ &= \frac{1}{2}\left(D^*Du\otimes u^* - u\otimes (D^*Du)^*\right).\end{aligned}$$

Each connection $D$ defines a Hodge Laplacian $\triangle = D^*D + DD^*$ both on $\Omega^p(Ad\,\eta)$ and on $\Omega^p(\eta)$, $p\geq 0$. There is another rough (or crude) Laplacian $\nabla^*\nabla$. The relation between these operators are the Weitzenböck formula

$$\nabla^*\nabla\phi = \triangle\phi + F\#\phi + Rm\#\phi, \tag{2.5}$$

where $\phi\in\Omega^p(Ad\,\eta)$ or $\Omega^p(\eta)$, $F = F(D)$ and $Rm$ is the Riemannian curvature on $(M,g)$ (see for instance [11, Appendix]). Here and in the following, # denotes any multi-linear map with smooth coefficients.

## §3. Basic Estimates

First we make a convention: if the structure group of $\eta$ is $SU(r)$, then we take the inner product as $\Re\langle\cdot,\cdot\rangle$ and still write it as $\langle\cdot,\cdot\rangle$. It is a real inner product and defines the same norm as original one. The advantage is that we do not need to distinguish between the real and the complex cases. Note that our Euler-Lagrange equation (1.2) does not change under this convention.

**Lemma 3.1.** *Let $D(t)$ and $u(t)$ be weak solutions to* (1.4) *on* $[0,T]\times M$. *Assume that* $|u(0)|\leq m$ a.e. *on $M$ for a constant $m>0$. Then for all $t\in[0,T]$,*

$$|u(t,\cdot)|\leq \max\{m,1\}\quad \text{a.e.on}\quad M.$$

**Proof.** Without loss of generality, assume that $m\geq 1$. Define

$$\phi(t,x) = 1 - \frac{\min\{m, |u(t,x)|\}}{|u(t,x)|}\geq 0. \tag{3.1}$$

Then

$$d\phi = \begin{cases} 0, & \text{on } \{|u(t,\cdot)|\leq m\}, \\ m|u|^{-3}\langle u, Du\rangle, & \text{on } \{|u(t,\cdot)| > m\}. \end{cases} \tag{3.2}$$

Note that since $u \in \Omega^0(\eta)$, $Du \in \Omega^1(\eta)$ and $\langle u, Du\rangle \in \Lambda^1(M)$ is a 1-form. Choosing $\phi^2 u$ as the test function in (1.7) and using (3.2), we obtain

$$\begin{aligned}\Big(\frac{\partial u}{\partial t}, \phi^2 u\Big) &= -\big(Du, u\otimes d\phi^2\big) - \big(Du, \phi^2 Du\big) + \frac{\lambda}{2}\int_M \phi^2(1-|u|^2)|u|^2\\ &= -2m\int_M \phi|u|^{-3}\left\langle Du, \langle u, Du\rangle\otimes u\right\rangle - \int_M \phi^2|Du|^2 + \frac{\lambda}{2}\int_M \phi^2(1-|u|^2)|u|^2. \end{aligned}\tag{3.3}$$

Using (1.4), we have

$$\frac{\partial\phi}{\partial t} = \begin{cases} 0, & \text{on } \{|u(t,\cdot)|\le m\},\\ -m|u|^{-3}\left\langle D^*Du, u\right\rangle + \frac{\lambda}{2}m|u|^{-1}(1-|u|^2), & \text{on } \{|u(t,\cdot)|> m\},\end{cases}$$

and thus

$$\begin{aligned}\int_M |u|^2\frac{\partial\phi^2}{\partial t} &= -2m\left(D^*Du, \phi|u|^{-1}u\right) + \int_M \lambda m\phi|u|(1-|u|^2)\\ &= 2m\int_M |u|^{-3}\phi\left\langle Du, \langle u, Du\rangle\otimes u\right\rangle - 2m^2\int_{\{|u|>m\}} |u|^{-4}\left\langle Du, \langle Du, u\rangle\otimes u\right\rangle\\ &\quad - 2m\int_M \phi|u|^{-1}|Du|^2 + \int_M \lambda m\phi|u|(1-|u|^2). \end{aligned}\tag{3.4}$$

Note that

$$\frac{\partial}{\partial t}\Big(u, \phi^2 u\Big) = 2\Big(\frac{\partial u}{\partial t}, \phi^2 u\Big) + \int_M |u|^2\frac{\partial\phi^2}{\partial t}.\tag{3.5}$$

From (3.3), (3.4) and (3.5) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_M |u|^2\phi^2 &= \Big(\frac{\partial u}{\partial t}, \phi^2 u\Big) + \frac{1}{2}\int_M |u|^2\frac{\partial\phi^2}{\partial t}\\ &\le -m\int_M \phi|u|^{-3}(1+m|u|^{-1})|\left\langle Du, u\right\rangle|^2\\ &\quad - \int_M (\phi^2 + m\phi|u|^{-1})|Du|^2 + \frac{\lambda}{2}\int_M (\phi|u|+m)\phi|u|(1-|u|^2) \le 0.\end{aligned}$$

Here we have used the fact that $\left\langle Du, \langle Du, u\rangle\otimes u\right\rangle = |\left\langle Du, u\right\rangle|^2$.

Thus the function

$$f_m(t) = \int_M |u(t,x)|^2\phi(t,x)^2 = \int_{|u(t,x)|>m} (|u(t,x)|-m)^2 \ge 0$$

is a decreasing function of $t$. Since $f_m(0)=0$, we know that $f_m\equiv 0$. This is equivalent to saying that $\{|u(t,\cdot)|>m\}$ has measure zero, or $|u(t,\cdot)|\le m$ a.e. on $M$.

Since $F(D+ta) = (D+ta)\circ(D+ta) = F + tDa + t^2 a\wedge a$ for any $a\in\Omega^1(Ad\,\eta)$, we have

$$\frac{dF(D+\epsilon a)}{d\epsilon}\Big|_{\epsilon=0} = Da.$$

In particular, taking $a = \partial D/\partial t$ and using (1.4), we have

$$\frac{dF(D(t))}{dt} = \frac{dF(D+\epsilon\partial D/\partial t)}{d\epsilon}\Big|_{\epsilon=0} = D\frac{\partial D}{\partial t} = -D(D^*F+J).\tag{3.6}$$

**Lemma 3.2.** *If $u$ and $D$ are weak solutions to* (1.4) *on* $[0,T]\times M$, *then*

$$\mathrm{YMH}(u(T), D(T)) + 2\int_0^T\int_M \Big(\Big|\frac{\partial u}{\partial t}\Big|^2 + |D^*F+J|^2\Big) = \mathrm{YMH}(u(0), D(0)).\tag{3.7}$$

**Proof.** By (1.4) and (2.1), note that for any $a \in \Omega^1(Ad\,\eta)$,

$$\langle au, Du\rangle = \langle a, Du \otimes u^*\rangle = \langle a, J(u, D)\rangle\,. \tag{3.8}$$

We have

$$\begin{aligned}\int_M \left|\frac{\partial u}{\partial t}\right|^2 &= -\Big(\frac{\partial u}{\partial t},\, D^*Du - \frac{\lambda}{2}\left(1-|u|^2\right)u\Big)\\ &= \Big(\frac{\partial D}{\partial t}u, Du\Big) - \Big(\frac{\partial (Du)}{\partial t}, Du\Big) - \frac{\lambda}{8}\int_M \frac{\partial}{\partial t}\left(1-|u|^2\right)^2\\ &= -\frac{1}{2}\frac{d}{dt}\int_M \Big[|Du|^2 + \frac{\lambda}{4}\left(1-|u|^2\right)^2\Big] - (D^*F + J, J)\,.\end{aligned}$$

Applying (3.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_M |F|^2 = \Big(\frac{\partial F}{\partial t}, F\Big) = -\left(D(D^*F+J), F\right) = -(D^*F + J, D^*F).$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_M \Big[|F|^2 + |Du|^2 + \frac{\lambda}{4}\left(1-|u|^2\right)^2\Big] + \int_M \left|\frac{\partial u}{\partial t}\right|^2 = -\int_M |D^*F + J|^2.$$

Integrating on $[0, T]$ gives (3.7).

**Lemma 3.3.** *Let $u$ and $D$ be weak solutions to* (1.4) *on $[0,T]\times M$. Then for any $x_0 \in M$ and geodesic ball $B_{2R}(x_0)$, there is a constant $C = C(M)$ such that*

$$\begin{aligned}&\sup_{0\le t\le T}\int_{B_R(x_0)} |F(t,x)|^2 + |Du(t,x)|^2 + \frac{\lambda}{4}\left(1-|u(t,x)|^2\right)^2\\ \le &\int_{B_{2R}(x_0)} |F_0|^2 + |D_0u_0|^2 + \frac{\lambda}{4}\left(1-|u_0|^2\right)^2 + CTR^{-2}[\mathrm{YMH}(u(0), D(0))].\end{aligned}$$

**Proof.** Let $\phi$ be a cut off function with support inside $B_{2R}(x_0)$ and $\phi \equiv 1$ on $B_R(x_0)$. We can arrange that $0 \le \phi \le 1$ and $|d\phi| \le C_1R^{-1}$, where $C_1$ only depends on $M$. By (1.4), (2.2) and (3.8), we have

$$\begin{aligned}&\int_M \phi^2\Big|\frac{\partial u}{\partial t}\Big|^2 = -\Big(\phi^2 D\frac{\partial u}{\partial t}, Du\Big) - \Big(d\phi^2 \otimes \frac{\partial u}{\partial t}, Du\Big) + \frac{\lambda}{2}\Big(\phi^2\frac{\partial u}{\partial t}, \left(1-|u|^2\right)u\Big)\\ =&\Big(\phi^2\frac{\partial D}{\partial t}u, Du\Big) - \Big(\phi^2\frac{\partial (Du)}{\partial t}, Du\Big) - \Big(d\phi^2\otimes\frac{\partial u}{\partial t}, Du\Big) - \frac{\lambda}{8}\int_M \phi^2\frac{\partial}{\partial t}\left(1-|u|^2\right)^2\\ =&-\frac{1}{2}\frac{d}{dt}\int_M \phi^2\Big[|Du|^2 + \frac{\lambda}{4}\left(1-|u|^2\right)^2\Big] - (\phi^2(D^*F+J), J) - \Big(d\phi^2\otimes\frac{\partial u}{\partial t}, Du\Big).\end{aligned}$$

Using (3.6) again, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_M \phi^2|F|^2 &= \Big(\phi^2\frac{\partial F}{\partial t}, F\Big) = -\Big(\phi^2 D(D^*F+J), F\Big)\\ &= -\left(\phi^2(D^*F+J), D^*F\right) + \left(d\phi^2\wedge(D^*F+J), F\right).\end{aligned}$$

Note that

$$\begin{aligned}-\Big(d\phi^2\otimes\frac{\partial u}{\partial t}, Du\Big) &\le 2\int_M \phi|d\phi|\Big|\frac{\partial u}{\partial t}\Big||Du| \le \int_M \phi^2\Big|\frac{\partial u}{\partial t}\Big|^2 + |d\phi|^2|Du|^2,\\ \left(d\phi^2\wedge(D^*F+J), F\right) &\le 2\int_M \phi|d\phi||D^*F+J||F| \le \int_M \phi^2|D^*F+J|^2 + |d\phi|^2|F|^2.\end{aligned}$$

We finally obtain

$$\frac{1}{2}\frac{d}{dt}\int_M \phi^2\Big[|F|^2+|Du|^2+\frac{\lambda}{4}\left(1-|u|^2\right)^2\Big]+\int_M \phi^2\left|\frac{\partial u}{\partial t}\right|^2+\int_M \phi^2|D^*F+J|^2$$
$$\leq \int_M \phi^2\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|D^*F+J|^2\Big)+CR^{-2}\mathrm{YMH}(u,D).$$

Integrate on $[0,t]$ for $0<t\leq T$. The result follows from Lemma 3.2.

**Lemma 3.4.** *There exists constants $K$, $R_0$ depending only on $M$ such that for any $R\in]0,R_0]$ there exists a cover of $M$ by geodesic balls $B_{R/2}(x_i)$ with the property that at any point $x\in M$ at most $K$ of the balls $B_R(x_i)$ meet.*

**Proof.** For the proof of this lemma, we refer to [15].

**Lemma 3.5.** *Let $D=D_{\mathrm{ref}}+A$ with curvature $F=F(D)$. Then there exist constants $C=C(\eta)$ and $\delta=\delta(\eta)>0$ such that for $a\in\Omega^p(Ad\,\eta)$, and $B_R$, the geodesic ball of radius $R$ and centred at $x_0\in M$,*

$$\|a\|^2_{H^{1,2}(B_{R/2})}\leq C(\|Da\|^2_{L^2(B_R)}+\|D^*a\|^2_{L^2(B_R)}+(1+R^{-2})\|a\|^2_{L^2(B_R)}),$$

*provided*

$$\int_{B_R}|F|^2\,dM<\delta. \tag{3.9}$$

*Moreover if there is an $R>0$ such that (3.9) is true for any $x\in M$, then*

$$\|a\|^2_{H^{1,2}}\leq C\left(\|Da\|^2_{L^2}+\|D^*a\|^2_{L^2}+(1+R^{-2})\|a\|^2_{L^2}\right).$$

*The same result also holds for $u\in\Omega^p(\eta)$.*

**Proof.** Let $\phi\in C_0^\infty(B_R)$ be a cut-off function with $0\leq\phi\leq 1$, $|d\phi|\leq C(M)R^{-1}$, and $\phi=1$ inside $B_{R/2}$. Then by (2.5), we find

$$\|\nabla a\|^2_{L^2(B_{R/2})}\leq\|\nabla(\phi a)\|^2_{L^2}=(\nabla^*\nabla(\phi a),\phi a)$$
$$=\|D(\phi a)\|^2_{L^2}+\|D^*(\phi a)\|^2_{L^2}+(F\#\phi a,\phi a)+(Rm\#\phi a,\phi a).$$

By Hölder and Sobolev inequalities, we find

$$(F\#\phi a,\phi a)\leq C_2(\eta)\|F\|_{L^2(B_R)}\|\phi a\|^2_{L^4}\leq\delta^{1/2}C_1(\eta)C_2(\eta)\|\phi a\|^2_{H^{1,2}}.$$

Thus

$$\|a\|^2_{H^{1,2}(B_{R/2})}\leq C_3\Big(\|Da\|^2_{L^2(B_R)}+\|D^*a\|^2_{L^2(B_R)}+(1+R^{-2})\|a\|^2_{L^2(B_R)}\Big),$$

by choosing $\delta^{1/2}<\frac{1}{2C_1(\eta)C_2(\eta)}$. This proves our local claim. With the help of Lemma 3.4, the global version is also true by choosing

$$\delta^{1/2}<\frac{1}{2KC_1(\eta)C_2(\eta)}.$$

The proof for $u\in\Omega^p(\eta)$ is the same as above.

**Lemma 3.6.** *Let $D$ and $u$ be weak solutions to (1.4) on $[0,T]\times M$. Suppose $u(0)$ is bounded. Then there exists a constant $\delta=\delta(\eta)>0$ such that*

$$D^*F+J\in L^2_{\mathrm{loc}}\left((0,T];H^{1,2}\left(\Omega^1(Ad\,\eta)\right)\right),\quad \frac{\partial}{\partial t}F\in L^2_{\mathrm{loc}}\left((0,T];L^2\left(\Omega^2(Ad\,\eta)\right)\right),$$
$$D^*Du\in L^2_{\mathrm{loc}}\left((0,T];H^{1,2}\left(\Omega^0(\eta)\right)\right),\quad D\frac{\partial u}{\partial t}\in L^2_{\mathrm{loc}}\left((0,T];L^2\left(\Omega^1(\eta)\right)\right),$$

*provided for some $R>0$,*

$$\sup_{0<t<T}\int_{B_R(x)}\left(|Du|^2+|F|^2\right)<\delta,\quad\forall x\in M. \tag{3.10}$$

**Proof.** Using (1.4) and (3.6), we have

$$\begin{aligned}
&\Big\|\frac{\partial F}{\partial t}\Big\|^2_{L^2}+\|D(D^*F+J)\|^2_{L^2}\\
&=-2\Big(D^*F+J,\,D^*\Big(\frac{\partial F}{\partial t}\Big)\Big)\\
&=-2\Big(D^*F+J,\,\frac{\partial D}{\partial t}\#F-\frac{\partial J}{\partial t}+\frac{\partial}{\partial t}\,(D^*F+J)\Big)\\
&\le-\frac{d}{dt}\,\|D^*F+J\|^2_{L^2}+C\int_M|D^*F+J|\Big(|D^*F+J||F|+\Big|\frac{\partial J}{\partial t}\Big|\Big).
\end{aligned}$$

Using (2.2) we get

$$\frac{\partial J}{\partial t}=(D^*F+J)\#u\#u+D\frac{\partial u}{\partial t}\#u+\frac{\partial u}{\partial t}\#Du.$$

By Lemma 3.1, $u(t)$ is bounded, so

$$\begin{aligned}
&\Big\|\frac{\partial}{\partial t}F\Big\|^2_{L^2}+\|D(D^*F+J)\|^2_{L^2}\\
&\le-\frac{d}{dt}\,\|D^*F+J\|^2_{L^2}+C\int_M(|F|+|Du|)\Big(|D^*F+J|^2+\Big|\frac{\partial u}{\partial t}\Big|^2\Big)\\
&\quad+C\|D^*F+J\|^2_{L^2}+\frac{1}{4}\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}.
\end{aligned}\tag{3.11}$$

By Lemma 3.1 and (2.2) we have

$$\begin{aligned}
|d|u|^2|&=|2\,\langle Du,u\rangle\,|\le2|u||Du|\le C|Du|,\\
|J|^2&\le C|Du|^2.
\end{aligned}$$

Using (1.4) and Lemma 3.1, we have

$$\begin{aligned}
&\|DD^*Du\|^2_{L^2}+\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}\\
&=\Big(D\frac{\partial u}{\partial t},\,\frac{\lambda}{2}D\left[u(1-|u|^2)\right]-DD^*Du\Big)-\Big(D\frac{\partial u}{\partial t}-\frac{\lambda}{2}D\left[u(1-|u|^2)\right],\,DD^*Du\Big)\\
&\le-2\Big(D^*\frac{\partial(Du)}{\partial t},\,D^*Du\Big)+C\int_M(|Du|+|D^*F+J|)\Big(|DD^*Du|+\Big|D\frac{\partial u}{\partial t}\Big|\Big)\\
&\le-\frac{d}{dt}\|D^*Du\|^2_{L^2}+\frac{1}{2}\|DD^*Du\|^2_{L^2}+\frac{1}{2}\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}\\
&\quad+C\int_M|Du|(|D^*F+J|^2+|D^*Du|^2)+C\|D^*F+J\|^2_{L^2}+C\|Du\|^2_{L^2}.
\end{aligned}$$

Above and (3.11) yield

$$\begin{aligned}
&\frac{1}{2}\Big(\Big\|\frac{\partial}{\partial t}F\Big\|^2_{L^2}+\|D(D^*F+J)\|^2_{L^2}+\|DD^*Du\|^2_{L^2}+\frac{1}{2}\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}\Big)\\
&\le-\frac{d}{dt}\left(\|D^*Du\|^2_{L^2}+\|D^*F+J\|^2_{L^2}\right)+C\|D^*F+J\|^2_{L^2}+C\|Du\|^2_{L^2}\\
&\quad+C\int_M(|F|+|Du|)\Big(|D^*F+J|^2+|D^*Du|^2+\Big|\frac{\partial u}{\partial t}\Big|^2\Big).
\end{aligned}$$

On the other hand, $D^*D^*Du=0$ and by Lemma 3.1 and (2.3),

$$|D^*J|^2\le C|D^*Du|^2.$$

Notting that $D^*D^*F=0$ and applying Lemmas 3.4, 3.5, and Sobolev inequality we obtain

$$\begin{aligned}&\int_M (|F|+|Du|)\Big(|D^*F+J|^2+\Big|\frac{\partial u}{\partial t}\Big|^2+|D^*Du|^2\Big)\\ \le&\sum_i\int_{B_R(x_i)}(|F|+|Du|)\left(|D^*F+J|^2+\Big|\frac{\partial u}{\partial t}\Big|^2+|D^*Du|^2\right)\\ \le&C\sup_{x\in M}\Big(\int_{B_R(x)}(|Du|^2+|F|^2)\Big)^{\frac12}\sum_i\Big(\int_{B_R(x_i)}\Big(|D^*F+J|^4+|D^*Du|^4+\Big|\frac{\partial u}{\partial t}\Big|^4\Big)\Big)^{\frac12}\\ \le& C\delta^{1/2}K\Big(\|D(D^*F+J)\|^2_{L^2}+\|DD^*Du\|^2_{L^2}+\|D\frac{\partial u}{\partial t}\|^2_{L^2}\Big)\\ &+C\delta^{1/2}K(1+R^{-2})\Big(\|D^*F+J\|^2_{L^2}+\|D^*Du\|^2_{L^2}+\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2}\Big).\end{aligned}$$

By choosing $\delta^{1/2}CK<\frac18$ we get

$$\begin{aligned}&\frac14\Big(\Big\|\frac{\partial}{\partial t}F\Big\|^2_{L^2}+\|D(D^*F+J)\|^2_{L^2}+\|DD^*Du\|^2_{L^2}+\frac12\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}\Big)\\ \le&-\frac{d}{dt}\left(\|D^*Du\|^2_{L^2}+\|D^*F+J\|^2_{L^2}\right)\\ &+C\Big(\|D^*F+J\|^2_{L^2}+\|Du\|^2_{L^2}+\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2}+\|D^*Du\|^2_{L^2}\Big).\end{aligned}\tag{3.12}$$

Now given $\tau>0$ we can find $t_0\in[0,\tau]$ such that

$$\|D^*F+J\|^2_{L^2}(t_0)\le 2\tau^{-1}\int_0^\tau\|D^*F+J\|^2_{L^2}\,dt\le\tau^{-1}\mathrm{YMH}(D(0),u(0)),$$

$$\|D^*Du\|^2_{L^2}(t_0)\le 2\tau^{-1}\int_0^\tau\|D^*Du\|^2_{L^2}\,dt\le 2\tau^{-1}\Big(\int_0^T\int_M\Big|\frac{\partial u}{\partial t}\Big|^2dt+CT\tau\Big).$$

By Lemmas 3.1 and 3.2,

$$\int_0^T\Big(\|D^*F+J\|^2_{L^2}+\|Du\|^2_{L^2}+\Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^4}+\|F\|^2_{L^2}+\|D^*Du\|^2_{L^2}\Big)\,dt<\infty.$$

Integrating both sides of (3.12) from $t_0$ to $T$ gives

$$\int_\tau^T\Big(\|D(D^*F+J)\|^2_{L^2}+\Big\|D\frac{\partial u}{\partial t}\Big\|^2_{L^2}+\Big\|\frac{\partial}{\partial t}F\Big\|^2_{L^2}+\|DD^*Du\|^2_{L^2}\Big)\,dt\le C(\tau,T).$$

Since $\tau$ is arbitrarily small, the last inequality and Lemma 3.5 prove Lemma 3.6.

**Lemma 3.7.** *Let $D=D_{\mathrm{ref}}+A$ and $u$ be weak solutions of* (1.4) *in the sense of Theorem* A (i). *If $u(0)$ is bounded and* (3.10) *is true, then there exist $A(T)\in H^{1,2}\left(\Omega^1(Ad\,\eta)\right)$ and $u(T)\in H^{1,2}\left(\Omega^0(\eta)\right)$ such that as $t\nearrow T$,*

$$A(t)\to A(T)\ in\ H^{1,2}\left(\Omega^1(Ad\,\eta)\right),\qquad u(t)\to u(T)\ in\ H^{1,2}\left(\Omega^0(\eta)\right).$$

**Proof.** By Lemma 3.6, we have $\frac{\partial A}{\partial t}\in L^2_{\mathrm{loc}}\left((0,T];H^{1,2}\left(\Omega^1(Ad\,\eta)\right)\right)$, whence $A(t)\to A(T)$ in $L^4\left(\Omega^1(Ad\,\eta)\right)$ for some $A(T)\in L^4\left(\Omega^1(Ad\,\eta)\right)$ as $t\nearrow T$. In fact, by Fubini's theorem, $A(t,x)$ is absolutely continuous for almost all $x\in M$, thus

$$A(t_1,x)-A(t_2,x)=\int_{t_2}^{t_1}\frac{\partial A}{\partial t}(t,x)dt$$

for almost all $x \in M$. Then for any $0 < s \le t_2 < t_1 \le T$,

$$\|A(t_1) - A(t_2)\|_{L^4} = \Big\| \int_{t_2}^{t_1} \frac{\partial A}{\partial t}(t)dt \Big\|_{L^4} \le C \Big\| \int_{t_2}^{t_1} \frac{\partial A}{\partial t}(t)dt \Big\|_{H^{1,2}}$$
$$\le C|t_1 - t_2|^{\frac{1}{2}} \Big( \int_s^T \int_M \Big|\frac{\partial A}{\partial t}(t)\Big|^2 + \int_s^T \int_M \Big|\nabla_{\rm ref} \frac{\partial A}{\partial t}\Big|^2 \Big)^{\frac{1}{2}} \le C(s,T)|t_1 - t_2|^{\frac{1}{2}}.$$

The last inequality comes from Lemma 3.6.

Thus there exists an $A(T) \in L^4\big(\Omega^1(Ad\,\eta)\big)$ such that $A(t) \to A(T)$ in $L^4\big(\Omega^1(Ad\,\eta)\big)$, as $t \nearrow T$. By Lemma 3.6 we see that

$$\frac{\partial}{\partial t}(D_{\rm ref}A) = \frac{\partial}{\partial t}\left(F(D) - F(D_{\rm ref}) - A\wedge A\right) = \frac{\partial}{\partial t}\left(F(D) - A\wedge A\right)$$
$$= \frac{\partial}{\partial t}F + \frac{\partial A}{\partial t} \# A \in L^2_{loc}\left((0,T]; L^2\left(\Omega^2(Ad\,\eta)\right)\right),$$

and $D_{\rm ref}A(t)$ converges in $L^2\big(\Omega^2(Ad\,\eta)\big)$ as $t \nearrow T$. Moreover, by $D^*D^*F = 0$, (2.3), Lemmas 3.1 and 3.6,

$$\frac{\partial}{\partial t}(D^*_{\rm ref}A) = D^*_{\rm ref}\Big(\frac{\partial A}{\partial t}\Big) = A\#(D^*F + J) - D^*J \in L^2_{\rm loc}\left((0,T]; L^2\left(\Omega^2(Ad\,\eta)\right)\right)$$

and $D^*_{\rm ref}A(t)$ converges in $L^2\big(\Omega^0(Ad\,\eta)\big)$ as $t \nearrow T$. By Lemma 3.5, $A(t)$ converges in $H^{1,2}\big(\Omega^1(Ad\,\eta)\big)$. By the uniqueness of limit, the $A(T) \in H^{1,2}\big(\Omega^1(Ad\,\eta)\big)$ and $A(t) \to A(T)$ in $H^{1,2}\big(\Omega^1(Ad\,\eta)\big)$ as $t \nearrow T$.

Since $u$ is bounded, we only need check that $Du(t)$ converges in $L^2$.

$$\|Du(t_1) - Du(t_2)\|_{L^2} \le \Big\| \int_{t_2}^{t_1} \frac{\partial A}{\partial t}(t)u(t)dt \Big\|_{L^2} + \Big\| \int_{t_2}^{t_1} A(t)\frac{\partial u}{\partial t}(t)dt \Big\|_{L^2}$$
$$\le C|t_1 - t_2|^{\frac{1}{2}} \Big( \int_{t_2}^{t_1} \int_M \Big|\frac{\partial A}{\partial t}(t)\Big|^2 dt\Big)^{\frac{1}{2}}$$
$$+ 2C|t_1 - t_2|^{\frac{1}{2}} \Big( \int_{t_2}^{t_1} \Big( \Big( \int_M |A(t)|^4 \Big)^{\frac{1}{2}} + \Big( \int_M \Big|\frac{\partial u}{\partial t}(t)\Big|^4 \Big)^{\frac{1}{2}} \Big) dt \Big)^{\frac{1}{2}}.$$

Again by Lemma 3.6 and Sobolev embedding, $u(t) \to u(T)$ in $H^{1,2}\big(\Omega^0(\eta)\big)$ as $t \nearrow T$.

## §4. The Proof of Theorem A

### 4.1. The Smooth Case

We consider a scheme of a version of De Turck's trick that gives solutions to (1.4) and (1.5) when $u_0$ and $D_0 = D_{\rm ref} + A_0$ are smooth.

Let $a(t,\cdot) \in \Omega^1(Ad\,\eta)$ and $\bar D(t,\cdot) = D_0 + a(t,\cdot)$. We solve the initial value problems for $a$ and $\bar u$ as follows

$$\begin{cases} \dfrac{\partial \bar D}{\partial t} = \dfrac{\partial a}{\partial t} = -\bar D^* \bar F - \dfrac{1}{2}\left(\bar D\bar u \otimes \bar u^* - \bar u \otimes (\bar D \bar u)^*\right) + \bar D(-\bar D^* a), \\ \dfrac{\partial \bar u}{\partial t} = -\bar D^* \bar D \bar u + \dfrac{\lambda}{2}\bar u(1 - |\bar u|^2) + (\bar D^* a)\bar u \end{cases} \tag{4.1}$$

with initial values $a(0,\cdot) = 0$ and $\bar u(0,\cdot) = u_0$. Since

$$\bar F = (D_0 + a)\circ(D_0 + a) = F(D_0) + D_0 a + a\wedge a = F(D_0) + \bar D a - a\wedge a$$

and $\bar{D}^*\bar{u} = 0$, (4.1) can be written as

$$\begin{cases} \dfrac{\partial a}{\partial t} + \bar{\triangle} a = -\bar{D}^* F(D_0) + \bar{D}^*(a\wedge a) - \dfrac{1}{2}\left(\bar{D}\bar{u}\otimes\bar{u}^* - \bar{u}\otimes(\bar{D}\bar{u})^*\right), \\ \dfrac{\partial \bar{u}}{\partial t} + \bar{\triangle}\bar{u} = \dfrac{\lambda}{2}\bar{u}(1-|\bar{u}|^2) + (\bar{D}^* a)\bar{u} \end{cases} \tag{4.2}$$

with initial values $a(0,\cdot) = 0$ and $\bar{u}(0,\cdot) = u_0$. The system (4.2) is a perturbation of the standard heat equation. We see that if $D_0$ and $u_0$ are smooth, for small $T > 0$, there are unique smooth $a$ and $\bar{u}$ satisfying (4.2) on $[0,T]\times M$. Now let $S(t) \in C^\infty(\mathcal{G})$ be the unique smooth solution to the linear initial value problem:

$$S^{-1}\circ\frac{dS}{dt} = -\bar{D}^* a, \qquad S(0) = id. \tag{4.3}$$

Let $D = (S^{-1})^*\bar{D} = S\circ\bar{D}\circ S^{-1}$, $u = S\bar{u}$. Since $u^* = (S\bar{u})^* = \bar{u}^* S^{-1}$ and $D^* = S\circ\bar{D}^*\circ S^{-1}$, we have

$$\begin{aligned} &S\circ\bar{D}^*\bar{F}\circ S^{-1} = S\circ\bar{D}^*\circ S^{-1}\circ S\bar{F}\circ S^{-1} = D^* F, \\ &S\circ\left(\bar{D}\bar{u}\otimes\bar{u}^* - \bar{u}\otimes(\bar{D}\bar{u})^*\right)\circ S^{-1} = Du\otimes u^* - u\otimes(Du)^*, \\ &\bar{D}(-\bar{D}^* a) = -\bar{D}\circ(\bar{D}^* a) + (\bar{D}^* a)\circ\bar{D} = -\Big(S^{-1}\circ\frac{dS}{dt}\circ\bar{D} + \bar{D}\circ\frac{dS^{-1}}{dt}\circ S\Big). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial D}{\partial t} &= S\circ\Big(S^{-1}\circ\frac{dS}{dt}\circ\bar{D} + \bar{D}\circ\frac{dS^{-1}}{dt}\circ S\Big)\circ S^{-1} \\ &\quad - S\circ\left(\bar{D}^*\bar{F} + \frac{1}{2}\Big(\bar{D}\bar{u}\otimes\bar{u}^* - \bar{u}\otimes(\bar{D}\bar{u})^*\Big)\right)\circ S^{-1} + S\circ(\bar{D}(-\bar{D}^* a))\circ S^{-1} \\ &= -D^* F - \frac{1}{2}\left(Du\otimes u^* - u\otimes(Du)^*\right). \end{aligned}$$

Moreover, since $|u| = |S\bar{u}| = |\bar{u}|$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{dS}{dt}\bar{u} - S\circ\bar{D}^*\circ S^{-1}\circ S\bar{D}\circ S^{-1}\circ S\bar{u} + \frac{\lambda}{2}u(1-|u|^2) - \frac{dS}{dt}\bar{u} \\ &= -D^* Du + \frac{\lambda}{2}u(1-|u|^2). \end{aligned}$$

Of course $D(0) = D_0 + a(0) = D_0$, $u(0) = S(0)\bar{u}(0) = u_0$. We get the unique smooth solutions to (1.4) and (1.5) on $[0,T]\times M$.

**4.2. Proof of Local Existence**

When $u_0$ and $A_0$ are meerly in $H^{1,2}$, then the above method would not work since the solution $a$ is only in $H^{1,2}$ and thus $\bar{D}^* a$ is only in $L^2$. This leads to that the solution $S$ of (4.3) is only measurable. Then $S\circ\bar{D}\circ S^{-1}$ is not necessarily an $H^{1,2}$ connection.

We use a method of Struwe[17] to round about this difficulty.

Since $A_0$ is in $H^{1,2}$, we can select smooth $A_1 \in \Omega^1(Ad\,\eta)$ such that $B_0 = A_0 - A_1 \in H^{1,2}\left(\Omega^1(Ad\,\eta)\right)$ has $H^{1,2}$ norm as small as we please. We can also make $u_0 = u_1 + v_0$, where $u_1$ is smooth and $v_0$ has $H^{1,2}$ norm as small as we please.

Let $D_1 = D_{\text{ref}} + A_1$ and $\triangle_1 = D_1 D_1^* + D_1^* D_1$. We solve the heat equation

$$\begin{cases} \dfrac{\partial A_{bg}}{\partial t} + \triangle_1 A_{bg} = 0, \\ \dfrac{\partial u_{bg}}{\partial t} + \triangle_1 u_{bg} = 0, \end{cases} \tag{4.4}$$

with initian values $A_{bg}(0) = B_0$ and $u_{bg}(0) = v_0$.

By general theory of PDE, for some $T>0$ there is a unique pair of solutions,

$$A_{bg}\in C^0\left([0,\infty);H^{1,2}(\Omega^1(Ad\,\eta))\right)\cap L^2\left([0,\infty);H^{2,2}(\Omega^1(Ad\,\eta))\right),$$
$$u_{bg}\in C^0\left([0,\infty);H^{1,2}(\Omega^0(\eta))\right)\cap L^2\left([0,\infty);H^{2,2}(\Omega^0(\eta))\right),$$

such that they are smooth for $t>0$. In particular, there exist constants $C=C(\eta)$ and $T_1=T(\eta,D_1)$ such that on the interval $[0,T_1]$ we have

$$\|A_{bg}\|_{L^\infty(H^{1,2})}\le C\|B_0\|_{H^{1,2}},\qquad \|u_{bg}\|_{L^\infty(H^{1,2})}\le C\|v_0\|_{H^{1,2}}. \tag{4.5}$$

Using the $A_{bg}$ and $u_{bg}$ we can have data which are smooth for $t>0$. We first establish solution $(a,\bar u)$ to (4.1) which is smooth for $0<t<T$ for some $T>0$.

We define $D_{bg}=D_1+A_{bg}$, $\bar D=D_{bg}+a$, and define $\bar u=u_1+u_{bg}+v$ such that $a$ and $\bar u$ satisfy the equations in (4.2).

Using $Da=da+A\wedge a-(-1)^p a\wedge A$ and $D^*a=-*D*a$ for $D=d+A$ and $a\in\Omega^p(Ad\,\eta)$, we can compute that

$$\begin{aligned}\triangle_A a=&\triangle a-*d*(A\wedge a+a\wedge A)-d*[A\wedge *a+(*a)\wedge A]\\&-*\{A\wedge *(da+A\wedge a+a\wedge A)-[*(da+A\wedge a+a\wedge A)]\wedge A\}\\&-A*[d*a+A\wedge *a+(*a)\wedge A]+\{*[d*a+A\wedge *a+(*a)\wedge A]\}A,\end{aligned}\tag{4.6}$$

where $\triangle a=-*d*da-d*d*a$. In particular, we have

$$\bar\triangle a=\triangle_1 a+\nabla_1 A_{bg}\#a+\nabla_1 a\#A_{bg}+\nabla_1 a\#a+A_{bg}\#A_{bg}\#a+A_{bg}\#a\#a+a\#a\#a.$$

By the above notation, (4.2) is equivalent to

$$\begin{cases}\dfrac{\partial a}{\partial t}+\triangle_1 a=f_1+f_2+g_1(a,\nabla_1 a)+g_2(a,v,D_1 v),\\[2mm]\dfrac{\partial v}{\partial t}+\triangle_1 v=h_1+h_2(a,v,\nabla_1 a,D_1 v),\end{cases}\tag{4.7}$$

with initial values $a(0)=0$ and $v(0)=0$, where

$$\begin{aligned}f_1=&-\frac{\partial A_{bg}}{\partial t}-D_1^*F_{bg}+A_{bg}\#F_{bg},\\f_2=&D_1u_2\#u_2+A_{bg}\#u_2\#u_2,\\g_1(a,\nabla_1 a)=&F_{bg}\#a+A_{bg}\#\nabla_1 a+\nabla_1 A_{bg}\#a+A_{bg}\#A_{bg}\#a\\&+\nabla_1 a\#a+A_{bg}\#a\#a+a\#a\#a,\\g_2(a,v,D_1v)=&D_1v\#u_2+D_1u_2\#v+D_1v\#v\\&+a\#u_2\#u_2+a\#v\#u_2+A_{bg}\#u_2\#v+a\#v\#v,\\h_1=&D_1u_2\#v+\nabla_1A_{bg}\#u_2+A_{bg}\#A_{bg}\#u_2+u_2\#u_2\#u_2-\triangle_1u_1,\\h_2(a,v,\nabla_1 a,D_1v)=&D_1v\#A_{bg}+\nabla_1A_{bg}\#v+D_1v\#a+\nabla_1a\#v+D_1u_2\#a\\&+\nabla_1a\#u_2+A_{bg}\#A_{bg}\#v+A_{bg}\#a\#v+A_{bg}\#a\#u_2\\&+a\#a\#u_2+a\#a\#v+u_2\#u_2\#v+u_2\#v\#v+v\#v\#v,\end{aligned}$$

for $u_2=u_1+u_{bg}\in L^2(H^{2,2})\cap C^0(H^{1,2})\cap H^{1,2}(L^2)$ and $F_{bg}=F(D_{bg})$.

As in [17], we introduce the following spaces:

$$V=V_T(\Omega^p(Ad\,\eta))=L^2\left([0,T];H^{2,2}(\Omega^p(Ad\,\eta))\right)\cap H^{1,2}\left([0,T];L^2(\Omega^p\,(Ad\,\eta))\right),\ p\ge 0,$$
$$W=W_T\left(\Omega^p(\eta)\right)=L^2\left([0,T];H^{2,2}\left(\Omega^p(\eta)\right)\right)\cap H^{1,2}\left([0,T];L^2\left(\Omega^p(\eta)\right)\right),\quad p\ge 0.$$

The norms in these spaces are defined as

$$\|\phi\|_V^2:=\left\|\frac{d}{dt}\phi\right\|^2_{L^{2,2}}+\|\phi\|^2_{L^2(H^{2,2})},$$

similarly for $W_T$. Here we denote the space-time $L^p - L^q$-norms

$$\|\phi\|_{L^{q,p}} = \Big( \int_0^T \|\phi\|_{L^p}^q dt \Big)^{1/q}, \quad 1 \le p,\, q < \infty,$$

etc. $V_T$ and $W_T$ are continuously embedded in

$$L^\infty \left([0,T]; H^{1,2} \left(\Omega^p(Ad\,\eta)\right)\right), \quad L^\infty \left([0,T]; H^{1,2} \left(\Omega^p(\eta)\right)\right),$$

and as pointed in [17], with

$$\sup_{0\le t\le T} \|\phi\|_{H^{1,2}}^2 \le \|\phi(0)\|_{H^{1,2}}^2 + 2\|\phi\|_V^2, \tag{4.8}$$

$$\sup_{0\le t\le T} \|u\|_{H^{1,2}}^2 \le \|u(0)\|_{H^{1,2}}^2 + 2\|u\|_V^2. \tag{4.9}$$

We will use Lemmas 3.1–3.3 in [17]. They are also true for $u \in \Omega^p(\eta)$ since the Weitzenböck formula (2.5) also holds in this case.

Since $a(0) = 0$ and $v(0) = 0$, choosing suitable $B_0$ and $v_0$ and $T > 0$ small enough we can make $\|a\|_{L^\infty(H^{1,2})} + \|v\|_{L^\infty(H^{1,2})} + \|B_0\|_{H^{1,2}} + \|v_0\|_{H^{1,2}}$ small enough. Then we can find $0 < \theta < 1$ such that

$$\|g_1\|_{L^{2,2}} \le \theta \|a\|_{V_T}, \qquad \|g_2\|_{L^{2,2}} \le \theta\|v\|_{W_T}, \qquad \|h_2\|_{L^{2,2}} \le \theta \left(\|a\|_{V_T} + \|v\|_{W_T}\right).$$

The estimates are similar to that in [17]. For example, since $|u_0|$ is bounded and $u_2(t)$ is smooth for $t > 0$, for $0 \le t \le T$ there is $0 < N(t) < \infty$ such that $|u_2(t)| \le N(t)$ almost everywhere on $M$. Then writing $V$ and $W$ for $V_T$ and $W_T$, we have

$$\begin{aligned}\|D_1 v \# u_2\|_{L^{2,2}} &= \Big( \int_0^T \int_M |D_1 v \# u_2|^2 dM dt \Big)^{1/2} \\ &\le C \Big( \int_0^T N^2(t) dt \Big)^{1/2} \|v\|_{L^\infty(H^{1,2})} \le C\epsilon(T) \|v\|_W.\end{aligned}$$

We can shrink $T$ to make $\epsilon(T)$ as small as we please. Similarly,

$$\|D_1 u_2 \# v\|_{L^{2,2}} \le C \Big( \int_0^T \|u_2\|_{H^{2,2}}^2 dt \Big)^{1/2} \|v\|_{L^{\infty,4}} \le C \|u_2\|_{L^2(H^{2,2})} \|v\|_W.$$

Again when $T > 0$ small, we can make $\|u_2\|_{L^2(H^{2,2})}$ small.

By Lemma 3.2 of [17] and its analogue for $v$, we have

$$\|a\|_V \le C_2 \Big\| \frac{da}{dt} + \triangle_1 a \Big\|_{L^{2,2}} \le C_1 \|f_1 + f_2\|_{L^{2,2}} + \theta \left(\|a\|_V + \|v\|_W\right), \tag{4.10}$$

$$\|v\|_V \le C_2 \Big\| \frac{dv}{dt} + \triangle_1 v \Big\|_{L^{2,2}} \le C_1 \|h_1\|_{L^{2,2}} + \theta \left(\|a\|_V + \|v\|_W\right). \tag{4.11}$$

Select $T = T(\epsilon) > 0$ so small that

$$C_1(\|f_1 + f_2\|_{L^{2,2}} + \|h_1\|_{L^{2,2}}) < (1-\theta)\epsilon.$$

Fix these $T$ and $\epsilon$. Let

$$U_\epsilon^T := \{(a,v) \in V_T \times W_T \mid \|a\|_{V_T} + \|v\|_{W_T} < \epsilon\}. \tag{4.12}$$

Given any $(a,v) \in U_\epsilon^T$, there is a unique weak solution $(b,w)$ to the equations

$$\begin{cases} \dfrac{\partial b}{\partial t} + \triangle_1 b = f_1 + g_1(a, \nabla_1 a) + f_2 + g_2(a, v, D_1 v), \\ \dfrac{\partial w}{\partial t} + \triangle_1 w = h_1 + h_2(a, v, \nabla_1 a, D_1 v) \end{cases}$$

with initial values $b(0) = 0$ and $w(0) = 0$. Write $(b,w) = L(a,v)$. Then (4.10), (4.11), and (4.12) give us $\|b\|_{V_T} + \|w\|_{W_T} = \|L(a,v)\|_{V_T \times W_T} < \epsilon$. Thus $L : U_\epsilon^T \to U_\epsilon^T$.

Now since the terms in $g_1(a, \nabla_1 a)$, $g_2(a, v, D_1 v)$, and $h_2(a, v, \nabla_1 a, D_1 v)$ are either linear or multiplications of at most order 3 of $(a, v)$, or at most order 2 of $(a, v, \nabla_1 a, D_1 v)$, with similar estimates and may replace smaller $T > 0$ and $\epsilon > 0$, we have

$$\|L(a, v) - L(c, z)\|_{V_T \times W_T} \le \theta \|(a - c, v - z)\|_{V_T \times W_T}, \quad 0 < \theta < 1.$$

Thus by contraction mapping theorem on $U_\epsilon^T \subset V_T \times W_T$ we have unique weak solution $(a, \bar{u}) \in V_T \times W_T$ of (4.7). By general theory of quasi-linear parabolic equations, $(a, \bar{u})(t)$ is smooth for $t > 0$ since $f_1 + f_2$ and $h_1$ are smooth for $t > 0$ (for example, see [10]).

Choose $t_k \searrow 0$ and solve the equation for $S_k(t) \in C^\infty(CalG)$, $t > 0$,

$$S_k^{-1} \circ \frac{dS_k}{dt} = -\bar{D}^* a, \qquad S_k(t_k) = id. \tag{4.13}$$

Clearly, $S_k = S_l^{-1}(t_k) \circ S_l$. Let $D_k = (S_k^{-1})^* \bar{D} = [S_l(t_k)]^* D_l$ be the corresponding connections. For each $k$, $D_k(t) = [S_k^{-1}(t)]^* \bar{D}(t)$ and $u_k(t) = S_k(t)\bar{u}(t)$ are smooth for $0 < t \le T$ and are classical solutions to (1.4).

We first construct the $\hat{D}$ and $\hat{u}$ mentioned in Theorem A (i).

Since $dS_k/dt = -S_k \bar{D}^* a$ and $\bar{D}^* a \in L^2(H^{1,2})$, we have

$$\Big\| \frac{d}{dt} S_k \Big\|_{L^{2,4}}^2 = \|\bar{D}^* a\|_{L^{2,4}}^2 \le C \|\bar{D}^* a\|_{L^2(H^{1,2})}^2.$$

Let $|M|$ be the volume of $(M, g)$. Since $S_k$ is smooth for $t > 0$, for $0 < t_1 < t_2 < T$, when $t_2 - t_1$ is so small that

$$\int_{t_1}^{t_2} \Big| \frac{dS_k}{dt}(t) \Big|^2 dt = \int_{t_1}^{t_2} |\bar{D}^* a(t)|^2 dt < 1,$$

we have

$$\begin{aligned}\int_M |S_k(t_2) - S_k(t_1)|^4 &= \int_M \Big| \int_{t_1}^{t_2} \frac{dS_k}{dt}(t) dt \Big|^4 \le \int_M (t_2 - t_1)^2 \left( \int_{t_1}^{t_2} \Big| \frac{dS_k}{dt}(t) \Big|^2 dt \right)^2 \\ &\le C(t_2 - t_1)^2 |M|^{\frac{1}{2}} \|\bar{D}^* a\|_{L^2(H^{1,2})}^2.\end{aligned}$$

Thus, $S_k(0) = \lim_{t \searrow 0} S_k(t) \in L^4$ exists for any $k$.

Let $D_k(t) = D_{ref} + A_k(t)$. Then $|F_k(0)| = |S_k(0) F_0 S_k(0)^{-1}| = |F_0|$, $|u_k(0)| = |S_k(0) u_0| = |u_0|$, and $|D_k(0) u_k(0)| = |S_k(0) D_0 u_0| = |D_0 u_0|$. Since $D_k$ and $u_k$ satisfy (1.4), by Lemma 3.2, we have

$$\Big\| \frac{\partial A_k}{\partial t} \Big\|_{L^{2,2}}^2 = \|D_k^* F_k + J_k\|_{L^{2,2}}^2 \le \mathrm{YMH}(u_k(0), D_k(0)) = \mathrm{YMH}(u_0, D_0). \tag{4.14}$$

Note that by Lemma 3.2 and (4.14),

$$\sup_t \mathrm{YMH}(u_k(t), D_k(t)) = \mathrm{YMH}(u_k(0), D_k(0)) = \mathrm{YMH}(u_0, D_0) \tag{4.15}$$

for any $k$. Thus

$$\int_M |A_k(t_2) - A_k(t_1)|^2 \le (t_2 - t_1) \mathrm{YMH}(u_0, D_0).$$

For each $k$ the limit $A_k(0) = \lim_{t \searrow 0} A_k(t)$ exists in $L^2$. Moreover, since $S_k(t_k) = id$, $A_k(t_k) = \bar{A}(t_k) \to A_0$ in $H^{1,2}$, $\lim_{k \to \infty} D_k(0) = D_0$ in $L^2$.

By Lemma 3.2 and (4.15),

$$\Big\| \frac{\partial u_l}{\partial t} \Big\|_{L^{2,2}}^2 \le \mathrm{YMH}(u_l(0), D_l(0)) = \mathrm{YMH}(u_0, D_0),$$

whence $u_l(0) = \lim_{t\searrow 0} u_l(t)$ in $L^2$ exists for any $l$. Fix some $l = \hat{l}$ and let $\hat{S} = S_{\hat{l}}$, $\hat{D} = D_{\hat{l}}$, $\hat{D}_0 = \hat{D}(0)$, $\hat{u} = u_{\hat{l}}$, $\hat{S}_k = \hat{S}(t_k)$. $\hat{D}$ and $\hat{u}$ satisy (1.4). Moreover,

$$\hat{S}_k^*(\hat{D})(t_k) = \bar{D}(t_k) \to D_0, \qquad \hat{S}_k^{-1}\hat{u} = \bar{u}(t_k) \to u_0 \text{ in } H^{1,2}\left([0,T);L^2\right).$$

Since $S_k(t) = S_{\hat{l}}(t_k)^{-1}S_{\hat{l}}(t) = \hat{S}_k^{-1}\hat{S}(t)$,

$$\lim_{k\to\infty} \hat{S}_k^*(\hat{D}_0) = \lim_{k\to\infty} D_k(0) = D_0 \text{ in } L^2, \quad \lim_{t\searrow 0} \hat{S}_k^{-1}\hat{u}(t) = \lim_{t\searrow 0} u_k(t) = u_0 \text{ in } L^2.$$

We next prove that $\hat{S}_k \to \hat{S}_0$ in $H^{1,2}$. If we let $\hat{D}_0 = D_{\rm ref} + \hat{A}_0$ with $\hat{A}_0 \in H^{1,2}(\Omega^1(Ad\,\eta))$, we find

$$\hat{S}_k^*(\hat{D}_0) - \hat{D}_0 = \hat{S}_k^{-1} \circ (D_{\rm ref}\hat{S}_k) + \hat{S}_k^{-1} \circ \hat{A}_0 \circ \hat{S}_k - \hat{A}_0 \to A_0 - \hat{A}_0 \quad \text{in} \quad L^2.$$

Thus

$$\lim_{k\to\infty} D_{\rm ref}\hat{S}_k = \lim_{k\to\infty} (\hat{S}_k A_0 - \hat{A}_0\hat{S}_k) \quad \text{in} \quad L^2$$

exists and necessarily coincides with the distributional limit $D_{\rm ref}\hat{S}_0$; that is, $\hat{S}_k \to \hat{S}_0$ in $H^{1,2}$. But this implies that $D_k = \hat{S}_k^*(\hat{D})$ converges uniformly to some $D = \hat{S}_0^*(\hat{D}) \in C^0(L^2)$ with $D(0) = D_0$ and $\partial D/\partial t \in L^{2,2}$ by (4.12). Then we have $u(t) = \hat{S}_0^{-1}\hat{u}(t) = \lim_{k\to\infty} \hat{S}_k^{-1}\hat{u}(t)$ in $L^2$ for any $t > 0$ and $u(0) = u_0$. Moreover,

$$Du(t) = \hat{S}_0^{-1} \circ \hat{S}(t) \circ \bar{D}\bar{u}(t) = \lim_{k\to\infty} \hat{S}_k^{-1} \circ \hat{S}(t) \circ \bar{D}\bar{u}(t) \ \text{ in } \ L^2,$$

for $0 < t \le T$, since $\bar{u} \in H^{1,2}$. Thus $u \in C^0\left([0,T); H^{1,2}\left(\Omega^0(\eta)\right)\right)$.

It only remains to prove that $F(D) \in C^0\left([0,T); L^2\left(\Omega^2(Ad\,\eta)\right)\right)$. It is clear that $F(D_k) = \hat{S}_k^*(F(\hat{D}))$ converges in $L^2$, locally uniformly for $t > 0$, as $k \to \infty$. Since $D_k \to D$ in $C^0(L^2)$, $F(D_k) \to F(D)$ in the sense of distributions. Together, these results imply $F(D_k) \to F(D)$ in $C^0((0,T]; L^2(\Omega^2(Ad\,\eta)))$.

Let $D(t) = D_{\rm ref} + A(t)$. Since $A \in C\left([0,T]; L^2\left(\Omega^1(Ad\,\eta)\right)\right)$ and $u$ is bounded and is in $L^2\left([0,T]; H^{1,2}\left(\Omega^0(\eta)\right)\right)$, we know that $J(D,u)$ is uniformly bounded in $L^2\left(\Omega^1(\eta)\right)$. By (4.12) and $D \in C^0((0,T]; L^2)$, we also obtain that $F(D(t))$ converges to $F(D_0)$ weakly in $L^2$ as $t \to 0$.

Note that since $(a,v) \in V_T \times W_T$, $\bar{D}\bar{u} \in C^0\left([0,T]; L^2(\Omega^1(\eta))\right)$, etc., we have

$$\begin{aligned}\|Du(t)\|_{L^2}^2 + \frac{\lambda}{4}\left\|1 - |u(t)|^2\right\|_{L^2}^2 &= \left\|\hat{S}_0^{-1}\hat{S}(t)\bar{D}\bar{u}(t)\right\|_{L^2}^2 + \frac{\lambda}{4}\left\|1 - |\hat{S}_0\hat{u}(t)|^2\right\|_{L^2}^2 \\ &\to \|D_0u_0\|_{L^2}^2 + \frac{\lambda}{4}\left\|1 - |u_0|^2\right\|_{L^2}^2 \quad \text{as } t \to 0.\end{aligned} \tag{4.16}$$

Finally by (4.15), (4.16), and Lemma 3.2, we have

$$\limsup_{t\to 0} \|F(D(t))\|_{L^2}^2 \le \|F(D_0)\|_{L^2}^2.$$

We obtain that $F(D(t)) \to F(D_0)$ in $L^2$ as $t \to 0$; that is, $F(D) \in C^0([0,T]; L^2)$. Hence $D$ and $u$ in fact are weak solutions to (1.4) and (1.5) in the sense of Theorem A (i).

### 4.3. Proof of Local Uniqueness

First note that as pointed out in [17] the uniqueness of $s = S^{-1}dS/dt$ and so $S$ by solving (4.3) depends on whether or not $D$ is irreducible. A form of describing irreducibility is (1.8). As pointed out in [17], if $D \in H^{1,2}$ satisfies (1.8) for $C = C(D)$, then in an $H^{1,2}$ neighbourhood of $D$ there is a $C$ such that (1.8) is true.

The idea in [17] of proving the uniqueness is to fix a global analogue of Uhlenbeck's theorem[18−19] on the existence of local Coulomb gauges, depending smoothly on the connection.

Let $D_0$ be a connection of class $H^{1,2}$ satisfying (1.8), and let $D_{bg} = D_1 + A_{bg}$, $0 \le t \le T$, be a family of background connections such that $D_{bg}(0) = D_0$, $A_{bg} \in C^\infty$ for $t > 0$ and $A_{bg} \in L^2(H^{2,2}) \cap H^{1,2}(L^2)$, as determined by (4.4). As in [17], we have

**Proposition 4.1.** *Let* $(D, u)$ *be a weak solution to* (1.4) *and* (1.5) *on* $[0,T] \times M$ *as in Theorem* A (i). *There exist* $T_0 > 0$ *and a family of gauge transformations*

$$S = S(t) \in C^0\left([0,T_0]; H^{1,2}(\mathcal{G})\right)$$

*with*

$$s = S^{-1} \circ \frac{dS}{dt} \in L^2\left([0,T]; H^{1,2}\left(\Omega^0(Ad\,\eta)\right)\right), \quad S(0) = id,$$

*such that* $\bar{D} = S^*(D) = D_{bg} + \bar{a}$ *satisfies*

$$\bar{a} \in L^\infty\left([0,T_0]; H^{1,2}\left(\Omega^1(Ad\,\eta)\right)\right) \cap H^{1,2}\left([0,T_0]; L^2\left(\Omega^1(Ad\,\eta)\right)\right),$$

$\bar{a}(t) \to 0$ *in* $H^{1,2}$ *as* $t \to 0$, *and* $\bar{D}^*\bar{a} = 0$.

The proof of Proposition 4.1 is almost the same as the proof of Proposition 5.2 in [17]. The only difference is in the proof of Claim 4 of Lemma 5.3 in [17]. Instead of the estimate

$$\mathrm{I} := \frac{1}{2}\left\|F_{\bar{a}} - F_{bg}\right\|^2_{L^{\infty,2}} + \left\|D_{\bar{a}}(F_{\bar{a}} - F_{bg})\right\|^2_{L^{2,2}} + \left\|D^*_{\bar{a}}(F_{\bar{a}} - F_{bg})\right\|^2_{L^{2,2}} \le \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

we have $\mathrm{I} \le \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}$, where

$$\begin{aligned}\mathrm{V} &= -\int_0^{T_0} (D_{\bar{a}} J_{\bar{a}}, F_{\bar{a}} - F_{bg})\, dt \le \int_0^{T_0} \|J_{\bar{a}}\|_{L^2} \|D^*_{\bar{a}}(F_{\bar{a}} - F_{bg})\|_{L^2} dt \\ &\le \frac{1}{2}\|J_{\bar{a}}\|^2_{L^{2,2}} + \frac{1}{2}\|D^*_{\bar{a}}(F_{\bar{a}} - F_{bg})\|^2_{L^{2,2}} = \frac{1}{2}\|J_a\|^2_{L^{2,2}} + \frac{1}{2}\|D^*_{\bar{a}}(F_{\bar{a}} - F_{bg})\|^2_{L^{2,2}}.\end{aligned}$$

Here we have used the fact that

$$J_a = \frac{1}{2}\left(Du \otimes u^* - u \otimes (Du)^*\right) \in L^2\left([0,T]; L^2\left(\Omega^1(Ad\,\eta)\right)\right),$$

$$J_{\bar{a}} = S^{-1} \circ J_a \circ S, \quad \text{and} \quad \|J_{\bar{a}}\|^2_{L^{2,2}} = \|J_a\|^2_{L^{2,2}}.$$

By the estimates of II, III, and IV in [17], we have

$$\frac{1}{2}\left\|F_{\bar{a}} - F_{bg}\right\|^2_{L^{\infty,2}} + \left\|D_{\bar{a}}(F_{\bar{a}} - F_{bg})\right\|^2_{L^{2,2}} + \frac{1}{2}\left\|D^*_{\bar{a}}(F_{\bar{a}} - F_{bg})\right\|^2_{L^{2,2}} \le C\epsilon(T). \tag{4.17}$$

The final proof of uniqueness then is also a slightly modified version of [17].

Given $D_0 \in H^{1,2}$, a family of background connections and sections $D_{bg}$, $u_{bg}$ as in (4.4), let $D_a = D_{bg} + a$ and $u$ be a pair of local weak solutions to (1.4) and (1.5) and $S^* D_a = D_{\bar{a}} = D_{bg} + \bar{a}$ and $\bar{u} = S^{-1}u = u_1 + u_{bg} + v$ the corresponding family of normalized connections according to Proposition 4.1.

Let $s = S^{-1} \circ dS/dt$. It is easy to see that $D_{\bar{a}}$ and $\bar{u}$ weakly solves the problem:

$$\frac{d}{dt} D_{\bar{a}} = -D^*_{\bar{a}} F_{\bar{a}} - \frac{1}{2}\left(D_{\bar{a}}\bar{u} \otimes \bar{u}^* - \bar{u} \otimes (D_{\bar{a}}\bar{u})^*\right) + D_{\bar{a}} s, \tag{4.18}$$

$$D^*_{\bar{a}} \bar{a} = 0, \tag{4.19}$$

$$\bar{a}(0) = 0, \tag{4.20}$$

$$\frac{d\bar{u}}{dt} = -D^*_{\bar{a}} D_{\bar{a}} \bar{u} + \frac{\lambda}{2}\bar{u}\left(1 - |\bar{u}|^2\right) - s\bar{u}, \tag{4.21}$$

$$\bar{u}(0) = S^{-1}(0) u_0, \tag{4.22}$$

where $F_{\bar a} = F(D_{\bar a})$, and

$$\begin{cases} \bar a \in L^\infty\left([0,T]; H^{1,2}\left(\Omega^1(Ad\,\eta)\right)\right) \cap H^{1,2}\left([0,T]; L^2\left(\Omega^1(Ad\,\eta)\right)\right), \\ \bar u \in L^\infty\left([0,T]; H^{1,2}\left(\Omega^0(\eta)\right)\right) \cap H^{1,2}\left([0,T]; L^2\left(\Omega^0(\eta)\right)\right), \\ F_{\bar a} \in C^0\left([0,T]; L^2\left(\Omega^2(Ad\,\eta)\right)\right), \quad s \in L^2\left([0,T]; H^{1,2}\left(\Omega^0(Ad\,\eta)\right)\right), \end{cases} \tag{4.23}$$

on some interval $[0,T]$. The $\bar a$ and $\bar u$ attain their initial data in the $H^{1,2}$-sense. The following result, analogous to Proposition 6.1 of [17], shows that—provided $D_0$ is irreducible—the solution $D_{\bar a}$, $\bar u$ and $s$ above is unique.

**Proposition 4.2.** *For any $D_0 \in H^{1,2}$ satisfying* (1.8)*, $u_0 \in H^{1,2}$, there exists $T > 0$ and a unique solution $(\bar a, \bar u, s)$ of* (4.18)–(4.22) *on $[0,T] \times M$ satisfying* (4.23).

*In addition, $\bar a \in L^2(H^{2,2})$, $\bar u \in L^2(H^{2,2})$, and $\bar a$, $\bar u$ and $s$ are smooth for $t > 0$. Finally, if $D_0$ is smooth, $\bar a$, $\bar u$ and $s$ are smooth up to $t = 0$.*

**Proof.** Our proof is almost exactly the proof of Proposition 6.1 in [17], hence we only point out the difference.

(i) The existence has been shown.

(ii) Estimates for $s$: From (4.17) and $D^*D^*F = 0$ we have

$$D^*_{\bar a} D_{\bar a} s = D^*_{\bar a}\Big(\frac{d}{dt} D_{\bar a}\Big) + D^*_{\bar a} J_{\bar a} = D^*_{\bar a}\Big(\frac{d}{dt} A_{bg}\Big) + \bar a \# \frac{d\bar a}{dt} + D^*_{\bar a} J_{\bar a}.$$

Using (1.8) to estimate as in [17], we see that the final estimate for $\|s\|_{L^2(H^{1,2})}$ then is

$$\|s\|^2_{L^2(H^{1,2})} \le C\Big\|\frac{d}{dt} A_{bg}\Big\|^2_{L^{2,2}} + C\|\bar a\|^2_{L^\infty(H^{1,2})}\Big\|\frac{d}{dt}\bar a\Big\|^2_{L^{2,2}} + C\,\|J_{\bar a}\|^2_{L^{2,2}}$$

with an extra term

$$C\,\|J_{\bar a}\|_{L^{2,2}} \le \theta\left(\|\bar a\|_{V_T} + \|v\|_{W_T}\right) + C_1\|f\|_{L^{2,2}}$$

if $\|B_0\|_{H^{1,2}} + \|v_0\|_{H^{1,2}} + \|\bar a\|_{L^\infty(H^{1,2})} + \|\bar v\|_{L^\infty(H^{1,2})}$ is small enough. The estimate is similar to the estimates in the proof of local existence. Here $f \in L^{2,2}$ is a function not depending on $(\bar a, v, s)$. Hence we have

$$\|s\|_{L^2(H^{1,2})} \le C\Big(\Big\|\frac{d}{dt} A_{bg}\Big\|_{L^{2,2}} + \|f\|_{L^{2,2}}\Big) + \theta(\|\bar a\|_{V_T} + \|v\|_{W_T}).$$

(iii) Estimates for $\bar a$, $v$: We observe that using (4.17) we can write the equations as

$$\frac{d\bar a}{dt} + \triangle_1 \bar a = f + g_1(a, \nabla_1 a) + g_2(a, v, \nabla_1 a, D_1 v) + D_{\bar a} s, \tag{4.18'}$$

$$D^*_{\bar a}\bar a = 0, \tag{4.19'}$$

$$\bar a(0) = 0, \tag{4.20'}$$

$$\frac{dv}{dt} + \triangle_1 v = h_1 + h_2(a, v, \nabla_1 a, D_1 v) - s\bar u, \tag{4.21'}$$

$$v(0) = 0. \tag{4.22'}$$

Let $L(\bar a, v, s) = (\bar b, w)$ be the unique weak solution to

$$\begin{cases} \dfrac{d\bar b}{dt} + \triangle_1 \bar b = f + g_1(a, \nabla_1 a) + g_2(a, v, \nabla_1 a, D_1 v) + D_{\bar a} s, \\ \dfrac{dw}{dt} + \triangle_1 w = h_1 + h_2(a, v, \nabla_1 a, D_1 v) - s\bar u \end{cases}$$

with initial values $\bar b(0) = 0$ and $w(0) = 0$.

Using the estimate for $\|s\|_{L^2(H^{1,2})}$, we have similar estimates for $(\bar b, w)$ as in the proof of local existence.

(iv) Estimates for differences: Now suppose $L(\bar a_1, v_1, s_1) = (\bar a_1, v_1)$, $L(\bar a_2, v_2, s_2) = (\bar a_2, v_2)$ are two solutions to (4.18')–(4.22'). Then

$$\|(\bar a_1 - \bar a_2, v_1 - v_2)\|_{V_T\times W_T} \le \theta_1(\|\bar a_1 - \bar a_2\|_{V_T} + \|v_1 - v_2\|_{W_T}) + C\|D_{\bar a_1}s_1 - D_{\bar a_2}s_2\|_{L^{2,2}} + C\|(s_1 - s_2)v_1\|_{L^{2,2}} + C\|s_2(v_1 - v_2)\|_{L^{2,2}}. \tag{4.24}$$

By shrinking $T > 0$, we can make $\theta_1$ as small as we please. Since

$$D_{\bar a_1}s_1 - D_{\bar a_2}s_2 = D_1(s_1 - s_2) + A_{bg}(s_1 - s_2) - (s_1 - s_2)A_{bg} + (\bar a_1 - \bar a_2)s_1 + \bar a_2(s_1 - s_2) - (s_1 - s_2)\bar a_1 + s_2(\bar a_2 - \bar a_1),$$
$$\|D_{\bar a_1}s_1 - D_{\bar a_2}s_2\|_{L^{2,2}} \le C\|s_1 - s_2\|_{L^2(H^{1,2})}. \tag{4.25}$$

Exactly as in the of proof of Proposition 6.1 of [17], "Estimate for $\sigma$", we have

$$\begin{aligned}\|s_1 - s_2\|^2_{L^2(H^{1,2})} &\le C\epsilon\Big(\Big\|\frac{d(\bar a_1 - \bar a_2)}{dt}\Big\|^2_{L^{2,2}} + \|\bar a_1 - \bar a_2\|^2_{L^\infty(H^{1,2})}\Big)\\ &\quad + C\,\|J_{\bar a_1} - J_{\bar a_2}\|^2 + \frac{1}{2}\,\|D_{\bar a}(s_1 - s_2)\|^2_{L^{2,2}} \\ &\le C\epsilon\|\bar a_1 - \bar a_2\|^2_{V_T} + C\,\|J_{\bar a_1} - J_{\bar a_2}\|^2 + \frac{1}{2}\|s_1 - s_2\|^2_{L^2(H^{1,2})}.\end{aligned} \tag{4.26}$$

The last two terms come from the estimate of $(D^*_{\bar a}(J_{\bar a_1} - J_{\bar a_2}),\ s_1 - s_2)$. Here $\bar a$ is any convex linear combination of $\bar a_1$ and $\bar a_2$ and $\epsilon \to 0$ as $T \to 0$.

As in the estimate of local existence, we can make

$$\|J_{\bar a_1} - J_{\bar a_2}\| \le \theta_2(\|\bar a_1 - \bar a_2\|_{V_T} + \|v_1 - v_2\|_{W_T})$$

for $0 < \theta_2 < 1$. By shrinking $T > 0$ so that $\theta = C(\theta_1 + \theta_2 + \epsilon) < 1$ and by (1.8), (4.24), (4.25), and (4.26), we will have

$$\|\bar a_1 - \bar a_2\|_{V_T} + \|v_1 - v_2\|_{W_T} \le \theta(\|\bar a_1 - \bar a_2\|_{V_T} + \|v_1 - v_2\|_{W_T}).$$

Thus $(\bar a_1, v_1) = (\bar a_2, v_2)$. By (4.26), $(\bar a_1, v_1, s_1) = (\bar a_2, v_2, s_2)$.

The smoothness comes from the classical quasi-linear parabolic system of equations (see for example [10]). So the proof of Proposition 4.2 is completed.

Thus if $(D_1, u_1)$ and $(D_2, u_2)$ are two solutions to (1.4), (1.5), then there will be $s_1$ and $s_2$ such that $(S_1^*D_1, S_1^{-1}u_1, s_1)$ and $(S_2^*D_2, S_2^{-1}u_2, s_2)$ are two solutions to (4.18)–(4.22), contradicting Proposition 4.2.

Now we complete the proof of Theorem A.

**Proof of Theorem A.** By the local existence and uniqueness, we can get a unique maximal solution to (1.4) and (1.5) on $[0, T)$, $0 < T \le \infty$. If $T < \infty$, then (1.4) and (1.5) possesses a weak solution $(D, u)$ which is gauge-equivalent to a smooth $(\hat D, \hat u) = ((\hat S_0^{-1})^*D, \hat S_0 u)$ on $(0, T)$, and assume (by contradiction) that there exists $R > 0$ such that (3.10) holds. Then by Lemma 3.7, $\lim_{t\nearrow T}\hat D(t) = \hat D(T)$ and $\lim_{t\nearrow T}\hat u(t) = \hat u(T)$ exist in $H^{1,2}\left(\Omega^1(Ad\eta)\right)$ and $H^{1,2}\left(\Omega^0(\eta)\right)$ respectively.

Thus for $t_0 < T$ sufficiently near $T$ the local solution $(\hat D^{t_0}, \hat u^{t_0})$ to the initial value problem (1.4) with initial data $(\hat D(t_0), \hat u(t_0))$ at time $t = t_0$ constructed in the existence proof extends to an interval $[t_0, t_1)$, $t_1 > T$. By uniqueness of weak solutions to (1.4) and equivalence of (1.4) under time-independent gauge transformations, necessarily $D(t) = \hat S_0^*(\hat D^{t_0}(t))$ on $[t_0, T)$. Hence $\hat S_0^*(\hat D^{t_0})$ extends the solution $D(t)$ to the interval $[t_0, t_1)$, contradicting the maximality of $T$.

Let $M = \bigcup B_i$, be a covering of $M$ by geodesic balls of very small radii such that the conclusion of Lemma 3.4 is true. Then for any $0 < t < T$,

$$\sum \int_{B_i} |F(t)|^2 + |Du(t)|^2 \le K \int_M |F(t)|^2 + |Du(t)|^2 \le K \left(\mathrm{YMH}(u_0, D_0)\right).$$

Let $x_j$, $j = 1, \cdots, N$, be the first $N$ singular points, and $x_j \in B_{i_j}$. Then

$$N\delta \le \limsup_{t \nearrow T} \sum_{j=1}^{N} \int_{B_{i_j}} |F(t)|^2 + |Du(t)|^2 \le K \left(\mathrm{YMH}(u_0, D_0)\right).$$

We know that there are only finite many singular points $\{x_1^1, \cdots, x_1^{l_1}\}$ and

$$l_1 \le \frac{K}{\delta} \mathrm{YMH}(u_0, D_0).$$

This proves Theorem A.

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