Chin. Ann. of Math.
21B: 4(2000),499-510.

ON THE EXISTENCE OF POSITIVE SOLUTIONS OF AN ELLIPTIC BOUNDARY VALUE PROBLEM***

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Abstract

Using variational method, the authors get an existence result for positive solutions of a superlinear elliptic boundary value problem without assuming the P.S. condition. To prove the results in this paper, the authors adopt the method of gradient flow and use a new class of truncation functions.

Keywords Superlinear elliptic BVP, Positive solution, Variational method, Gradient flow
1991 MR Subject Classification 35J20
Chinese Library Classification O175.25, O176.3 **Document code** A
Article ID 0252-9599(2000)04-0499-12

§1. Introduction and the Main Theorem

In this paper, we consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f : R^+ \to R$ satisfies locally Lipschitz condition and $f(0) \geq 0$. By a solution u of (1.1) we mean a classical solution $u \in C^2(\bar{\Omega})$ which satisfies (1.1) pointwise.

Denote by $\lambda_1 < \lambda_2 < \cdots$ the eigenvalues of $-\Delta$ with 0-Dirichlet boundary condition. It is known that $\lambda_1 > 0$ and any eigenfunction corresponding to λ_1 does not change sign on Ω.

We will assume that $f(t)$ is superlinear and subcritical at infinity. That is,

(F_1) $\liminf\limits_{t\to+\infty} f(t)t^{-1} > \lambda_1$,

(F_2) $\lim\limits_{t\to+\infty} f(t)t^{-(N+2)/(N-2)} = 0$ if $N \geq 3$, $\lim\limits_{t\to+\infty} f(t)e^{-t^{\alpha}} = 0$ for some $\alpha < 2$ if $N = 2$.

Manuscript received January 4, 1999. Revised January 21, 2000.

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*** Project supported by the National Natural Science Foundation of China and the Research Fund for the Doctoral Program of Higher Education of China.

It is well known that, under condition (F_2), u is a solution of (1.1) if and only if u is a nontrivial critical point of J defined by

$$J(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 - F(u)\Big] dx, \qquad u \in H_0^1(\Omega),$$

where $F(t) = \int_0^t \tilde{f}(s) ds$ and

$$\tilde{f}(t) = \begin{cases} f(0), & t < 0, \\ f(t), & t \geq 0. \end{cases}$$

We say J satisfies the P.S. condition if any sequence $\{u_n\} \subset H_0^1(\Omega)$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. We say J satisfies condition (J_*) if θ, the origin of $H_0^1(\Omega)$, is a local minimizer of J. When (F_1), (F_2), (J_*) and the P.S. condition are satisfied, the Mountain Pass Lemma guarantees a nontrivial critical point, which is also a solution of (1.1) (see e.g. [2, 9]). On the other hand, in order to get a nontrivial critical point by the Mountain Pass Lemma, the four conditions are also necessary in some sense. When the domain Ω is convex and (F_1), (F_2) and (J_*) are satisfied and the P.S. condition, however, is not satisfied, Figueiredo, Lions, and Nussbaum proved an existence theorem for (1.1) by an approximate argument (see [6, Theorem 2.2]). They modified J by defining a sequence of functionals $\{J_n\}$, in which every J_n satisfies the P.S. condition. Then they obtained nontrivial critical point u_n for every J_n by the Mountain Pass Lemma and proved that $\|u_n\|_{C(\bar{\Omega})} \leq C$ for some constant C independent of n. Hence for large n, u_n is a solution of (1.1). In this argument, the condition (J_*) is crucial. If (J_*) is not satisfied, yet each J_n has a local minimizer distinct from θ, the Mountain Pass Lemma can be used to get a critical point of the Mountain Pass type of J_n. In this case, the critical point of the Mountain Pass type can not be distinguished from θ if we do not impose any condition upon the local behaviour of J_n or J around θ. Therefore, we can not get nontrivial critical point in general. Under the following condition

(F_3) there is a $\beta > 0$ such that $f(\beta) = 0$,

if that (J_*) is satisfied is not sure, the Mountain Pass Lemma ceases to be effective. If Ω is convex and (F_1), (F_2), (F_3), and $f(0) \geq 0$ are satisfied, Lions proved that (1.1) has a solution u satisfying $\max\limits_\Omega u > \beta$ by using a topological degree argument under the following additional condition (see [7, Theorem 3.1])

(F_*) $\limsup\limits_{t\to+\infty} \frac{tf(t)-\theta F(t)}{t^2 f(t)^{2/N}} \leq 0$ for some $0 < \theta < \frac{2N}{N-2}$ (if $N \geq 3$).

This condition was introduced by Figueiredo, Lions, and Nussbaum[6] to get a priori bounds for all solutions of (1.1), and it was used only to get a priori bounds. It was conjectured by Lions that [7, Theorem 1.2, Theorem 2.1, Theorem 2.2, and Theorem 3.1] are also true when the convexity of Ω and the condition (F_*) (i.e. ($7'$) in [7]) is taken out from these theorems (see [7, Remark 1.3, Remark 1.6, Remark 2.1, Remark 2.3, and Remark 3.1]).

It should be noticed that, as indicated in [7], the convexity of Ω and the assumption ($7'$) in these theorems can be replaced with other conditions which imply a priori bounds of solutions of (1.1) (see [3, 6]). Nevertheless, if the convexity of Ω and the assumption ($7'$) is taken off completely, whether these theorems are still true is left as an open problem by now. If the convexity of Ω is retained, it has been proved that the assumption ($7'$) can be dropped (see [5, 8]). In [5], two positive solutions were obtained with one being a minimum and the other being a Mountain Pass point, and it was shown that the Mountain Pass point is larger than the minimum. We would like to mention that [5] did not give further information on the Mountain Pass point, for example, in the case of (F_3), whether there exists a solution u satisfying $\max_\Omega u > \beta$ was unknown. In [8], a new kind of argument based on descending flow was introduced and some interesting properties on the solutions of (1.1) were given. In this paper, we present another argument which is more transparent and more concise than the one in [8], and we will prove that if the convexity of Ω is retained, the assumption ($7'$) can be taken out from the theorems in [7] mentioned above thoroughly.

For reasons of simplicity, here we only give the extension counterpart of [7, Theorem 3.1] ([7, Theorem 1.2, Theorem 2.1, and Theorem 2.2] can be handled in a similar way). Our main theorem is

Theorem 1.1. *Assume that f satisfies* (F_1), (F_2), (F_3), *and* $f(0) \geq 0$, *and that Ω is convex. Then there exists a solution u of* (1.1) *satisfying* $\max\limits_{\Omega} u > \beta$.

For proving Theorem 1.1, we will modify problem (1.1) by defining a sequence of functionals $\{J_n\}$ as in the proof of [6, Theorem 2.2] or as in [8]. But we adopt a new technique of truncation here which enables us to deal with the problem in a much simpler way. We will see that each J_n satisfies the P.S. condition and it, however, does not satisfy (J_*). Therefore, we can not use the Mountain Pass Lemma as in the proof of Theorem 2.2 in [6] to get a nontrivial critical point. Meanwhile, since y(F_*) is not assumed, the proof of Theorem 3.1 in [7], which depends on a priori estimates, no longer has any effect. Instead of using the Mountain Pass Lemma, we will study directly the gradient flow of J_n and prove that along a certain curve of the flow we can arrive at a nontrivial critical point u_n of J_n. Finally, we prove that $\|u_n\|_{C(\bar{\Omega})} \leq C$, which shows that u_n is a solution of (1.1) when n is large enough.

§2. Proof of Theorem 1.1

We will give the proof only in the case $N \geq 3$, the case $N = 2$ is treated by similar arguments.

Choose a number $M > 0$ such that $\inf\limits_{t \geq M} f(t) > \sup\limits_{0 \leq t \leq \beta} f(t)$, and choose a number $m > 0$ such that $f(t) + mt$ is strictly increasing on $[0,\ M]$; such numbers exist because $f(t)$ satisfies (F_1) and satisfies locally Lipschitz condition. The inner product of the Hilbert space $H_0^1(\Omega)$ is taken to be

$$(u,\ v) = \int_\Omega (\nabla u \cdot \nabla v + muv)dx, \quad u,\ v \in H_0^1(\Omega).$$

The norms of the spaces $H_0^1(\Omega)$ and $C_0^1(\bar{\Omega})$ are denoted by $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{C_0^1(\bar{\Omega})}$ respectively. Choose an increasing sequence $\{s_n\}$ with $s_1 > M$ and $s_n \to +\infty$ as $n \to +\infty$. Fix a number $\gamma:\ \lambda_1 < \gamma < \lambda_2$. Then we can define a sequence of approximate functionals as follows.

$$f_n(t) = \begin{cases} f(0), & t < 0, \\ f(t), & 0 \leq t \leq s_n, \\ f(s_n) + \gamma(t - s_n), & s_n < t, \end{cases}$$

$$F_n(t) = \int_0^t f_n(s)ds, \quad t \in R^1,$$

$$J_n(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - F_n(u)\right] dx, \quad u \in H_0^1(\Omega).$$

It is known that $J_n \in C^{2-0}(H_0^1(\Omega))$ and

$$J_n'(u) = u - (-\Delta + m)^{-1}(f_n(u) + mu), \quad u \in H_0^1(\Omega).$$

Lemma 2.1. *J_n satisfies the P.S. condition for each n.*

Proof. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in $H_0^1(\Omega)$ such that

$$C_1 \le J_n(u_j) \le C_2, \quad J_n'(u_j) \to 0 \text{ as } j \to +\infty. \tag{2.1}$$

In this proof, each C_i means a constant number independent of j. To accomplish the proof, we need only to get a uniform bound of u_j in $H_0^1(\Omega)$. Denote

$$u_j^+(x) = \max\{u_j(x),\ 0\}, \quad u_j^-(x) = \min\{u_j(x),\ 0\}, \quad u_j = v_j + w_j, \quad v_j \in V,\ w_j \in W,$$

where $V = \text{span}\{\phi\}$, $W = V^\perp$ is the orthogonal complement of V in $H_0^1(\Omega)$, and ϕ is the eigenfunction corresponding to λ_1. Note that

$$u_j = u_j^+ + u_j^-, \quad \int_\Omega \nabla v_j \cdot \nabla w_j = \int_\Omega v_j w_j = 0,$$
$$\int_\Omega v_j^2 = \frac{1}{\lambda_1}\int_\Omega |\nabla v_j|^2, \quad \int_\Omega w_j^2 \le \frac{1}{\lambda_2}\int_\Omega |\nabla w_j|^2$$

for $j = 1,\ 2,\ \cdots$. For any $u \in H_0^1(\Omega)$,

$$(J_n'(u_j), u) = \int_\Omega \nabla u_j \cdot \nabla u - \int_\Omega f_n(u_j)u. \tag{2.2}$$

Taking u to be u_j^- in (2.2), we have

$$\begin{aligned}(J_n'(u_j),\ u_j^-) &= \int_\Omega \nabla u_j \cdot \nabla u_j^- - \int_\Omega f_n(u_j)u_j^- = \int_\Omega |\nabla u_j^-|^2 - \int_\Omega f_n(u_j^-)u_j^- \\ &= \int_\Omega |\nabla u_j^-|^2 - \int_\Omega f_n(0)u_j^-.\end{aligned}$$

By (2.1), the Holder inequality, the Schwartz inequality, and the Sobolev inequality, it follows that

$$\int_\Omega |\nabla u_j^-|^2 \le C_3\|u_j^-\|_{L^2} + C_4 \le C_5\left(\int_\Omega |\nabla u_j^-|^2\right)^{1/2} + C_4.$$

Hence

$$\int_\Omega |\nabla u_j^-|^2 \le C_6. \tag{2.3}$$

Taking u to be v_j in (2.2), we have

$$\begin{aligned}(J_n'(u_j), v_j) &= \int_\Omega \nabla u_j \cdot \nabla v_j - \int_\Omega f_n(u_j)v_j = \int_\Omega |\nabla v_j|^2 - \int_\Omega f_n(u_j^+)v_j \\ &= \int_\Omega |\nabla v_j|^2 - \int_\Omega \gamma u_j^+ v_j - \int_\Omega (f_n(u_j^+) - \gamma u_j^+)v_j \\ &= \int_\Omega |\nabla v_j|^2 - \int_\Omega \gamma u_j v_j + \int_\Omega \gamma u_j^- v_j - \int_\Omega (f_n(u_j^+) - \gamma u_j^+)v_j \\ &= \Big(1 - \frac{\gamma}{\lambda_1}\Big)\int_\Omega |\nabla v_j|^2 + \int_\Omega \gamma u_j^- v_j - \int_\Omega (f_n(u_j^+) - \gamma u_j^+)v_j \\ &\le \Big(1 - \frac{\gamma}{\lambda_1}\Big)\int_\Omega |\nabla v_j|^2 + \gamma\|u_j^-\|_{L^2}\|v_j\|_{L^2} + C_7\|v_j\|_{L^2}.\end{aligned}$$

By (2.1) and (2.3), it follows that

$$-C_8\Big(\int_\Omega |\nabla v_j|^2\Big)^{1/2} \le \Big(1 - \frac{\gamma}{\lambda_1}\Big)\int_\Omega |\nabla v_j|^2 + C_9\|v_j\|_{L^2}.$$

In view of $\lambda_1 < \gamma$ we see that

$$\int_\Omega |\nabla v_j|^2 \le C_{10}. \tag{2.4}$$

Finally, taking u to be w_j in (2.1), we have in a similar way

$$\begin{aligned}(J_n'(u_j), w_j) &= \int_\Omega |\nabla w_j|^2 - \gamma \int_\Omega w_j^2 + \gamma \int_\Omega u_j^- w_j - \int_\Omega (f_n(u_j^+) - \gamma u_j^+) w_j \\ &\ge \Big(1 - \frac{\gamma}{\lambda_2}\Big) \int_\Omega |\nabla w_j|^2 - C_{11} \|w_j\|_{L^2}\end{aligned}$$

from which and the fact that $\gamma < \lambda_2$ we get

$$\int_\Omega |\nabla w_j|^2 \le C_{12}. \tag{2.5}$$

By (2.4) and (2.5) we get the boundedness of $u_j = v_j + w_j$ in $H_0^1(\Omega)$.

Lemma 2.2. *$f_n(t)$ satisfies the conditions* (F_1) *and* (F_2) *uniformly in n, that is to say,*

(i) *there exist $\delta > 0$ and $T > 0$ such that for $t \ge T$ and $n = 1, 2, \cdots$, $f_n(t)t^{-1} > \lambda_1 + \delta$;*

(ii) *for any $\varepsilon > 0$, there exists $T^* > 0$ such that for $t \ge T^*$ and $n = 1, 2, \cdots$,*

$$f_n(t)t^{-l} < \varepsilon.$$

Proof. (i) From (F_1) we can take $\delta > 0$ such that $\lambda_1 + \delta < \gamma$, $\lambda_1 + \delta < \liminf_{t\to+\infty} f(t)t^{-1}$. Take $T > 0$ such that for $t \ge T$, $f(t)t^{-1} > \lambda_1 + \delta$. Without loss of generality, we can assume that $s_n > T$ for all n. For $t \ge T$ and $n = 1, 2, \cdots$, if $t \ge s_n$, then

$$f_n(t)t^{-1} = f(s_n)t^{-1} + \gamma(1 - s_n t^{-1}) > (\lambda_1 + \delta)s_n t^{-1} + \gamma(1 - s_n t^{-1}) > \lambda_1 + \delta.$$

(ii) For any $\varepsilon > 0$, since $s_n \to +\infty$ as $n \to +\infty$, we can assume without loss of generality that $\gamma s_n^{1-l} < \varepsilon/2$ for all $n = 1, 2, \cdots$. From (F_2), there exists $T^* > 0$ such that for $t \ge T^*$, $f(t)t^{-l} < \varepsilon/2$. Without loss of generality, we can also assume that $s_n > T^*$. For $t \ge T^*$ and $n = 1, 2, \cdots$, if $t \ge s_n$, then

$$\begin{aligned}f_n(t)t^{-l} &= (f(s_n) + \gamma(t - s_n))t^{-l} < \Big(\frac{\varepsilon}{2} s_n^l + \gamma(t - s_n)\Big) t^{-l} \\ &< \frac{\varepsilon}{2} + \gamma t^{1-l} \le \frac{\varepsilon}{2} + \gamma s_n^{1-l} < \varepsilon.\end{aligned}$$

This finishes the proof.

Denote $A_n u = (-\Delta + m)^{-1}(f_n(u) + mu)$. Then A_n maps $H_0^1(\Omega)$ into $H_0^1(\Omega)$ and there exists a constant $L > 0$ such that

$$\|A_n u - A_n v\|_{H_0^1(\Omega)} \le L \|u - v\|_{H_0^1(\Omega)}, \quad u, v \in H_0^1(\Omega). \tag{2.6}$$

Indeed, such an L exists because $K \overset{\triangle}{=} (-\Delta + m)^{-1}$ is a bounded linear operator from $L^{\frac{2N}{N+2}}(\Omega)$ to $W^{2, \frac{2N}{N+2}}(\Omega) \cap W_0^{1, \frac{2N}{N+2}}(\Omega)$, because of the imbeddings $W^{2, \frac{2N}{N+2}}(\Omega) \cap W_0^{1, \frac{2N}{N+2}}(\Omega) \to H_0^1(\Omega)$ and $H_0^1(\Omega) \to L^{\frac{2N}{N+2}}(\Omega)$, and because $f_n(t) + mt$ satisfies Lipschitz condition on all of R^1.

Consider the initial value problem of ordinary differential equations in $H_0^1(\Omega)$

$$\begin{cases} \frac{du(t)}{dt} = -u(t) + A_n u(t), \\ u(0) = u_0. \end{cases} \tag{2.7}$$

Let $u(t, u_0)$ be the unique solution with maximal right interval of existence $[0, \eta(u_0))$.

Lemma 2.3. *For any* $u_0 \in H_0^1(\Omega)$, $\eta(u_0) = +\infty$.

Proof. For any $0 \le t_1 < t_2 < \eta(u_0)$, let h denote $t_2 - t_1$. Then by using (2.6) and (2.7) we have, for $0 \le t < \eta(u_0) - h$,

$$\begin{aligned}\|u(t+h,\ u_0) - u(t,\ u_0)\|_{H_0^1(\Omega)} &\le \int_t^{t+h} \|u'(s,\ u_0)\|_{H_0^1(\Omega)} ds \\ &\le (L+1)\int_t^{t+h} \|u(s,\ u_0)\|_{H_0^1(\Omega)} ds + \|A_n\theta\|_{H_0^1(\Omega)} h.\end{aligned}$$

Denote $K(t,\ h) = (L+1)\int_t^{t+h} \|u(s,\ u_0)\|_{H_0^1(\Omega)} ds + \|A_n\theta\|_{H_0^1(\Omega)} h$. Then, for $0 \le t < \eta(u_0) - h$,

$$|K_t'(t,\ h)| \le (L+1)\|u(t+h,\ u_0) - u(t,\ u_0)\|_{H_0^1(\Omega)} \le (L+1)K(t,\ h).$$

This implies that, for $0 \le t < \eta(u_0) - h$, $K(t,\ h) \le K(0,\ h)e^{(L+1)t}$. Hence we have, for $0 \le t < \eta(u_0) - h$,

$$\|u(t+h,\ u_0) - u(t,\ u_0)\|_{H_0^1(\Omega)} \le K(0,\ h)e^{(L+1)t}.$$

If $\eta(u_0) < +\infty$, choose $t = t_1$ in the last inequality, it would be that, for $0 \le t_1 < t_2 < \eta(u_0)$,

$$\|u(t_2,\ u_0) - u(t_1,\ u_0)\|_{H_0^1(\Omega)} \le K(0,\ t_2 - t_1)e^{(L+1)\eta(u_0)}.$$

Therefore

$$\lim_{t_1,\ t_2 \to \eta(u_0)-0} \|u(t_2,\ u_0) - u(t_1,\ u_0)\|_{H_0^1(\Omega)} = 0.$$

It follows that there exists $u^* \in H_0^1(\Omega)$ such that

$$\lim_{t \to \eta(u_0)-0} \|u(t,\ u_0) - u^*\|_{H_0^1(\Omega)} = 0.$$

Hence the solution curve $u(t,\ u_0)$ can be extended to $[0,\ \eta(u_0) + \eta(u^*))$. This contradicts the maximality of the interval $[0,\ \eta(u_0))$.

Now we regard A_n as an operator from $C_0^1(\bar\Omega)$ to $C_0^1(\bar\Omega)$. Because $K = (-\Delta + m)^{-1}$ is a bounded linear operator from $L^N(\Omega)$ to $W^{2,\ N}(\Omega) \cap W_0^{1,\ N}(\Omega)$, because of the imbeddings $W^{2,\ N}(\Omega) \cap W_0^{1,\ N}(\Omega) \to C_0^1(\bar\Omega)$ and $C_0^1(\bar\Omega) \to L^N(\Omega)$, and because $f_n(t) + mt$ satisfies Lipschitz condition on all of R^1, there exists a number $L_1 > 0$ such that

$$\|A_n u - A_n v\|_{C_0^1(\bar\Omega)} \le L_1\|u - v\|_{C_0^1(\bar\Omega)}, \quad u,\ v \in C_0^1(\bar\Omega). \tag{2.8}$$

Therefore, we can consider the initial value problem (2.7) in $C_0^1(\bar\Omega)$. For $u_0 \in C_0^1(\bar\Omega)$, let $\tilde u(t,\ u_0)$ be the unique solution of (2.7) considered in $C_0^1(\bar\Omega)$ with maximal right interval of existence $[0,\ \tilde\eta(u_0))$. By the same argument as in Lemma 2.2, we have the following lemma.

Lemma 2.4. *For any* $u_0 \in C_0^1(\bar\Omega)$, $\tilde\eta(u_0) = +\infty$.

Lemma 2.5. *For any* $u_0 \in C_0^1(\bar\Omega)$ *and* $0 \le t < +\infty$, *we have* $\tilde u(t,\ u_0) = u(t,\ u_0)$.

Proof. Because of the imbedding $C_0^1(\bar\Omega) \to H_0^1(\Omega)$, $\tilde u(t,\ u_0)$ is also a solution of (2.7) considered in $H_0^1(\Omega)$. Then the uniqueness of the solution in $H_0^1(\Omega)$ gives the result.

Let $D = \{u \in C_0^1(\bar\Omega) \mid 0 \le u(x) \le \beta \text{ for all } x \in \Omega\}$ and $P = \{u \in C_0^1(\bar\Omega) \mid 0 \le u(x) \text{ for all } x \in \Omega\}$. The interior part of D in P is $D^\circ = \{u \in C_0^1(\bar\Omega) \mid 0 \le u(x) < \beta \text{ for all } x \in \Omega\}$. Since $f_n(t) = f(t)$ for $0 \le t \le \beta$, $f(0) \ge 0$, $f(\beta) = 0$ and $f(t) + mt$ is strictly increasing on $[0,\ \beta]$, by Hopf's strong maximum principle we have the following lemma.

Lemma 2.6. *For each* n,

$$A_n(D) \subset D^\circ. \tag{2.9}$$

Proof. We have $A_n u = Au$ for $\forall u \in D$, since $s_1 > M > \beta$. For any $u \in D$, let $v = Au$. Note that

$$-\Delta v + mv = f(u) + mu \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

First, since $f(t) + mt$ is increasing in $[0,\ \beta]$, we see that

$$-\Delta v + mv \geq 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

By the maximum principle, we have $v \geq 0$ in Ω. Second, we have

$$-\Delta v + mv \leq f(\beta) + m\beta = m\beta \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Rewrite the last inequality in the following way

$$\Delta(\beta - v) - m(\beta - v) \leq 0 \quad \text{in } \Omega, \quad \beta - v = \beta \quad \text{on } \partial\Omega.$$

By Hopf's strong maximum principle, $\beta - v$ can not reach a non-positive value in Ω except that $\beta - v \equiv$constant. Hence $\min\limits_{\Omega}(\beta - v) > 0$ and $\max\limits_{\Omega} v < \beta$. Then we have $v \in D^\circ$.

Lemma 2.7. *If* $u_0 \in D$, $u(t,\ u_0) \in D^\circ$ *for all* $t > 0$.

Since $A_n(D) \subset D$ and $A_n(P) \subset P$, a result of [10] gives the following lemma.

Lemma 2.8. (i) *If* $u_0 \in D$, $u(t,\ u_0) \in D$ *for all* $t > 0$.

(ii) *If* $u_0 \in P$, $u(t,\ u_0) \in P$ *for all* $t > 0$.

Proof of Lemma 2.7. Since $u(t,\ u_0)$ is the solution of (2.7) in $C_0^1(\bar{\Omega})$, it satisfies

$$u(t, u_0) = e^{-t}u_0 + \int_0^t e^{-t+s} A_n u(s,\ u_0) ds, \quad 0 \leq t < +\infty. \tag{2.10}$$

For any $t > 0$, let $F_t = \{A_n u(s,\ u_0) \mid 0 \leq s \leq t\}$. Then F_t is a compact subset of $C_0^1(\bar{\Omega})$ since $A_n u(s,\ u_0)$ is continuous from $[0,\ t]$ to $C_0^1(\bar{\Omega})$. In view of (2.9) and Lemma 2.8 (i), we have $F_t \subset D^\circ$. Hence $\overline{\text{co}} F_t \subset D^\circ$, where $\overline{\text{co}} F_t$ is the closed convex hull of F_t in $C_0^1(\bar{\Omega})$. From (2.10), we see that

$$\begin{aligned} u(t,\ u_0) &= e^{-t}u_0 + e^{-t}\int_1^{e^t} A_n u(\ln s,\ u_0) ds \\ &= e^{-t}u_0 + (1 - e^{-t}) \lim_{m \to +\infty} \frac{1}{m} \sum_{i=1}^m A_n u\Big(\ln \Big(1 + \frac{i}{m}(e^t - 1)\Big),\ u_0\Big), \end{aligned}$$

where the integral and the limit are taken in the $C_0^1(\bar{\Omega})$ topology. Since $u_0 \in D$ and $t > 0$, and since

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{i=1}^m A_n u\Big(\ln \Big(1 + \frac{i}{m}(e^t - 1)\Big),\ u_0\Big) \in \overline{\text{co}} F_t \subset D^\circ,$$

we see that $u(t,\ u_0) \in D^\circ$. The proof is finished.

By Lemma 2.2, there exist constants $\alpha > 0$ and $t^* > 0$ independent of n such that

$$f_n(t) \geq (\lambda_1 + \alpha)t \quad \text{for all } t \geq t^* \text{ and } n = 1,\ 2,\ \cdots.$$

Therefore

$$F_n(t) \geq \frac{1}{2}(\lambda_1 + \alpha)t^2 - C_1 \quad \text{for all } t \geq 0 \text{ and } n = 1,\ 2,\ \cdots.$$

Here and in the sequel, we use C_i to denote a positive constant independent of n. Let ϕ be the eigenfunction of $-\Delta$ corresponding to λ_1 satisfying $\phi(x) > 0$ in Ω. For $t > 0$, we have

$$J_n(t\phi) = \int_\Omega \Big[\frac{1}{2}t^2|\nabla\phi|^2 - F_n(t\phi)\Big] dx \leq -\frac{\alpha}{2}t^2 \int_\Omega \phi^2 dx + C_1|\Omega|,$$

where $|\Omega|$ means the volume of Ω. Since $s_1 > M > \beta$, from the definition of J_n we see that $J_n(u) = J(u)$ for all $u \in D$ and $n = 1, 2, \cdots$. Therefore, there exists a number $T > 0$ independent of n such that

$$J_n(T\phi) < \inf_{u\in D} J_n(u) = \inf_{u\in D} J(u). \tag{2.11}$$

Define

$$\tau^* = \sup\{\tilde\tau \mid 0 \le \tilde\tau \le T \text{ and for any } 0 \le \tau \le \tilde\tau \text{ there exists } t_\tau > 0 \text{ such that } u(t_\tau, \tau\phi) \in D^\circ\}.$$

Lemma 2.9. *$0 < \tau^* < T$ and $u(t, \tau^*\phi) \notin D^\circ$ for all $t > 0$.*

Proof. Take a $\delta_0 :\ 0 < \delta_0 < T$ such that $\delta_0\phi \in D$. By Lemma 2.7, we have $u(t, \tau\phi) \in D^\circ$ for all $t > 0$ and all $0 \le \tau \le \delta_0$. On the other hand, (2.11) implies that $u(t, \tau\phi) \notin D^\circ$ for all $t > 0$ if $\tau < T$ and τ is sufficiently close to T, since J_n is decreasing along the gradient flow curve $u(t, \tau\phi)$. Therefore, $0 < \tau^* < T$.

By Lemma 2.8 (ii), the definition of τ^*, and the theory of ordinary differential equations, we see that $u(t, \tau^*\phi) \notin D^\circ$ for all $t > 0$.

Lemma 2.10. *For all $t \ge 0$,*

$$\inf_{u\in D} J(u) \le J_n(u(t, \tau^*\phi)) \le \sup_{0\le\tau\le T} J_n(\tau\phi).$$

Proof. The second inequality is valid since $0 < \tau^* < T$ and since J_n is decreasing along the curve $u(t, \tau^*\phi)$. The reason for the first inequality is as follows.

For any $t_0 \ge 0$ and any $\varepsilon > 0$, by the theory of ordinary differential equations, there exists a number $\tilde\tau :\ 0 < \tilde\tau < \tau^*$ such that

$$J_n(u(t_0, \tau^*\phi)) \ge J_n(u(t_0, \tilde\tau\phi)) - \varepsilon.$$

By the definition of τ^*, there is a $t_{\tilde\tau} > 0$ such that $u(t_{\tilde\tau}, \tilde\tau\phi) \in D^\circ$. Then Lemma 2.7 implies that $u(t, \tilde\tau\phi) \in D^\circ$ for all $t \ge t_{\tilde\tau}$. Hence we can take a number $t_1 > t_0$ such that $u(t_1, \tilde\tau\phi) \in D^\circ$. Since J_n is decreasing along $u(t, \tau^*\phi)$, we have

$$J_n(u(t_0, \tau^*\phi)) \ge J_n(u(t_0, \tilde\tau\phi)) - \varepsilon \ge J_n(u(t_1, \tilde\tau\phi)) - \varepsilon \ge \inf_{u\in D} J_n(u) - \varepsilon = \inf_{u\in D} J(u) - \varepsilon.$$

Let $\varepsilon \to 0+$, then we get the result.

Since T is independent of n, Lemma 2.10 implies that there are two constants C_2 and C_3 independent of n such that

$$C_2 \le J_n(u(t, \tau^*\phi)) \le C_3 \quad \text{for all } t \ge 0 \text{ and } n = 1, 2, \cdots. \tag{2.12}$$

Therefore, for a fixed integer n, there exists an increasing sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that

$$\frac{d}{dt} J_n(u(t, \tau^*\phi))|_{t=t_k} \to 0 \quad \text{as} \quad k \to +\infty.$$

That is,

$$\|J_n'(u(t_k, \tau^*\phi))\|^2_{H_0^1(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.13}$$

Since J_n satisfies the P.S. condition, (2.12) and (2.13) imply that $\{u(t_k, \tau^*\phi)\}$ has a convergent subsequence in the $H_0^1(\Omega)$ topology, which we also denote by $\{u(t_k, \tau^*\phi)\}$ without loss of generality. Hence, there is $u_n \in H_0^1(\Omega)$ such that

$$\|u(t_k, \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.14}$$

(2.13) and (2.14) imply that u_n is a critical point of J_n. Since f_n satisfies the Lipschitz condition, it follows that $u_n \in C^2(\bar{\Omega})$ and u_n satisfies

$$\begin{cases} -\Delta u_n = f_n(u_n), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases}$$

By maximum principle, $u_n \in P$. From (2.12) and (2.14), we have

$$C_2 \le J_n(u_n) \le C_3, \quad n = 1,\ 2,\ \cdots.$$

Using the same argument as in [6, Theorem 2.2] (see [6, p.61, Step 2]), we get that $\|u_n\|_{C(\bar{\Omega})} \le C_4$ for some number C_4 independent of n. Note that in this argument the result of Lemma 2.1 is necessary. Note also that in this argument the assumption that Ω is convex is crucial since in this case the Pohozaev identity

$$(N-2)\int_\Omega |\nabla u_n|^2 = 2N\int_\Omega F_n(u_n) - \int_{\partial\Omega} (x - x_0,\ n(x))|\nabla u_n|^2 ds$$

together with the boundedness of ∇u_n on $\partial\Omega$ plays important roles in getting a uniform bound of u_n in $H_0^1(\Omega)$ norm, while the boundedness of ∇u_n on $\partial\Omega$ is derived from the convexity of Ω.

Now, we are going to prove that $\max\limits_\Omega u_n > \beta$ for all n.

Lemma 2.11. *$\{u(t,\ \tau^*\phi) \mid t > 0\}$ is bounded in the $H_0^1(\Omega)$ norm.*

Proof. Since J_n satisfies the P.S. condition, there exist two numbers $R > 0$ and $\mu > 0$ such that $\|J_n'(u(t,\ \tau^*\phi))\|_{H_0^1(\Omega)} \ge \mu$ if $\|u(t,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \ge R$, $(\forall t > 0)$. Now two cases may occur.

Case (i) there exists $\tilde{t} > 0$ such that

$$\|u(t,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \le R \quad \text{for} \quad t > \tilde{t}.$$

The result is obvious.

Case (ii) such a number $\tilde{t}$ does not exist. In view of (2.14), we see that there exists a sequence of mutually disjoint intervals $\{[S_j,\ T_j]\}_{j=1}^\infty$ such that

$$\begin{aligned} &\|u(S_j,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} = \|u(T_j,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} = R, \quad j = 1,\ 2,\ 3,\ \cdots, \\ &\|u(t,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} > R \ \text{ if and only if } \ t \in (S_j,\ T_j) \ \text{ for some } \ j. \end{aligned}$$

If $t \in (S_j,\ T_j)$ for some j,

$$\begin{aligned} \|u(t,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} &\le \|u(t,\ \tau^*\phi) - u(S_j,\ \tau^*\phi)\|_{H_0^1(\Omega)} + \|u(S_j,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \\ &\le \int_{S_j}^t \|\frac{d}{ds}u(s,\ \tau^*\phi)\|_{H_0^1(\Omega)} ds + R \\ &\le \int_{S_j}^t \|J_n'(u(s,\ \tau^*\phi))\|_{H_0^1(\Omega)} ds + R \\ &\le (T_j - S_j)^{\frac{1}{2}} \Big(\int_{S_j}^t \|J_n'(u(s,\ \tau^*\phi))\|^2_{H_0^1(\Omega)} ds\Big)^{\frac{1}{2}} + R \\ &\le (T_j - S_j)^{\frac{1}{2}} \Big(-\int_{S_j}^t \frac{d}{ds} J_n(u(s,\ \tau^*\phi)) ds\Big)^{\frac{1}{2}} + R. \end{aligned}$$

Using (2.12), we see that, if $t \in (S_j,\ T_j)$ for some j,

$$\|u(t,\ \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \le (T_j - S_j)^{\frac{1}{2}} (C_3 - C_2)^{\frac{1}{2}} + R. \tag{2.15}$$

Meanwhile,

$$-\int_{\bigcup\limits_{j=1}^{\infty}[S_j,\ T_j]} \frac{d}{ds} J_n(u(s,\ \tau^*\phi))ds \geq \int_{\bigcup\limits_{j=1}^{\infty}[S_j,\ T_j]} \|J_n'(u(s,\ \tau^*\phi))\|^2_{H_0^1(\Omega)} ds \geq \mu^2 \sum_{j=1}^{\infty}(T_j - S_j).$$

Again using (2.12), we have

$$C_3 - C_2 \geq \mu^2 \sum_{j=1}^{\infty}(T_j - S_j). \tag{2.16}$$

Combining (2.15) and (2.16) leads us to, for $t \in (S_j,\ T_j)$ for some j,

$$\|u(t, \tau^*\phi) - u_n\|_{H_0^1(\Omega)} \leq \mu^{-1}(C_3 - C_2) + R.$$

This proves the result.

Lemma 2.12. *For any $\alpha \in (0,\ 1)$, $\{u(t,\ \tau^*\phi) \mid t > 0\}$ is bounded in the $C_0^{1,\ \alpha}(\bar{\Omega})$ norm.*

Proof. We will use an argument similar to the proof of Lemma 4.1 in [4]. Since $f_n(t)+mt$ satisfies $|f_n(t) + mt| \leq a(|t| + 1)$ for some constant a, A_n is a bounded linear operator from $L^p(\Omega)$ to $W^{2,\ p}(\Omega) \cap W_0^{1,\ p}(\Omega)$ for any $p > 1$ by the L^p theory of elliptic operators. Let $q = \frac{N}{1-\alpha}$ and choose an integer i such that $4i - 2 < N \leq 4(i + 1) - 2$. Then we have the following chain of Sobolev imbeddings and bounded linear operators

$$\begin{array}{ccccc}
H_0^1(\Omega) & \to & L^{\frac{2N}{N-2}}(\Omega) & \overset{A_n}{\to} & W^{2,\ \frac{2N}{N-2}}(\Omega) \cap W_0^{1,\ \frac{2N}{N-2}}(\Omega) \\
 & \to & L^{\frac{2N}{N-6}}(\Omega) & \overset{A_n}{\to} & W^{2,\ \frac{2N}{N-6}}(\Omega) \cap W_0^{1,\ \frac{2N}{N-6}}(\Omega) \\
\cdots & & \cdots & \cdots & \cdots \\
 & \to & L^{\frac{2N}{N-4i+2}}(\Omega) & \overset{A_n}{\to} & W^{2,\ \frac{2N}{N-4i+2}}(\Omega) \cap W_0^{1,\ \frac{2N}{N-4i+2}}(\Omega) \\
 & \to & L^q(\Omega) & \overset{A_n}{\to} & W^{2,\ q}(\Omega) \cap W_0^{1,\ q}(\Omega) \\
 & \to & C_0^{1,\ \alpha}(\bar{\Omega}). & &
\end{array} \tag{2.17}$$

For any $p > 1$ and any $\alpha \in (0,\ 1)$, since $f_n(t)$ satisfies the Lipschitz condition on all of R^1, A_n, when considered as an operator either from $L^p(\Omega)$ to $L^p(\Omega)$ or from $C_0^{1,\ \alpha}(\bar{\Omega})$ to $C_0^{1,\ \alpha}(\bar{\Omega})$, satisfies the Lipschitz condition uniformly. That is, in $L^p(\Omega)$ and $C_0^{1,\ \alpha}(\bar{\Omega})$, inequalities similar to (2.7) and (2.8) are also hold. Therefore, by the argument of Lemma 2.5, solutions of (2.7), when considered in these spaces, are all the same. Therefore (2.10) is valid when the integral in it is taken in $H_0^1(\Omega)$, in $L^p(\Omega)$, in $C_0^1(\bar{\Omega})$, and in $C_0^{1,\ \alpha}(\bar{\Omega})$.

These comments combined with (2.17) imply that, if $u_0 \in C_0^{1,\ \alpha}(\bar{\Omega})$ for some $\alpha \in (0,\ 1)$ and if $\{u(t,\ u_0) \mid t > 0\}$ is bounded in the $H_0^1(\Omega)$ norm, $\{u(t,\ u_0) \mid t > 0\}$ is bounded in turn in the $L^{\frac{2N}{N-2}}(\Omega)$ norm, $\cdots$, in the $L^q(\Omega)$ norm, and in the $C_0^{1,\ \alpha}(\bar{\Omega})$ norm. Then the result follows from Lemma 2.11.

In view of (2.14) and Lemma 2.12, there is a subsequence of $\{u(t_k,\ \tau^*\phi)\}$ (Arzela-Ascoli Theorem), which is also denoted by $\{u(t_k,\ \tau^*\phi)\}$, such that

$$\|u(t_k,\ \tau^*\phi) - u_n\|_{C_0^1(\bar{\Omega})} \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.18}$$

Now, Lemma 2.8 (ii), Lemma 2.9, and (2.18) together imply that $\max\limits_{\Omega} u_n \geq \beta$. The Hopf's strong maximum principle shows that $\max\limits_{\Omega} u_n > \beta$, since $f_n(\beta) = 0$. In fact, if $\max\limits_{\Omega} u_n = \beta$, then the same argument as in the proof of Lemma 2.6 shows that $\max\limits_{\Omega} u_n < \beta$, which is impossible.

Proof of Theorem 1.1. For a large n, denote u_n by u. Then u is a solution of (1.1) and satisfies $\max\limits_{\Omega} u > \beta$. The proof is finished.

Remark 2.1. The assumption that Ω is convex can be replaced with other conditions (see, e.g., [6] or [7]). In particular, Ω is permitted to be $\Omega = \Omega_1 - \bar{\Omega}_2$, where Ω_1 is strictly convex, Ω_2 is star-shaped and $\bar{\Omega}_2 \subset \Omega_1$.

Remark 2.2. If we assume that there is a positive supersolution $\psi(x)$ of (1.1) instead of assuming (F_3), a similar argument shows that (1.1) possesses a solution u satisfying $u(x_0) > \psi(x_0)$ for some $x_0 \in \Omega$.

§3. Some Multiplicity Results

Theorem 3.1. *Assume that f satisfies* (F_1), (F_2), (F_3)*, and* $f(0) > 0$*, and that Ω is convex. Then there exist two solutions u_1 and u_2 of* (1.1) *such that*

$$0 < u_1(x) < \beta < \max_{\Omega} u_2 \quad \text{for all} \ \ x \in \Omega.$$

Theorem 3.2. *We assume that f satisfies* (F_1), (F_2), (F_3)*, and*

(F_4) $f(0) = 0, \ \ \liminf_{t\to 0+} f(t)t^{-1} > \lambda_1,$

and that Ω is convex. Then there exist two solutions u_1 and u_2 of (1.1) *such that*

$$0 < u_1(x) < \beta < \max_{\Omega} u_2 \quad \textit{for all} \ \ x \in \Omega.$$

These two theorems are just combinations of known results with Theorem 1.1 (see, e.g., [7, Example 3.1 and Example 3.2]). In fact, define

$$Au = (-\Delta + m)^{-1}(f(u) + mu) \quad \text{as} \ \ u \in D.$$

Since $f(t) + mt$ is increasing in $t \in [0, \ \beta]$, A is an increasing operator from D to D. Define ψ as

$$-\Delta\psi + m\psi = m\beta \ \text{ in } \Omega, \quad \psi = 0 \ \text{ on } \partial\Omega.$$

Then $0 < \psi < \beta$ in Ω by Hopf's strong maximum principle and $A\psi \le \psi$. If either $f(0) > 0$ or (F_4) is satisfied, then $\delta\phi < A(\delta\phi)$ and $\delta\phi < \psi$ for $\delta > 0$ sufficiently small, where ϕ is the first eigenfunction. According to [1, Corollary 6.2], there exists u_1 satisfying $Au_1 = u_1$ and $0 < u_1(x) < \beta$ in Ω and u_1 is a solution of (1.1).

Consider the counterpart of (1.1)

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $\lambda > 0$ is a parameter.

Theorem 3.3. *Assume that f satisfies* (F_2), (F_3),

(F_5) $\lim_{t\to+\infty} f(t)t^{-1} = +\infty,$

(F_6) $f(0) = 0, \ \ f'(0) = 1,$

and that Ω is convex. Then, if α is the first zero of $f(t)$ in $(0, \ \beta]$ and λ^* is the infimum of all $\lambda > 0$ such that there exists a solution of $(1.1)_\lambda$ *less than α, we have*

(i) $0 < \lambda^* \le \lambda_1$ *and for $\lambda^* \le \lambda$ there exists a solution $u_{1,\ \lambda}$ of* $(1.1)_\lambda$ *which is the maximal one among all solutions of* $(1.1)_\lambda$ *less than α.*

(ii) $u_{1,\ \lambda}$ *is strictly increasing with respect to λ in Ω and $u_{1,\ \lambda}$ is continuous on the right with respect to λ from $[\lambda^*, \ +\infty)$ to $C^2(\bar{\Omega})$.*

(iii) *If* $f''(t) < 0$ *for* $0 \le t \le \alpha$, *then* $u_{1,\ \lambda}$ *is the unique solution of* $(1.1)_\lambda$ *less than* α *and* $u_{1,\ \lambda}$ *is continuous with respect to* λ *from* $[\lambda^*,\ +\infty)$ *to* $C^2(\bar{\Omega})$.

(iv) *For all* $\lambda > 0$, *there exists a solution* $u_{2,\ \lambda}$ *of* $(1.1)_\lambda$ *satisfying* $\max\limits_{\Omega} u_{2,\ \lambda} > \beta$.

Proof. (i) is due to [7] and (iv) is just Theorem 1.1.

(ii) For any $\lambda^* \le \lambda < \mu < +\infty$, since $f(t) > 0$ for $0 < t < \alpha$, $u_{1,\ \lambda}$ is a strict subsolution of $(1.1)_\mu$, while β is a strict supersolution of $(1.1)_\mu$. Therefore $u_{1,\ \mu} > u_{1,\ \lambda}$ in Ω (see [1]). We have

$$\lim_{\mu\to\lambda+} \|u_{1,\ \mu} - u_{1,\ \lambda}\|_{C^2(\bar{\Omega})} = 0,$$

since $\lim\limits_{\mu\to\lambda+} u_{1,\ \mu} \ge u_{1,\ \lambda}$, $\lim\limits_{\mu\to\lambda+} u_{1,\ \mu}$ is a solution of $(1.1)_\lambda$ less than α, and $u_{1,\ \lambda}$ is the maximal solution of $(1.1)_\lambda$ less than α.

(iii) If $f''(t) < 0$ for $0 \le t \le \alpha$, a conventional argument shows that $(1.1)_\lambda$ has a unique solution less than α. If

$$\lim_{\mu\to\lambda-} u_{1,\ \mu} \ne \lim_{\mu\to\lambda+} u_{1,\ \mu},$$

there would be at least two solutions $\lim\limits_{\mu\to\lambda-} u_{1,\ \mu}$ and $\lim\limits_{\mu\to\lambda+} u_{1,\ \mu}$ of $(1.1)_\lambda$ less than α, which is impossible.

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