Chin. Ann. of Math.
21B: 4(2000),479-488.

MULTIPLICITY OF PERIODIC SOLUTIONS OF DUFFING'S EQUATIONS WITH LIPSCHITZIAN CONDITION

WANG Zaihong*

Abstract

This paper deals with the existence and multiplicity of periodic solutions of Duffing equations $\ddot{x} + g(x) = p(t)$. The author proves an infinity of periodic solutions to the periodically forced nonlinear Duffing equations provided that $g(x)$ satisfies the globally lipschitzian condition and the time-mapping satisfies the weaker oscillating property.

Keywords Periodic solution, Time mapping, Poincaré-Birkhoff twist theorem
1991 MR Subject Classification 34C25, 34B15
Chinese Library Classification O175.1 **Document code** A
Article ID 0252-9599(2000)04-0477-10

§1. Introduction

We deal with the Duffing equation

$$\ddot{x} + g(x) = p(t), \tag{1.1}$$

where $g(x)$, $p(t) \in C(R, R)$ and $p(t)$ is periodic, whose least period is 2π. The multiplicity of periodic solutions of Equation (1.1) has been widely studied since the 50s. In [1], T. Ding studied the multiplicity of periodic solutions of Equation (1.1) under the following conditions.

(g_1) Let $g(x) \in C^1(R, R)$, and let K be a positive constant, such that

$$|g'(x)| \le K, \quad x \in R.$$

(g_2) There exist two constants $A_0 > 0$ and $M_0 > 0$ such that

$$\frac{g(x)}{x} \ge A_0, \quad |x| \ge M_0.$$

(τ_0) There exist a constant $\sigma > 0$, an integer $m > 0$, and two sequences $\{a_k\}$ and $\{b_k\}$ ($k \in N$), such that $a_k \to \infty$ and $b_k \to \infty$ as $k \to \infty$; and moreover

$$\tau(a_k) < \frac{2\pi}{m} - \sigma, \quad \tau(b_k) > \frac{2\pi}{m} + \sigma,$$

where $\tau(e)$ denotes the least positive period of the solution $x(t)$ $(x(0) = 0, \dot{x}(0) = \sqrt{2e})$ for the equation $\ddot{x} + g(x) = 0$.

Manuscript received March 28, 1999. Revised January 12, 2000.

*Department of Mathematics, Capital Normal University, Beijing 100037, China.
E-mail: Wangzh@mail.cnu.edu.cn

By using a generalized form of the Poincaré-Birkhoff twist theorem, T. Ding proved that Equation (1.1) has infinitely many 2π-periodic solutions. Lately, T. Ding, R. Iannaci and F. Zanolin[2] further generalized the condition (τ_0) to the following condition

(τ_1) $$\Delta\tau = \limsup_{e\to+\infty}\tau(e) - \liminf_{e\to+\infty}\tau(e) > 0.$$

Qian Dingbian[3] improved the results in [1,2] by replacing the conditions (g_1) and (g_2) with the condition,

$$\limsup_{|x|\to+\infty}\frac{G(x)}{g^2(x)} < +\infty, \quad G(x) = \int_0^x g(s)ds.$$

Hao Dunyuan and Ma Shiwang[4] generalized the condition (τ_0) to the condition

(τ_2) $$\liminf_{e\to+\infty}\sqrt{e}(\tau(e) - \tfrac{2\pi}{n}) = -\infty, \quad \limsup_{e\to+\infty}\sqrt{e}(\tau(e) - \tfrac{2\pi}{n}) = +\infty,$$

where n is a positive integer. They also obtained the multiplicity of periodic solutions of Equation (1.1) under conditions (g_1), (g_2) and (τ_2)

In the present paper, we study the existence of 2π-periodic solutions for Equation (1.1) under conditions (τ_2) and

(H_1) $\lim\limits_{|x|\to+\infty}\operatorname{sgn}(x)g(x) = +\infty$,

(H_2) $g(x)$ satisfies the globally lipschitzian condition. That is, there exists a positive constant a such that $|g(x) - g(y)| \le a|x-y|$.

By developing an idea in [1], we obtain the following result.

Theorem 1.1. *Assume that conditions* (H_1)*,* (H_2) *and* (τ_2) *hold. Then Equation* (1.1) *possesses infinitely many* 2π*-periodic solutions* $\{x_k(t)\}_{k=1}^{\infty}$ *which satisfy*

$$\lim_{k\to\infty}\big(\min_{t\in R}(|x_k(t)| + |\dot{x}_k(t)|)\big) = +\infty.$$

If condition (H_2) is replaced by the following condition,

(H_3) there exist two positive constants a and b such that $|g(x) - g(y)| \le a|x-y| + b$, then we also have

Theorem 1.2. *Assume that conditions* (H_1)*,* (H_3) *and* (τ_2) *hold. Then Equation* (1.1) *possesses infinitely many* 2π*-periodic solutions* $\{x_k(t)\}_{k=1}^{\infty}$ *which satisfy*

$$\lim_{k\to\infty}\big(\min_{t\in R}(|x_k(t)| + |\dot{x}_k(t)|)\big) = +\infty.$$

In Section 4, we construct an example for an application of the above theorems. This example also shows that the above theorems are not contained in the results of the previously quoted articles.

§2. Priliminary Lemmas

At first, we consider the auxiliary autonomous equation

$$\ddot{x} + g(x) = 0, \tag{2.1}$$

or, its equivalent system

$$\dot{u} = w, \qquad \dot{w} = -g(u). \tag{2.2}$$

The orbits Γ_e of the autonomous system (2.2) are curves determined by the equation

$$\frac{1}{2}w^2 + G(u) = e, \tag{2.3}$$

where e is an arbitrary constant.

Then, we can easily prove the following lemma.

Lemma 2.1. *If* (H_1) *holds, then there is a constant* $e_0 > 0$*, such that , for each* $e \geq e_0$*,* Γ_e *is a closed curve which is star-shaped with respect to the origin* O*.*

It follows from Lemma 2.1 that each curve $\Gamma_e (e \geq e_0)$ intersects the u-axis at two points $(d(e), 0)$ and $(c(e), 0)$, where $d(e) < 0$ and $c(e) > 0$ are uniquely determined by the formula

$$G\big(d(e)\big) = G(c(e)) = e.$$

Let $\big(u(t), w(t)\big)$ be any solution of (2.2) whose orbit is $\Gamma_e (e \geq e_0)$. Clearly, this solution is periodic. Let $\tau(e)$ denote the least positive period of this solution. It follows from (2.2) and (2.3) that

$$\tau(e) = \sqrt{2} \int_{d(e)}^{c(e)} \frac{du}{\sqrt{e - G(u)}}.$$

By the definition, it follows that $\tau(e)$ is continuous for $e \geq e_0$.

Now we perform some phase-plane analysis for Equation (1.1). First of all, we write the Duffing equation (1.1) in the equivalent system:

$$\dot{x} = y, \qquad \dot{y} = -g(x) + p(t). \tag{2.4}$$

Let $(x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of (2.4) through the initial point $(x(0), y(0)) = (x_0, y_0)$. It is not hard to show that every solution satisfying the initial vlaue problem exists uniquely on the whole t-axis under conditions (H_1) and (H_2). Then the Poincaré mapping $P : R^2 \mapsto R^2$ is well defined by

$$(x_0, y_0) \mapsto \big(x(2\pi, x_0, y_0), y(2\pi, x_0, y_0)\big).$$

It is well known that P is an area-preserving homeomorphism.

By applying the transformation $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$ to the system (2.4), we get the equation for $r(t)$ and $\theta(t)$,

$$\begin{aligned} \dot{r} &= r\sin\theta\cos\theta - g(r\cos\theta)\sin\theta + p(t)\sin\theta, \\ \dot{\theta} &= -\sin^2\theta - \frac{1}{r}(g(r\cos\theta)\cos\theta - p(t)\cos\theta). \end{aligned} \tag{2.5}$$

Let $\big(r(t, r_0, \theta_0), \theta(t, r_0, \theta_0)\big)$ be the solution of (2.5) through the initial point $(r(0), \theta(0)) = (r_0, \theta_0)$. Then the mapping P can also be written in the polar coordinate form

$$r^* = r(2\pi, r_0, \theta_0), \qquad \theta^* = \theta(2\pi, r_0, \theta_0) + 2l\pi,$$

where l is an arbitrary integer. It can be easily seen that if (r_0, θ_0) is such that

$$r(t, r_0, \theta_0) > 0, \quad t \in [0, 2\pi],$$

then $\theta(2\pi, r_0, \theta_0)$ is well defined and continuous in (r_0, θ_0), and moreover,

$$\theta(2\pi, r_0, \theta_0 + 2\pi) = \theta(2\pi, r_0, \theta_0) + 2\pi.$$

Next, we take the transformation $u(t) = \rho(t)\cos\varphi(t)$, $w(t) = \rho(t)\sin\varphi(t)$ to the system (2.2). The resulting equations for $\rho(t)$ and $\varphi(t)$ are

$$\begin{aligned} \dot{\rho} &= \rho\cos\varphi\sin\varphi - g(\rho\cos\varphi)\sin\varphi, \\ \dot{\varphi} &= -\sin^2\varphi - \frac{1}{\rho} g(\rho\cos\varphi)\cos\varphi. \end{aligned} \tag{2.6}$$

Let $(\rho(t, \rho_0, \varphi_0)$,$\varphi(t, \rho_0, \varphi_0)$ be the solution of (2.6) through the initial point $(\rho(0), \varphi(0)) = (\rho_0, \varphi_0)$. Using (2.5) and (2.6), we can easily prove the following

Lemma 2.2. *If* (H_1) *and* (H_2) *hold and* $T>0$ *is a given constant, then there exist positive constants* $\alpha>1$ *and* $\Lambda_0>1$ *such that*

(1) $\rho_0/\alpha \le \rho(t,\rho_0,\varphi_0) \le \alpha\rho_0$ *and* $\dot\varphi(t,\rho_0,\varphi_0)<0$ *for* $t\in[0,T]$ *and* $\rho_0 \ge \Lambda_0$;

(2) $r_0/\alpha \le r(t,r_0,\theta_0) \le \alpha r_0$ *and* $\dot\theta(t,r_0,\theta_0)<0$ *for* $t\in[0,T]$ *and* $r_0\ge\Lambda_0$.

Remark 2.1. From Lemma 2.2 we know that if ρ_0 is large enough, then

$$\frac{\sqrt{2e}}{\alpha} \le \rho_0 \le \alpha\sqrt{2e}, \quad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0)\in\Gamma_e.$$

Lemma 2.3. *Assume that conditions* (H_1), (H_2) *hold and* $T>0$ *is a given constant. Then there exist positive constants* c_0, Γ_0 *and* M *such that*

(1) $|\theta(t,r_0,\theta_0)-\varphi(t,r_0,\theta_0)| \le \frac{c_0}{r_0}$, $r_0\ge\Gamma_0$, $t\in[0,T]$;

(2) $|r(t,r_0,\theta_0)-\rho(t,r_0,\theta_0)|\le M$, $t\in[0,T]$.

Proof. (1). The proof follows the arguements in [1].

Let $\big(u(t,x_0,y_0),\, w(t,x_0,y_0)\big)$ be the solution of the system (2.2) through the initial point $\big(u(0),w(0)\big)=(x_0,y_0)$ with $x_0=r_0\cos\theta_0, y_0=r_0\sin\theta_0$. Set

$$s(t)=s(t,x_0,y_0)=x(t,x_0,y_0)-u(t,x_0,y_0),$$
$$v(t)=v(t,x_0,y_0)=y(t,x_0,y_0)-w(t,x_0,y_0).$$

Then we have

$$\frac{ds(t)}{dt}=v(t), \quad \frac{dv(t)}{dt}=p(t)+g(u(t,x_0,y_0))-g(x(t,x_0,y_0)).$$

Let $\eta(t)=(s^2(t)+v^2(t))^{\frac12}$. Then we have

$$\eta(t)\frac{d\eta(t)}{dt}=s(t)v(t)+p(t)v(t)+[g(u(t,x_0,y_0))-g(x(t,x_0,y_0))]v(t).$$

It follows from (H_2) that

$$\Big|\eta(t)\frac{d\eta(t)}{dt}\Big| \le |p(t)v(t)|+(1+a)|s(t)v(t)|.$$

Furthermore,

$$\Big|\frac{d\eta(t)}{dt}\Big| \le \frac12(1+a)\eta(t)+B, \tag{2.7}$$

where $B=\max\limits_{t\in[o,2\pi]}|p(t)|$. The differential inequality (2.7) together with $\eta(0)=0$ yields

$$\eta(t)\le\frac{2B}{1+a}[e^{\frac{(1+a)T}{2}}-1]\equiv H_0$$

for $t\in[0,T]$. Write

$$\psi(t)=\psi(t,r_0,\theta_0)=\varphi(t,r_0,\theta_0)-\theta(t,r_0,\theta_0).$$

It is clear that if $|\psi(t)|<\pi$, then $\psi(t)$ is just the angle between the vectors $(x(t),y(t))$ and $(u(t),w(t))$. Therefore, we have

$$\cos\psi(t)=\frac{r^2(t)+\rho^2(t)-\eta^2(t)}{2r(t)\rho(t)} \ge 1-\frac{H_0^2}{2r(t)\rho(t)}, \quad t\in[0,T].$$

On the other hand, we have $\rho(t)\ge r(t)-H_0$. It follows from Lemma 2.2 that, for r_0 sufficiently large, we have

$$\frac{1}{\alpha^2}-\frac{H_0}{\alpha r_0}>\frac{1}{4\alpha^2}, \tag{2.8}$$
$$r(t)-H_0>0 \quad \text{for} \quad t\in[0,T].$$

Therefore

$$1-\cos\psi(t)\le \frac{H_0^2}{2r(t)(r(t)-H_0)}, \qquad t\in[0,T].$$

This results in

$$\sin^2\Big(\frac{1}{2}\psi(t)\Big)\le \frac{H_0^2}{4r(t)(r(t)-H_0)} \quad \text{for} \quad t\in[0,T]. \tag{2.9}$$

It follows from Lemma 2.2 and the inequality (2.9) that

$$\Big|\sin\Big(\frac{1}{2}\psi(t)\Big)\Big|\le \frac{H_0}{2r_0\sqrt{\frac{1}{\alpha^2}-\frac{H_0}{\alpha r_0}}}. \tag{2.10}$$

Since $\psi(0)=0$ and $\psi(t)$ varies continuously as t increases from 0 to T, we can see from (2.10) that

$$|\psi(t)|\le 4\Big|\sin\Big(\frac{1}{2}\psi(t)\Big)\Big| \quad \text{for} \quad t\in[0,T] \tag{2.11}$$

for r_0 large enough.

Combining (2.8), (2.10) and (2.11) yields

$$|\psi(t)|\le \frac{4\alpha H_0}{r_0} \quad \text{for} \quad t\in[0,T].$$

Write $c_0=4\alpha H_0$. Then we have

$$|\theta(t,r_0,\theta_0)-\varphi(t,r_0,\theta_0)|\le \frac{c_0}{r_0} \quad \text{for } t\in[0,T],$$

for r_0 large enough.

(2) It follows from the proof of (1) that there exists a constant $C>0$ such that

$$|x(t,x_0,y_0)-u(t,x_0,y_0)|\le C, \qquad |y(t,x_0,y_0)-w(t,x_0,y_0)|\le C, \quad t\in[0,T].$$

Hence, for $t\in[0,T]$,

$$\begin{aligned}|r(t,r_0,\theta_0)-\rho(t,r_0,\theta_0)| &= \Big|\sqrt{x^2(t,x_0,y_0)+y^2(t,x_0,y_0)}-\sqrt{u^2(t,x_0,y_0)+w^2(t,x_0,y_0)}\Big|\\ &\le \frac{|x^2(t,x_0,y_0)-u^2(t,x_0,y_0)|+|y^2(t,x_0,y_0)-w^2(t,x_0,y_0)|}{\sqrt{x^2(t,x_0,y_0)+y^2(t,x_0,y_0)}+\sqrt{u^2(t,x_0,y_0)+w^2(t,x_0,y_0)}}\\ &\le |x(t,x_0,y_0)-u(t,x_0,y_0)|+|y(t,x_0,y_0)-w(t,x_0,y_0)|\\ &\le 2C.\end{aligned}$$

Taking $M=2C$, we complete the proof of this lemma.

Let θ_0 and $\theta_1(\theta_0>\theta_1)$ be two given constants. Assume that it takes time $\Delta T_1(\theta_0,\theta_1)$ for $\theta(t)=\theta(t,r_0,\theta_0)$ to change from θ_0 to θ_1. The required time for $\varphi(t)=\varphi(t,r_0,\theta_0)$ to change from θ_0 to θ_1 is $\Delta T_2(\theta_0,\theta_1)$.

Lemma 2.4. *Assume that conditions* (H_1), (H_2) *hold and A,B are two given positive constants such that $\theta_0-\theta_1\le A$ and $\tau(e)\le B$. Then, for $(r_0\cos\theta_0, r_0\sin\theta_0)\in\Gamma_e$,*

$$|\Delta T_1(\theta_0,\theta_1)-\Delta T_2(\theta_0,\theta_1)|=O\Big(\frac{1}{r_0}\Big), \quad r_0\to+\infty.$$

Proof. For simplicity, we write

$$(r(t),\theta(t))=(r(t,r_0,\theta_0),\theta(t,r_0,\theta_0)), \quad (\rho(t),\varphi(t))=(\rho(t,r_0,\theta_0),\varphi(t,r_0,\theta_0)).$$

It follows from (2.5) and (2.6) that

$$\Delta T_1(\theta_0,\theta_1)=\int_{\theta_0}^{\theta_1}\frac{d\theta}{-\sin^2\theta(t)-\frac{1}{r(t)}g(r(t)\cos\theta(t))\cos\theta(t)+\frac{1}{r(t)}p(t)\cos\theta(t)},$$

$$\Delta T_2(\theta_0,\theta_1)=\int_{\theta_0}^{\theta_1}\frac{d\varphi}{-\sin^2\varphi(t)-\frac{1}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)}.$$

Set $T=([\frac{A}{2\pi}]+1)B$. Obviously, $0<\Delta T_2(\theta_0,\theta_1)<T$. By Lemma 2.2 and Lemma 2.3 we can obtain that there exists a positive constant M_0 such that

$$|r(t)\cos\varphi(t)-r(t)\cos\theta(t)|\le M_0,\quad t\in[0,2T], \tag{2.12}$$

for r_0 large enough. It follows from Lemma 2.3 and (2.12) that there exists a positive constant M^0 such that, for $t\in[0,2T]$,

$$\begin{aligned}|g(\rho(t)\cos\varphi(t))-g(r(t)\cos\theta(t))|&\le a|\rho(t)\cos\varphi(t)-r(t)\cos\theta(t)|\\&\le a|\rho(t)-r(t)|+r(t)|\cos\varphi(t)-\cos\theta(t)|\\&\le M^0.\end{aligned}$$

If $t\in[0,2T]$, then we have

$$\begin{aligned}&\Big|\frac{1}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)-\frac{1}{r(t)}g(r(t)\cos\theta(t))\cos\theta(t)\Big|\\&=\frac{1}{r(t)}\Big|\frac{r(t)}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)-g(r(t)\cos\theta(t))\cos\theta(t)\Big|\\&=\frac{1}{r(t)}|g(\rho(t)\cos\varphi(t))\cos\varphi(t)-g(r(t)\cos\theta(t))\cos\theta(t)|+O\Big(\frac{1}{r_0}\Big)\\&=\frac{1}{r(t)}|g(\rho(t)\cos\varphi(t))\cos\varphi(t)-g(r(t)\cos\theta(t))\cos\varphi(t)|\\&\quad+\frac{1}{r(t)}|g(r(t)\cos\theta(t))\cos\varphi(t)-g(r(t)\cos\theta(t))\cos\theta(t)|+O\Big(\frac{1}{r_0}\Big)\\&=\frac{1}{r(t)}|g(r(t)\cos\theta(t))||\cos\varphi(t)-\cos\theta(t)|+O\Big(\frac{1}{r_0}\Big)\\&=O\Big(\frac{1}{r_0}\Big).\end{aligned}$$

Therefore, if $\Delta T_1(\theta_0,\theta_1)\le 2T$, from $\tau(e)\le B$ we get

$$\begin{aligned}\Delta T_1(\theta_0,\theta_1)&=\int_{\theta_0}^{\theta_1}\frac{d\theta}{-\sin^2\theta(t)-\frac{1}{r(t)}g(r(t)\cos\theta(t))\cos\theta(t)+\frac{1}{r(t)}p(t)\cos\theta(t)}\\&=\int_{\theta_0}^{\theta_1}\frac{d\varphi}{-\sin^2\varphi(t)-\frac{1}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)+O\Big(\frac{1}{r_0}\Big)}\\&=\int_{\theta_0}^{\theta_1}\Big(\frac{1}{-\sin^2\varphi(t)-\frac{1}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)}\\&\qquad\times\frac{1}{\Big(1+O\Big(\frac{1}{r_0}\Big)\Big(-\sin^2\varphi(t)-\frac{1}{\rho(t)}g(\rho(t)\cos\varphi(t))\cos\varphi(t)\Big)^{-1}\Big)}\Big)d\varphi\end{aligned}$$

$$
\begin{aligned}
&= \int_{\theta_0}^{\theta_1} \Big(\frac{1}{-\sin^2\varphi(t) - \frac{1}{\rho(t)} g(\rho(t)\cos\varphi(t))\cos\varphi(t)} + O\Big(\frac{1}{r_0}\Big)\Big) d\varphi \\
&= \int_{\theta_0}^{\theta_1} \frac{d\varphi}{-\sin^2\varphi(t) - \frac{1}{\rho(t)} g(\rho(t)\cos\varphi(t))\cos\varphi(t)} + O\Big(\frac{1}{r_0}\Big) \\
&= \Delta T_2(\theta_0,\theta_1) + O\Big(\frac{1}{r_0}\Big).
\end{aligned}
$$

This shows that

$$|\Delta T_1(\theta_0,\theta_1) - \Delta T_2(\theta_0,\theta_1)| = O\Big(\frac{1}{r_0}\Big), \quad r_0 \to +\infty.$$

If $\Delta T_1(\theta_0,\theta_1) > 2T$, then there exists $\tilde{\theta}_1$ such that $\theta_1 < \tilde{\theta}_1 < \theta_0$ and $\Delta T_1(\theta_0,\tilde{\theta}_1) = 2T$. Using the same method, we can prove

$$|\Delta T_1(\theta_0,\tilde{\theta}_1) - \Delta T_2(\theta_0,\tilde{\theta}_1)| = O\Big(\frac{1}{r_0}\Big), \quad r_0 \to +\infty.$$

This is a contradiction.

Lemma 2.5. *Assume* (H_1), (H_2) *and* (τ_2) *hold. Then there exist two sequences* $\{a_k\}$, $\{b_k\}$ *such that* $a_k < b_k$, $k \in N$ *and*

$$
\begin{aligned}
&\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi, \quad (r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{a_k}, \\
&\theta(2\pi, r_0, \theta_0) - \theta_0 > -2n\pi, \quad (r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{b_k},
\end{aligned}
$$

where n *is a positive integer given in condition* (τ_2).

Proof. From (H_1) we know that there exist two positive constants R_1 and A_1 such that

$$\dot{\theta}(t, r_0, \theta_0) \le -\frac{A_1}{r_0}, \quad r_0 \ge R_1, \quad t \in [0, 2\pi].$$

Write $\theta(2\pi, r_0, \theta_0) - \theta_0 = -2j\pi - \eta$, where $j \ge 0$ is an integer, $0 \le \eta < 2\pi$. Denote by t_η the time for $\theta(t, r_0, \theta_0)$ to decrease from $\theta_0 - 2j\pi$ to $\theta_0 - 2j\pi - \eta$. Obviously,

$$2\pi = T_j + t_\eta \le T_{j+1}, \tag{2.13}$$

where T_j and T_{j+1} are the required time for the solution $(r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ to complete j and $j+1$ turns around the origin O, where $(r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_e : \frac{1}{2}w^2 + G(u) = e$. By Lemma 2.2 and Lemma 2.4 we have that there exists a constant $c_{(j+1)} > 0$ such that

$$|T_{j+1} - (j+1)\tau(e)| \le \frac{c_{j+1}}{\sqrt{e}} \tag{2.14}$$

provided that $\tau(e) < +\infty$. From the first inequality of (τ_2) we see that there exists a sequence $\{a_k\}$ such that $\sqrt{a_k} \to +\infty$, $\frac{k}{\sqrt{a_k}} \to 0$, as $k \to +\infty$ and

$$\sqrt{a_k}\Big(\tau(a_k) - \frac{2\pi}{n}\Big) \le -k. \tag{2.15}$$

From (2.13), (2.14) and (2.15) we have that, for $(r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{a_k} : \frac{1}{2}w^2 + G(u) = a_k$ and k large enough, $j \ge n$.

If $j \ge n+1$, then

$$\theta(2\pi, r_0, \theta_0) - \theta_0 \le -2(n+1)\pi < -2n\pi. \tag{2.16}$$

Now we assume $j = n$. Then for k large enough,

$$t_\eta = 2\pi - T_n \ge 2\pi - n\Big(\frac{2\pi}{n} - \frac{k}{\sqrt{a_k}}\Big) - \frac{c_n}{\sqrt{a_k}} = \frac{nk - c_n}{\sqrt{a_k}} > 0.$$

Therefore

$$-\eta = \int_{T_n}^{T_n+t_\eta} \dot{\theta}(t, r_0, \theta_0)dt \le -\frac{(nk - c_n)A_1}{r_0\sqrt{a_k}}.$$

Furthermore,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi - \frac{(nk - c_n)A_1}{r_0\sqrt{a_k}}. \tag{2.17}$$

It follows from (2.16) and (2.17) that, for k large enough,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi, \quad (r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{a_k}.$$

Similarly, we can prove that the second inequality of this lemma holds.

§3. The Proof of Main Results

At first, we restate a generalized form of the Poincaré-Birkhoff fixed point theorem.

Let D denote an annular region in the (x, y)-plane. The boundary of D consists of two simple closed curves: the inner boundary curve C_1 and the outer boundary C_2. Let D_1 denote the simple connected open set bounded by C_1. Consider an area-preserving mapping $T : R^2 \mapsto R^2$. Suppose that $T(D) \subset R^2 - \{O\}$, where O is the origin. Let (r_0, θ_0) be the polar coordinate of (x_0, y_0), that is,

$$x_0 = r_0\cos\theta_0, y_0 = r_0\sin\theta_0.$$

Write the mapping T in the form

$$r^* = f(r_0, \theta_0), \quad \theta^* = \theta_0 + h(r_0, \theta_0),$$

where f and h are continuous in (r_0, θ_0) and 2π-periodic in θ_0.

A Generalized Form of the Poincaré-Birkhoff Fixed Point Theorem.[5] *Besides the above-mentioned assumptions, we assume that*

(1) *C_1 is star-shaped about the origin O;*

(2) *$O \in T(D_1)$;*

(3)

$$h(r_0, \theta_0) > 0(< 0), \quad (r_0\cos\theta_0, r_0\sin\theta_0) \in C_1;$$
$$h(r_0, \theta_0) < 0(> 0), \quad (r_0\cos\theta_0, r_0\sin\theta_0) \in C_2.$$

Then T has at least two fixed points in D.

Proof of Theorem. From Lemma 2.5 we have

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi, \qquad (r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{a_k},$$
$$\theta(2\pi, r_0, \theta_0) - \theta_0 > -2n\pi, \qquad (r_0\cos\theta_0, r_0\sin\theta_0) \in \Gamma_{b_k},$$

with k large enough.

Thus we have proved that the area-preserving homeomorphism P is twisting on the annulus $A_k = \{(u, v) \in \Gamma_e : a_k \le e < b_k\}$. Moreover, we also have that, for k sufficiently large,

$$O \in P(D_k),$$

where $D_k \subset R^2$ is an open bounded set with boundary Γ_{a_k}. Finally, by Lemma 2.1, Γ_{a_k} is star-shaped with respect to the origin O (for k large), so that all the assumptions of the generalized Poincaré-Birkhoff fixed point theorem are fulfilled. Therefore, we have proved that, for each k large enough, the mapping P has at least two fixed points in A_k. Thus we have obtained the existence of a sequence $\{u_k(t)\}_{k=1}^{+\infty}$ of periodic solutions of Equation (1.1), with minimal period 2π, such that

$$\lim_{k\to+\infty}\big(\min_{t\in R}(|x_k(t)|+|\dot{x}_k(t)|)\big)=+\infty.$$

The proof of the theorem is thus completed.

Remark 3.1. With a slight modification of the proof of Theorem 1.1, we can get the conclusion of Theorem 1.2.

§4. An Example

Example. Let $g: R\mapsto R$ be an odd function and $n\in N$.

$$g(x)=\begin{cases} n^2x-34x^{\frac{2}{3}}\sin\ln x+\sin 3(x-1)^3, & x\geq 1,\\ n^2x, & 0<x<1.\end{cases}$$

Obviously, $g(x)$ satisfies the condition (H_3). It is easy to check that

$$\lim_{x\to+\infty} g(x)=+\infty \quad \text{and} \quad \lim_{x\to+\infty}\frac{g(x)}{x}=n^2.$$

In order to check (τ_2) for g, from [6] we know that it is sufficient to check condition (τ_2) for an odd function

$$g_0(x)=n^2x-34x^{\frac{2}{3}}\sin\ln x, \quad x>0.$$

By a direct calculation, we have that

$$G_0(x)=\frac{1}{2}n^2x^2+9x^{\frac{5}{3}}\cos\ln x-15x^{\frac{5}{3}}\sin\ln x, \quad x>0.$$

Set

$$\begin{aligned} &P(c,s)=n^2(c^2-s^2)+18(c^{\frac{5}{3}}\cos\ln c-s^{\frac{5}{3}}\cos\ln s)-30(c^{\frac{5}{3}}\sin\ln c-s^{\frac{5}{3}}\sin\ln s),\\ &Q(c,s)=n^2(c^2-s^2). \end{aligned}$$

Then

$$\tau_0(e)=2\sqrt{2}\int_0^{c(e)}\frac{ds}{\sqrt{e-G_0(s)}}=2\sqrt{2}\int_0^{c(e)}\frac{ds}{\sqrt{G_0(c(e))-G_0(s)}},$$

where $c(e)$ satisfies $G_0(c(e))=e$. From $\lim\limits_{x\to+\infty}\frac{2G_0(x)}{x^2}=n^2$ we have that there exist constants $0<v_1<v_2$ such that $v_1c(e)\leq\sqrt{e}\leq v_2c(e)$ for e large enough. Write $c=c(e)$. Therefore

$$\begin{aligned} \sqrt{e}(\tau_0(e)-\frac{2\pi}{n}) &= 4\sqrt{e}\int_0^c\Big(\frac{1}{\sqrt{P(c,s)}}-\frac{1}{\sqrt{Q(c,s)}}\Big)ds\\ &= 4\sqrt{e}\int_0^c\frac{30(c^{\frac{5}{3}}\sin\ln c-s^{\frac{5}{3}}\sin\ln s)-18(c^{\frac{5}{3}}\cos\ln c-s^{\frac{5}{3}}\cos\ln s)}{\sqrt{P(c,s)}\sqrt{Q(c,s)}(\sqrt{P(c,s)}+\sqrt{Q(c,s)})}ds\\ &= 4l\sqrt{e}\int_0^c\frac{c^{\frac{5}{3}}\sin(\ln c-\alpha)-s^{\frac{5}{3}}\sin(\ln s-\alpha)}{\sqrt{P(c,s)}\sqrt{Q(c,s)}(\sqrt{P(c,s)}+\sqrt{Q(c,s)})}ds, \end{aligned}$$

where $\alpha=\arctan\frac{3}{5}$, $l=\sqrt{30^2+18^2}$.

Take a sequence $\{e_k\}$ such that $\sin(\ln c(e_k)-\alpha)=1$, $\forall k\in N$ and $e_k\to+\infty$, as $k\to+\infty$.

Set $c_k = c(e_k)$. Obviously, $c_k \to +\infty$ as $k \to +\infty$. Then we have

$$\begin{aligned}\sqrt{e_k}(\tau_0(e_k) - \frac{2\pi}{n}) &\geq 4lv_1c_k \int_0^{\frac{c_k}{2}} \frac{(c_k^{\frac{5}{3}} - s^{\frac{5}{3}}\sin(\ln s - \alpha))}{\sqrt{P(c_k,s)}\sqrt{Q(c_k,s)}(\sqrt{P(c_k,s)} + \sqrt{Q(c_k,s)})} ds \\ &\geq 4ld_0v_1 \int_0^{\frac{c_k}{2}} \frac{c_k^{\frac{8}{3}}}{\sqrt{P(c_k,s)}\sqrt{Q(c_k,s)}(\sqrt{P(c_k,s)} + \sqrt{Q(c_k,s)})} ds \\ &\geq 2ld_0v_1 \int_0^{\frac{c_k}{2}} \frac{c_k^{\frac{8}{3}}}{\sqrt{P(c_k,s)Q(c_k,s)}} ds,\end{aligned}$$

where $d_0 = 1 - (\frac{1}{32})^{\frac{1}{3}}$.

Furthermore, for k large enough,

$$\begin{aligned}\sqrt{e_k}(\tau_0(e_k) - \frac{2\pi}{n}) &\geq 2ld_0v_1 \int_0^{\frac{c_k}{2}} \frac{c_k^{\frac{2}{3}}}{n^2\sqrt{n^2(c_k^2 - s^2)}} ds \\ &= \frac{2ld_0v_1c_k^{\frac{2}{3}}}{n^3} \int_0^{\frac{c_k}{2}} \frac{1}{\sqrt{c_k^2 - s^2}} ds = \frac{l\pi d_0 v_1 c_k^{\frac{2}{3}}}{3n^3}.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \sqrt{e_k}\Big(\tau_0(e_k) - \frac{2\pi}{n}\Big) = +\infty.$$

Consequently,

$$\limsup_{e\to+\infty} \sqrt{e}\Big(\tau_0(e) - \frac{2\pi}{n}\Big) = +\infty.$$

Similarly, we can check that

$$\liminf_{e\to+\infty} \sqrt{e}\Big(\tau_0(e) - \frac{2\pi}{n}\Big) = -\infty.$$

Therefore

$$\liminf_{e\to+\infty} \sqrt{e}\Big(\tau(e) - \frac{2\pi}{n}\Big) = -\infty, \quad \limsup_{e\to+\infty} \sqrt{e}\Big(\tau(e) - \frac{2\pi}{n}\Big) = +\infty.$$

It follows from above theorem that $\ddot{x} + g(x) = p(t)$ has infinitely many 2π-periodic solutions.

References

[1] Ding, T., An infinite class of periodic solutions of periodically perturbed Duffing equation at resonance [J], *Proc. Amer. Math. Soc.,* **86**(1982), 47–54.

[2] Ding, T., Iannacci, R. & Zanolin, F., Existence and multiplicity results for periodic solutions of semilinear Duffing equation [J], *J. Diff. Eqs.,* **105**(1993), 364–409.

[3] Qian Dingbian, Time-maps and Duffing equations with resonance-acrossing [J], *Scientia Sinica* (*Series A*), **23:**5(1993), 471–479.

[4] Hao Dunyuan & Ma Shiwang, Semilinear Duffing equations crossing resonance points [J], *J. Diff. Eqs.,* **133**(1997), 98–116.

[5] Ding, W. Y., A generalization of the Poincaré-Birkhoff theorem [J], *Proc. Amer. Math. Soc.,* **88**(1983), 341–346.

[6] Wang Zaihong, Multiplicity of periodic solutions of Duffing equations at resonance [J], *J. Math. Anal. Appl.*, **237**(1999), 166–187.