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EQUIFOCAL HYPERSURFACES IN SYMMETRIC SPACES**

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Abstract

This note investigates the multiplicity problem of principal curvatures of equifocal hypersurfaces in simply connected rank 1 symmetric spaces. Using Clifford representation theory, and the author also constructs infinitely many equifocal hypersurfaces in the symmetric spaces.

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§1. Introduction

By [11], a hypersurface in a symmetric space is called equifocal if every normal geodesic perpendicular to it is closed of constant length, say l, and contains $2g$ focal points for some positive integer g. This is a natural generalization of isoparametric hypersurfaces in spheres where the integer g is the number of distinct principal curvatures. In this note we consider equifocal hypersurfaces in simply connected rank one symmetric spaces, i.e. the complex projective space $\mathbb{C}P^n$, the quarternionic projective space $\mathbb{H}P^n$ and the Cayley plane $\mathbb{Q}P^2$. By using the Hopf fibration, equifocal hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ can be lifted to isoparametric hypersurfaces in spheres with $2g$ distinct principal curvatures and the length of every normal geodesic is $2l$. Using Münzner's remarkable theorem[8] we see that the integer g must be among $\{1, 2, 3\}$ if the ambient space is $\mathbb{C}P^n$ and $\mathbb{H}P^n$. For equifocal hypersurfaces in the Cayley plane, it is known that g is either 1 or 2 (see [12]).

Inspired by the construction of Ferus-Karcher-Münzner[5], we present infinitely many equifocal hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ with $g = 2$.

Let b_m be an integral function on m satisfying that $b_{m+8k} = 2^{4k} b_m$ and $b_1 = 1, b_2 = 2, b_3 = b_4 = 4$ and $b_5 = b_6 = b_7 = b_8 = 8$.

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Theorem 1.1. (1) *If $m \equiv 2,3,4,5,6 \pmod 8$ and $n \equiv 0 \pmod{b_m}$, then there is at least an equifocal hypersurface in $\mathbb{C}P^{n-1}$ with $g = 2$ and multiplicities m and $n-m-1$. The same result holds if $m = 0, \pm 1 \pmod 8$ and $n \equiv 0 \pmod{2b_m}$.*

(2) *If $m \equiv 3,4,5 \pmod 8$ and $n \equiv 0 \pmod{b_m}$, then there is at least an equifocal hypersurface in $\mathbb{H}P^{\frac{n}{2}-1}$ with $g = 2$ and multiplicities m and $n-m-1$. The same result holds if $m = \pm 2 \pmod 8$ and $n \equiv 0 \pmod{2b_m}$ and if $m = 0, \pm 1 \pmod 8$ and $n \equiv 0 \pmod{4b_m}$.*

By the work of Stolz[9] (compare [4]) it is easy to show the following

Theorem 1.2. (1) *If $m \equiv 2,3,4,5,6 \pmod 8$ and $m \neq 4$, then m and $n-m-1$ are the multiplicities of an equifocal hypersurface in $\mathbb{C}P^{n-1}$ if and only if $n \equiv 0 \pmod{b_m}$.*

(2) *If $m \equiv 3,4,5 \pmod 8$, then m and $n-m-1$ are the multiplicities of an equifocal hypersurface in $\mathbb{H}P^{\frac{n}{2}-1}$ if and only if $n \equiv 0 \pmod{b_m}$.*

By the above theorem we get a necessary and sufficient condition for the multiplicity problem of equifocal hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ in about half cases. However, it is still open whether the same result can be extended to the rest. We should like to make the following

Conjecture 1.1. *Let m and l be positive integers. If $m \leq l$ and $m = 0, \pm 1 \pmod 8$ (resp. $m = 0, \pm 1, \pm 2 \pmod 8$), then m, l are the multiplicities of an equifocal hypersurface in $\mathbb{C}P^n$ (resp. $\mathbb{H}P^n$) with $g = 2$ if and only if $m + l + 1 \equiv 0 \pmod{b_m}$.*

For the multiplicity problem of equifocal hypersurfaces in $\mathbb{Q}P^2$, we have to use some extra argument to settle the multiplicity problem of equifocal hypersurfaces in $\mathbb{Q}P^2$. Using the Leray-Serre spectral sequence we get

Theorem 1.3. *Let $M \subset \mathbb{Q}P^2$ be an equifocal hypersurface with multiplicities $m_1 \leq m_2$. Then either $(m_1, m_2) = (7, 15)$, M is diffeomorphic to S^{15} and $g = 1$ or $(m_1, m_2) = (4, 7)$, M is an S^4-bundle over homotopy 11-sphere as well as an S^7-bundle over $\mathbb{H}P^2$ and $g = 2$.*

Remark 1.1. The above theorem was originally contained in [4] when the focal manifolds are both orientable. In [10] Tang obtained the same result by a very different approach (see Theorem 5.1 and Proposition 5.7 in [10]).

§2. Equifocal Hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$

In this section we define infinitely many isoparametric functions in odd dimensional spheres as in [5] which are invariant with respect to the canonical circle or $Sp(1)$ actions on the spheres. As in [5], these produce isoparametric hypersurfaces in spheres admitting the canonical circle or $Sp(1)$ actions. Therefore the quotients are equifocal hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

Let $Cl^{0,m}$ be the Clifford algebra defined by the quadratic form $(\mathbb{R}^m, -(x_1^2 + \cdots + x_m^2))$ (see [6]). Note that $Cl^{0,m+8k} = Cl^{0,m} \otimes \mathbb{R}(2^{4k})$ and $Cl^{0,m}$ is as in the following table for $m \leq 8$:

m	1	2	3	4	5	6	7	8
$Cl^{0,m}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

Let $2b_m$ be the dimension of irreducible real representation of $Cl^{0,m+1}$. It is easy to verify that $b_{m+8k} = 2^{4k} b_m$, $b_1 = 1, b_2 = 2, b_3 = b_4 = 4$ and $b_5 = b_6 = b_7 = b_8 = 8$.

Proof of Theorem 1.1. Observe that when m is mod 8 congruent to $2, 3, 4, 5$ and 6, $Cl^{0,m+1}$ is a matrix algebra or the direct sum of two matrix algebra over $\mathbb{C}$ or $\mathbb{H}$. Hence its irreducible representation(s) is complex of $\mathbb{C}$-dimension b_m and for each positive integer $n \equiv 0 (\operatorname{mod} b_m)$ there is an n-dimensional complex representation. Let $e_0, \cdots, e_m$ be an orthonormal basis of $(\mathbb{R}^{m+1}, -x_0^2 - \cdots - x_m^2)$. Obviously, $e_0, \cdots, e_m$ give rise to unitary $n \times n$ matrices $P_0, \cdots, P_m$ so that $P_i^2 = \mathbb{I}$ and $P_iP_j + P_jP_i = 0$ if $i \neq j$.

Now we define isoparametric function $F : \mathbb{C}^n \to \mathbb{R}$ by

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i(x), x\rangle^2, \quad x \in \mathbb{C}^n.$$

Note that $F(e^{i\theta}x) = F(x)$, i.e., F is invariant with respect to the circle action. The restricting of F on the unit sphere S^{2n-1} gives rise to a function $F : S^{2n-1} \to [-1, 1]$ which is a Bott-Morse function with ± 1 the only critical values. For any $c \in (-1, 1)$, $F^{-1}(c)$ is a hypersurface in S^{2n-1} invariant under the standard circle action. The quotient $F^{-1}(0)/S^1 \subset \mathbb{C}P^{n-1}$ is an equifocal hypersurface because its lifting to S^{2n-1} is isoparametric. By [5] we conclude that the multiplicities of the equifocal hypersurface in $\mathbb{C}P^{n-1}$ just constructed are m and $n - m - 1$, where n is a multiple of b_m.

When $m \equiv 0, \pm 1 (\operatorname{mod} 8)$, the dimension of irreducible complex representation of $Cl^{0,m+1}$ is $2b_m$. The above argument proves that if $n \equiv 0 (\operatorname{mod} 2b_m)$, there is an equifocal hypersurface in $\mathbb{C}P^{n-1}$ with multiplicities m and $n - m - 1$. The same argument applies to give the required equifocal hypersurfaces in $\mathbb{H}P^{\frac{n}{2}-1}$.

Proof of Theorem 1.2. If $m \geq 8$, m and $n-m-1$ are the mulitiplicities of an equifocal hypersurface in $\mathbb{C}P^{n-1}$ or $\mathbb{H}P^{\frac{n}{2}-1}$, by using the Hopf fibration we get an isoparametric hypersurface in the unit sphere with the same multiplicities. By appealling to a result of Stolz[9] it follows that $n \equiv 0 (\operatorname{mod} b_m)$. Now the desired result follows from Theorem 1.1.

Our next result shows that there does not exist equifocal hypersurfaces in $\mathbb{C}P^n$ (resp. $\mathbb{H}P^n$) with $g = 3$ except $n = 3$ (resp. $n = 1$).

Proposition 2.1. *There does not exist equifocal hypersurface in* $\mathbb{C}P^n$ (*resp.* $\mathbb{H}P^n$) *with* $g = 3$ *if* $n \neq 3$ (*resp.* $n \neq 1$).

Proof. By a result of Abresch, the dimension of an isoparametric hypersurface in the unit sphere must be either 12 or 6 with multiplicities 2 or 1 respectively. If there is an equifocal hypersurface in $\mathbb{C}P^6$, using the Hopf fibration we get an isoparametric hypersurface in S^{13} with six distinct principal curvatures. By [3] one of the focal manifolds of the isoparametric hypersurface in S^{13} must be homeomorphic to $X_5(2)$, the Fermat complex hypersurface in $\mathbb{C}P^6$ defined by the equation $z_0^2 + \cdots + z_6^2 = 0$. In particular, its Euler characteristic is 6. Therefore this focal manifold does not admit any free S^1-action, a contradiction.

§3. Equifocal Hypersurfaces in Cayley Plane

By [11] we know that an equifocal hypersurface M with multiplicities m_1, m_2 must be an S^{m_1}-bundle over a focal manifold M_1 as well as an S^{m_2}-bundle over another focal manifold M_2. Let S, S_1 and S_2 denote the sequences

$$\{m_1, m_2, u = m_1 + m_2, u + m_1, u + m_2, 2u, \cdots\},$$
$$\{m_1, u, u + m_1, 2u, \cdots\} \quad \text{and} \quad \{m_2, u, u + m_2, 2u, \cdots\}.$$

Let $P = P(\mathbb{Q}P^2, M \times p)$ (resp. $P_1 = P(\mathbb{Q}P^2, M_1 \times p)$ and $P_2 = P(\mathbb{Q}P^2, M_2 \times p)$) denote the path spaces consisting of all paths from $p \in \mathbb{Q}P^2$ to M (resp. M_1 and M_2) in $\mathbb{Q}P^2$, where p is not a focal point. It is proved in [11] that the dimension of the reduced homology groups of P (resp. P_1 and P_2) with any coefficient field occur exactly nontrivially at the sequence S (resp. S_1 and S_2). Moreover, the rank of each reduced homology group of P (resp. P_1 and P_2) is exactly the number of the elements in the sequence in the given dimension. The geometry of equifocal hypersurface in $\mathbb{Q}P^2$ implies the following identity of Euler characteristic

$$\chi(\mathbb{Q}P^2) - \chi(M_1) - \chi(M_2) + \chi(M) = 0.$$

Indeed, this follows from the fact that $\mathbb{Q}P^2$ is the union of two disc bundles over M_1 and M_2 along the hypersurface M. In this section we are to prove Theorem 1.3 using this assertion and the Leray-Serre spectral sequence.

Proof of Theorem 1.3. In the proof all coefficients of homology or cohomology groups will be $\mathbb{Z}_2$. The proof is divided into the following steps.

Step I. $7, 15 \in S$.

Consider the Leray-Serre cohomology spectral sequence (LSSS) of the fibration

$$\Omega\mathbb{Q}P(2) \to P \to M,$$

where $\Omega\mathbb{Q}P(2)$ is the loop space of $\mathbb{Q}P^2$. Note that the rational cohomology groups of $\Omega\mathbb{Q}P^2$ occur only in the dimensions $\{0, 7, 22, \cdots\}$. By [11] the reduced cohomology groups of the path space P occur nontrivially only in the dimensions in the sequence S. The argument is to compare this fact with the spectral sequence.

If the differential $d_8 : E_8^{0,7} \to E_8^{8,0}$ vanishes, then the term $E_8^{0,7}$ survives to E_∞ and so $7 \in S$. Also the term $E_8^{15,0} \cong H^{15}(M)$ survives and therefore $15 \in S$. If d_8 is nontrivial, then this means that $E_8^{8,0}$ is nonzero. By Poincaré duality, $H^7(M)$ is nontrivial and it survives to E_∞. Thus $7 \in S$ and $E_8^{8,7} \cong H^8(M)$ is of rank at least 1. The differential $d_8 : E_8^{8,7} \to E_8^{16,0} = 0$ and so $E_\infty^{8,7} \cong E_8^{8,7}$. Thus $15 \in S$. This completes Step I.

By Step I above it is elementary to see that $m_1 \neq m_2$.

Step II. $2 \leq m_1 \leq m_2 \leq 6$ can not happen.

Suppose not, consider the fibration

$$\Omega\mathbb{Q}P^2 \to P_1 = P(\mathbb{Q}P^2, M_1 \times p) \to M_1$$

and its Leray Serre spectral sequence. Note $\dim M_1 = 15 - m_1 \leq 13$. As $\Omega\mathbb{Q}P^2$ is 6-connected, it is easy to see that $H^q(M) \cong \mathbb{Z}_2$ if $q = 0$ or $q \leq 6$ and $q \in S_2$. This implies that

$$d_8 : E_8^{m_2,7} \to E_8^{m_2+8,0} \cong H^{7-u}(M_1) = 0$$

vanishes if $7 - u \neq 0$ since $7 - u \leq 2 < m_2$. Hence $E_8^{m_2,7}$ survives and so $m_2 + 7 \in S_2$ or $u = 7$. In all cases, note that

$$H^8(M_1) \cong H^{7-m_1}(M_1) \cong \mathbb{Z}_2.$$

Thus $E_8^{8,7} \cong \mathbb{Z}_2$ surviving to E_∞. This proves $15 \in S_2$.

If $m_2 + 7 \in S_2$ and $u \neq 7$, by Step I we know that $u + m_1 = 7$. Comparing with the relations $15 \in S_2$, i.e., $15 = ku$ or $15 = ku + m_2$, we get that either $u|15$ or $22 = (k+1)u$. This implies that $u = 5$ and $(m_1, m_2) = (2, 3)$ or $u = 11$ and $(m_1, m_2) = (5, 6)$.

Note that the case $u = 7$ and $m_2 + 7 \in S_2$, i.e., $15 = ku$ or $15 = ku + m_2$, is impossible.

For the two possibilities above, the dimensions of the focal manifolds are $12, 13$ and $9, 10$ respectively. In the first case the Euler characteristics $\chi(M) = 0$, $\chi(M_1) = 0$. $H^6(M_2) = 0$ by an analogous spectral sequence for the path space $P_2 = P(QP^2, M_2 \times p)$. Thus $\chi(M_2)$ is even since $\dim M_2 = 12$ is even. In the second case $\chi(M) = 0$, $\chi(M_2) = 0$ and similarly $\chi(M_1)$ is even since $\dim M_1 = 10$. Note $\chi(\mathbb{Q}P^2) = 3$. This contradicts the following identity of the Euler characteristics

$$\chi(\mathbb{Q}P^2) - \chi(M_1) - \chi(M_2) + \chi(M) = 0.$$

This completes Step II.

Step III. $2 \le m_1 \le 6$ and $m_2 \ge 6$ implies $(m_1, m_2) = (4, 7)$.

Note that $\dim M_1 = 15 - m_1$ and $H^*(M_1) = 0$ if $* \ne 0$ and ≤ 6 by the spectral sequence as before. By the duality it follows easily that M_1 has the rational homotopy type of S^{15-m_1}(In fact it is a homotopy sphere). M_2 is a manifold of dimension $15 - m_2 \le 8$. For $q \le 6$, $H^q(M_2, \mathbb{Q}) = \mathbb{Q}$ only if $q = 0$ or $q = m_1$. By Poincare duality it follows that M_2 is either homologous to $\mathbb{C}P^2$ or $\mathbb{H}P^2$. These imply $m_1 = 2$ or 4 and $m_2 = 11$ or 7 respectively. Recall that the hypersurface M is sphere bundle over the two focal manifolds. In the former case, M is an S^{11}-bundle over $\mathbb{C}P^2$ as well as an S^2-bundle over S^{13}. Obviously it is impossible since they have different rational homology groups. In the latter case, M is an S^7-bundle over a homological $\mathbb{H}P^2$ as well as an S^4-bundle over a homotopy 11-sphere. It is easy to check the focal manifold with the homology of $\mathbb{H}P^2$ is actually diffeomorphic to $\mathbb{H}P^2$ (see [13]). The proof for Step III is completed.

Combining Steps I, II and III, we know that if $m_1 > 1$, then either $(m_1, m_2) = (4, 7)$ or $m_2 > m_1 \ge 7$ and thus $m_1 = 7$ by Step I. Furthermore, the latter case implies that either $m_2 = 8$ or 15. If $m_2 = 8$, the Euler characteristic $\chi(M_1) = 2$ and $\chi(M) = \chi(M_2) = 0$. This contradicts the Euler characteristic identity as above. Thus $(m_1, m_2) = (7, 15)$ if the cases of $m_1 = 1$ can be excluded.

Step IV. $m_1 \ne 1$.

Suppose not, then $\dim M_1 = 14$ and $\dim M_2 = 15 - m_2 \le 14$. This implies that (m_1, m_2) are among the four cases $(1, 1)$, $(1, 3)$, $(1, 5)$ and $(1, 7)$ by Step I. The dimension of M_2 is 14, 12, 10 and 8 respectively. We claim that $H^7(M_1, \mathbb{Z}_2) = 0$. Conisder the fibration

$$\Omega\mathbb{Q}P^2 \to P_1 \to M_1.$$

Recall that the reduced cohomology group of the loop space $\Omega\mathbb{Q}P^2$ only occurs nontrivially at dimension $7, 22, \cdots$. Clearly $H^q(M_1) = H^q(P_1)$ for $q \le 6$. In particular, $H^6(M_1) = \mathbb{Z}_2$. By Poincare duality $H^8(M_1) = \mathbb{Z}_2$.

Consider the Leray-Serre spectral sequence of the above fibration. Let $x \in H^7(\Omega\mathbb{Q}P^2)$ be the unique generator. We assert that $d_8(x) = 0$, where $d_8 : E_8^{0,7} \to E_8^{8,0}$ is the differential. Suppose not, then $H^8(P_1) = 0$. This contradicts since $8 \in S_1$. This shows that $\chi(M_1)$ is even by the Poincare duality. Similarly, one can prove that $\chi(M_2)$ is even. This is impossible since one of $\chi(M_1)$ and $\chi(M_2)$ must be odd by the identity

$$3 = \chi(\mathbb{Q}P^2) = \chi(M_1) + \chi(M_2).$$

This completes Step IV.

Combining these steps we see that $(m_1, m_2) = (7, 15)$ or $(4, 7)$ as mentioned above and consequently $M = S^{15}$ or a sphere bundle over $\mathbb{H}P^2$. Notice that, in the former case, there is a free action of the dihedral group D_{2g} on M (see [11]). By [7], it is possible only if $g = 1$ and then $D_{2g} = \mathbb{Z}_2$, where $2g$ is the number of focal points on the normal geodesic. This completes the proof.

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