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THE ACTION OF THE LINEAR CHARACTER GROUP ON THE SET OF NON-LINEAR IRREDUCIBLE CHARACTERS***

REN Yongcai* LI Xiangyang**

Abstract

Let G be a nonabelian finite group. Then $\mathrm{Irr}(G/G')$ is an abelian group under the multiplication of characters and acts on the set of non-linear irreducible characters of G via the multiplication of characters. The purpose of this paper is to establish some facts about the action of linear character group on non-linear irreducible characters and determine the structures of groups G for which either all the orbit kernels are trivial or the number of orbits is at most two. Using the established results on this action, it is very easy to classify groups G having at most three non-linear irreducible characters.

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§1. Notations and Basic Results

Let G be a finite nonabelian group. Then $\mathrm{Irr}(G/G')$ is an abelian group under the multiplication of characters and acts on the set of non-linear irreducible characters of G via the multiplication of characters. The purpose of this paper is to investigate this action. As an application of our theory, in the end of Section 3 we give the classification of groups having exactly three non-linear irreducible caracters.

All groups in the paper are finite. For a factor group T of a group G, we consider $\mathrm{Irr}(T)$ as a subset of $\mathrm{Irr}(G)$.

Notations.

$L(G) := \mathrm{Irr}(G/G')$.

$NL(G) := \mathrm{Irr}(G) - \mathrm{Irr}(G/G')$.

$\mathrm{Ob}(\chi)$: For $\chi \in NL(G), \mathrm{Ob}(\chi) := \{\lambda\chi|\lambda\in L(G)\}$; $\mathrm{Ob}(\chi)$ is called an $L(G)$-orbit of χ in $NL(G)$.

$\mathrm{Ob}_T(\chi)$: For a factor group T of G and $\chi \in NL(T) \subseteq NL(G)$, $\mathrm{Ob}_T(\chi) := \{\lambda\chi \mid \lambda \in L(T)\}$.

$\mathrm{Ob}(G) := \{\mathrm{Ob}(\chi)|\chi\in NL(G)\}$.

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*Department of Mathematics, Sichuan University, Chengdu 610064, China.

**Department of Basic Sciences, Chengdu Institute of Meteorology, Chengdu 610041, China.

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$\mathrm{Obl}(G) := (|\mathrm{Ob}(\chi_1)|, |\mathrm{Ob}(\chi_2)|, \cdots, |\mathrm{Ob}(\chi_n)|)$, where $\mathrm{Ob}(\chi_1), \mathrm{Ob}(\chi_2), \cdots, \mathrm{Ob}(\chi_n)$ are all of the distinct $L(G)$-orbits in $NL(G)$.

$m_G(N)$ and $m(N)$: For a normal subset N of G, $m_G(N)$ denotes the number of G-conjugacy classes contained in N. When no confusion occurs, we shall write $m(N)$ instead of $m_G(N)$.

$V_G(\chi)$ and $V(\chi)$: Let $\chi \in NL(G)$. $V_G(\chi) := \langle g \in G | \chi(g) \neq 0\rangle$. When no confusion occurs, we shall write $V(\chi)$ instead of $V_G(\chi)$.

$\ker(\mathrm{Ob}(\chi))$: For $\chi \in NL(G)$, $\ker(\mathrm{Ob}(\chi)) = \cap\{\ker\varphi | \varphi \in \mathrm{Ob}(\chi)\}$. $\ker(\mathrm{Ob}(\chi))$ is called the kernel of the $L(G)$-orbit $\mathrm{Ob}(\chi)$.

$\mathrm{Irr}^{\#}(G) := \mathrm{Irr}(G) - \{1_G\}$.

Except the definitions given above, the rest of our notations are standard and adapted from [1]. Throughout, all groups mentioned are assumed to be finite groups and all group characters are ordinary characters.

Now we establish some basic results. The proofs of Lemma 1.1, Lemma 1.2 and Theorem 1.3 are left as an exercise for the reader.

Lemma 1.1. *Let G be a nonabelian group, and let $\chi \in NL(G)$. Then,* $\ker(\mathrm{Ob}(\chi)) = \ker\chi \cap G' < G'$. *(In particular, $\cap\{\ker\chi | \chi \in NL(G)\} = 1$.)*

Lemma 1.2. *Let $G = T \times A$, where T is a nonabelian group and A is an ablian group. Then, $|\mathrm{Ob}(G)| = |\mathrm{Ob}(T)|$. Moreover, for each $\chi \in NL(G)$ there exists a $\mu \in NL(T)$ such that $|\mathrm{Ob}(\chi)| = |\mathrm{Ob}_T(\mu)||A|$.*

Theorem 1.1. *Let G be a nonabelian group. Then, $|\mathrm{Ob}(G)| = m(G') - 1$.*

Theorem 1.2. *Let G be a nonabelian group, and let $\chi \in NL(G)$. Then*

$$|\mathrm{Ob}(\chi)| = |V(\chi)/(G' \cap V(\chi))|.$$

Proof. Let $S = \{\lambda \in L(G) | \lambda\chi = \chi\}$, and let $B = \cap\{\ker\lambda | \lambda \in S\}$. Clearly, χ vanishes on $G - B$. Thus, if $\theta \in L(G)$ such that $B \leq \ker\theta$, then $\theta\chi = \chi$ and hence $\theta \in S$. So, we get $S = (B/G')^{\perp}$, that is, $S = \{\lambda | \lambda \in L(G)$ and $\lambda(B) = 1\}$. Thus, by [2, 6.4, p.488] we have

$$|\mathrm{Ob}(\chi)| = |L(G)/S| = |\mathrm{Irr}(G/G')/(B/G')^{\perp}| = |B/G'|.$$

Hence, in order to complete the proof, it remains to show that $V(\chi)G' = B$.

Since χ vanishes on $G - B$, we have $V(\chi) \leq B$, and so $V(\chi)G' \leq B$. Suppose $V(\chi)G' < B$. Let $S_1 = (V(\chi)G')/G')^{\perp}$. Then, by [2, 6.4, p.488] we obtain

$$|S_1| = |G/(V(\chi)G')| = |G/B||B/(V(\chi)G')| > |G/B| = |S|$$

and so $|S_1| > |S|$. For every $\lambda \in S_1$, $\lambda\chi$ and χ vanish on $G - V(\chi)G'$ and $(\lambda\chi)_{V(\chi)G'} = \chi_{V(\chi)G'}$. We therefore have $\lambda\chi = \chi$ for every $\lambda \in S_1$. This implies that $S_1 \subseteq S$ and thus $|S_1| \leq |S|$, a contradiction. So, we have $V(\chi)G' = B$. This proves the theorem.

Theorem 1.3. *Let $N \lhd G$ and $G' \not\leq N$. Then, the following statements hold*:

(1) $|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G/(N \cap G'))| \leq m(G') - m(N \cap G')$.

(2) *Let* $\mathrm{Obl}(G/(N \cap G')) = (a_1, a_2, \cdots, a_k)$, *and let* $t = |N/(N \cap G')|$. *Then,* $\mathrm{Obl}(G/N) = (a_1/t, \cdots, a_k/t)$.

(3) $|\mathrm{Ob}(G/N)| \leq |\mathrm{Ob}(G)|$, *and* $|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G)|$ *if and only if* $N \cap G' = 1$.

Proof . By Theorem 1.1 we have

$$|\mathrm{Ob}(G/N)| = m_{G/N}(G'N/N) - 1,$$
$$|\mathrm{Ob}(G/(N\cap G'))| = m_{G/(N\cap G')}(G'/(N\cap G')) - 1.$$

Note that $G'N/N$ and $G'/(N\cap G')$ are G-groups and the isomorphism $G'N/N\cong G'/(N\cap G')$ is a G-isomorphism. Hence

$$m_{G/N}(G'N/N) = m_{G/(N\cap G')}(G'/(N\cap G')).$$

It follows that

$$|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G/(N\cap G'))|.$$

It is obvious that $m(G') - m(N \cap G') \geq m_{G/N}(G'N/N) - 1$. Hence

$$|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G/(N\cap G'))| = m_{G/N}(G'N/N) - 1 \leq m(G') - m(N\cap G')$$

and this completes the proof of (1).

Now, let us prove (2). First, we assume that $N \cap G' = 1$. Then, $N \leq Z(G)$ and $t = |N|$. Let $\chi \in NL(G/N)$. Since $N \leq Z(G) \leq V(\chi)$, $V_{G/N}(\chi) = V(\chi)/N$. Hence, by Theorem 1.2, $|\mathrm{Ob}_{G/N}(\chi)| = |\mathrm{Ob}(\chi)|/|N| = |\mathrm{Ob}(\chi)|/t$. In addition, by (1), $|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G)|$. We therefore have

$$\mathrm{Obl}(G/N) = (a_1/|N|, a_2/|N|, \cdots, a_k/|N|) = (a_1/t, a_2/t, \cdots, a_k/t).$$

We now assume that $N \cap G' \neq 1$. We use the notation "$-$" for the group $G/(N \cap G')$. By the assumptions of (2), $\mathrm{Obl}(\overline{G}) = \mathrm{Obl}(G/(N \cap G')) = (a_1, a_2, \cdots, a_k)$ and $|\overline{N}| = |N/(N \cap G')| = t$. In addition, $\overline{N} \cap \overline{G'} = (N/(N \cap G')) \cap (G'/(N \cap G')) = 1$. So, applying the result of the above paragraph to the group $\overline{G} = |G/(N \cap G')|$, we obtain

$$\mathrm{Obl}(\overline{G}/\overline{N}) = (a_1/|\overline{N}|, a_2/|\overline{N}|, \cdots, a_k/|\overline{N}|) = (a_1/t, a_2/t, \cdots, a_k/t).$$

Then, since $G/N \cong \overline{G}/\overline{N}$ and this isomorphism is a G-isomorphism, we have $\mathrm{Obl}(G/N) = Obl(\overline{G}/\overline{N}) = (a_1/t, a_2/t, \cdots, a_k/t)$, completing the proof of (2).

Let χ_1, $\chi_2 \in NL(G/N)$. If $\mathrm{Ob}(\chi_1) = \mathrm{Ob}(\chi_2)$, then there exists $\lambda \in L(G)$ such that $\lambda\chi_1 = \chi_2$. It follows that $\lambda \in L(G/N)$ and $\mathrm{Ob}_{G/N}(\chi_1) = \mathrm{Ob}_{G/N}(\chi_2)$. So, the first part of (3) holds.

If $|\mathrm{Ob}(G/N)| = |\mathrm{Ob}(G)|$, then by Theorem 1.1 we have

$$m_{G/N}((G/N)') = m_{G/N}(G'N/N) = m(G'),$$

forcing that $N\cap G' = 1$. Clearly, this argument also shows that if $N\cap G' = 1$, then $|\mathrm{Ob}(G)| = |\mathrm{Ob}(G/N)|$. Hence, the second part of (3) is true. This completes the proof.

Let χ be a character of G, and let $g \in G$. The element g is called a zero point of χ if $\chi(g) = 0$. Clearly, for any non-linear irreducibe character χ of G, all the characters in the orbit $\mathrm{Ob}(\chi)$ have the same zero points. So, if $\chi(g) = 0$, we may say that g is a zero point of the orbit $\mathrm{Ob}(\chi)$. Note that if g is a zero point of the orbit $\mathrm{Ob}(\chi)$, then every conjugate element of g in G is also a zero point of the orbit $\mathrm{Ob}(\chi)$.

Theorem 1.4. *Let G be a nonabelian group. Then the following statements hold*:

(1) *For a given coset gG', if there are exactly n orbits* $\mathrm{Ob}(\chi_1), \cdots, \mathrm{Ob}(\chi_n)$ *which do not vanish on gG', then $m(gG') = n + 1$.*

(2) *For a given orbit* $\mathrm{Ob}(\chi)$, *the number of the cosets gG' on which χ does not vanish eqauls* $|V(\chi)G'/G'| = |\mathrm{Ob}(\chi)|$.

Proof. Assume that there are exactly n orbits $\mathrm{Ob}(\chi_1),\cdots,\mathrm{Ob}(\chi_n)$ which do not vanish on the coset gG'. Suppose that $m(gG')\geq n+2$, and let $g_1,\cdots,g_{n+2}$ be the representative elements of $n+2$ G-conjugacy classes contained in gG', respectively. In addition, put $\mu=-|G/G'|$, $\mu_i=|C_G(g_i)|+\mu$ for $i=1,2,\cdots,n+2$, and $t_i=|\mathrm{Ob}(\chi_i)|$ for $i=1,2,\cdots,n$. Then, by using the Second Orthogonality Relation we obtain that

$$\begin{pmatrix} \chi_1(g_1) & \chi_2(g_1) & \cdots & \chi_n(g_1) \\ \chi_1(g_2) & \chi_2(g_2) & \cdots & \chi_n(g_2) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_1(g_{n+2}) & \chi_2(g_{n+2}) & \cdots & \chi_n(g_{n+2}) \end{pmatrix} \cdot \begin{pmatrix} t_1\overline{\chi_1(g_1)} & t_1\overline{\chi_1(g_2)} & \cdots & t_1\overline{\chi_1(g_{n+2})} \\ t_2\overline{\chi_2(g_1)} & t_2\overline{\chi_2(g_2)} & \cdots & t_2\overline{\chi_2(g_{n+2})} \\ \vdots & \vdots & \ddots & \vdots \\ t_n\overline{\chi_n(g_1)} & t_n\overline{\chi_n(g_2)} & \cdots & t_n\overline{\chi_n(g_{n+2})} \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1 & \mu & \cdots & \mu \\ \mu & \mu_2 & \cdots & \mu \\ \vdots & \vdots & \ddots & \vdots \\ \mu & \mu & \cdots & \mu_{n+2} \end{pmatrix}.$$

Since $\mu_i\neq\mu$ for $i=1,2,\cdots,n+2$, it is easy to show that the rank of the matrix on the right hand of the above equality is not less than $n+1$. Hence, the rank of the first matrix on the left of the above equality is not less than $n+1$. However, this is impossible because any $n+1$ vectors in an n-dimenional vector space are linearly dependent. This proves $m(gG')\leq n+1$.

Now, put $m=|G/G'|$ and let $G=h_1G'\cup h_2G'\cup\cdots\cup h_mG'$. For each coset h_iG', let d_i be the number of the $L(G)$-orbits which do not vanish on h_iG'. Let $NL(G)=\bigcup\limits_{j=1}^{k}\mathrm{Ob}(\chi_j)$, where $\mathrm{Ob}(\chi_r)\neq\mathrm{Ob}(\chi_s)$ for $r\neq s$, and let c_j be the number of the cosets h_iG' on which χ_j does not vanish. By Theorem 1.2 we have $c_j\leq|\mathrm{Ob}(\chi_j)|$ for $j=1,2,\cdots k$. It is easy to see that $\sum\limits_{i=1}^{m}d_i=\sum\limits_{j=1}^{k}c_j$. Then, writing $Cl_G(G)$ to denote the number of conjugacy classes of G, by the result of the previous paragraph, we have

$$|Cl_G(G)|\leq\sum_{i=1}^{m}(d_i+1)=m+\sum_{j=1}^{k}c_j\leq|G/G'|+\sum_{j=1}^{k}|\mathrm{Ob}(\chi_j)|=|\mathrm{Irr}(G)|=|Cl_G(G)|.$$

It follows that $m(h_iG')=d_i+1$ for $i=1,2,\cdots,m$, and $c_j=|\mathrm{Ob}(\chi_j)|=|V(\chi_j)G'/G'|$ for $j=1,2,\cdots,k$. This establishes (1) and (2), completing the proof.

§2. The Case Where All the Orbit Kernels Are Trivial

In this section we consider a basic situation of the action of $L(G)$ on $NL(G)$, the situation that the kernel of every $L(G)$-orbit is trivial.

Lemma 2.1. *Let G be a nonabelian group. Then, the kernel of every $L(G)$-orbit is trivial if and only if G' is a minimal normal subgroup of G.*

Proof. Note that if $M\lhd G$ and $M<G'$, then M is contained in the kernel of some non-linear irreducible character of G. So, using Lemma 1.1 completes the proof.

Theorem 2.1. *Let G be a solvable nonabelian group with $|\mathrm{Ob}(G)|=n$. Assume that the kernel of every $L(G)$-orbit is trivial. Then, one of the following assertions is true*:

(1) *$G=PL$, where P is a normal Sylow p-subgroup of G for some prime p and L is an abelian p-complement in G. Moreover, P is abelian, $P=G'\times C_P(L)$ (hence $G=G'L\times C_P(L)$), G' is an elementary abelian minimal normal subgroup of G, and $G'L/C_L(G')$*

is a Frobenius group with the Frobenius kernel $G'C_L(G')/C_L(G') \cong G'$ and a Frobenius complement $L/C_L(G')$. (In particular, $Z(G) = C_P(L)C_L(P)$ and $|L/C_L(G')| = (|G'| - 1)/n$.)

$$\begin{aligned} &\text{c.d.}(G) = \text{c.d.}(G/Z(G)) = \{1, (|G'|-1)/n\}, \\ &\text{Obl}(G) = (\underbrace{|Z(G)|, |Z(G)|, \cdots, |Z(G)|}_{n}). \end{aligned}$$

(2) *$G = P \times A$, where P is a p-group for some prime p and A is an abelian p'-group, and $|P'| = p$. In particular, every $\chi \in NL(G)$ vanishes on $G - Z(G)$ and G is isoclinic to an extra-special p-group.*

$$\begin{aligned} &\text{c.d.}(G) = \{1, |G/Z(G)|^{1/2}\} = \{1, |P/Z(P)|^{1/2}\}, \\ &\text{Obl}(G) = (\underbrace{|Z(P)||A|/p, |Z(P)||A|/p, \cdots, |Z(P)||A|/p}_{n}). \end{aligned}$$

Proof. By the hypothesis and Lemma 2.1, G' is an elementary abelian minimal normal susbgroup of G. There are two cases to consider:

(a) G is not nilpotent.

In this case, , by [2, 3.7, p.270] we have $G = G'K$, $G' \cap K = 1$ and $Z(G) = C_K(G')$. It follows that $G'Z(G)/Z(G)$ is a unique minimal normal subgroup of the solvable group $G/Z(G)$. Clearly, $G/Z(G)$ is not nilpotent. So, according to [1, (12.3)], $G/Z(G)$ is a Frobenius group with the Frobenius kernel $(G/Z(G))' \cong G'$ and a Frobenius complement $K/Z(G) = K/C_K(G')$, all non-linear irreducible characters of $G/Z(G)$ have equal degree f and $|K/Z(G)| = f$. We therefore have $|(G/Z(G))'| - 1 = (m_{G/Z(G)}((G/Z(G))') - 1)f$. Noticing that $(G/Z(G))'$ is G-isomorphic to G', we get $|G'| - 1 = (m(G') - 1)f$, and hence by Theorem 1.3 we have $f = (|G'| - 1)/|\text{Ob}(G)|$. Since $G' \cap Z(G) = 1$ and $\text{Obl}(G/Z(G)) = (1, 1, \cdots, 1)$ (see [1, (6.34)] and [1, Problems (5.3), p.74]), by Theorem 1.3 we have $\text{Obl}(G) = (|Z(G)|, |Z(G)|, \cdots, |Z(G)|)$.

From Theorem 1.3(3) and the results in the above paragraph, we obtain $\text{c.d.}(G) = \text{c.d.}(G/Z(G)) = \{1, f\} = \{1, (|G'| - 1)/|\text{Ob}(G)|\} = \{1, (|G'| - 1)/n)\}$.

Since G' is an elementary abelian p-group for some prime p, G has a normal Sylow p-subgroup P and thus $G = PL$, where L is a p-complement in G. Clearly, we may assume that $K = C_P(L)L = C_P(L) \times L$, and hence $P = G'C_P(L)$. $P' = 1$; otherwise, $P' = G'$, $P = P'C_P(L) = C_P(L)$, and G would be nilpotent. Then, since $Z(G) = C_K(G')$, we get $Z(G) = C_P(L)C_L(G')$. This establishes (1).

(b) G is nilpotent.

In this case, $G = P \times A$, where P is a p-group for some prime p and A is an abelian p'-group, and $|G'| = |P'| = p$. It follows that xG' is a conjugacy class of G for every $x \in G - Z(G)$. So, by Theorem 1.4 we get that for every $\chi \in NL(G)$, χ vanishes on $G - Z(G)$, $|\text{Ob}(\chi)| = |Z(G)|/|G'| = |Z(P)||A|/p$ and $Z(\chi) = V(\chi) = Z(G)$. Then, by [1, (2.30)] we get $\chi(1) = |G/Z(G)|^{1/2}$ for every $\chi \in NL(G)$. Further, noticing that $|G'| = p$, by [3, Theorem 2.4, p.588] G is isoclinic to an extra-special p-group. This establishes (2), completing the proof of the theorem.

Corollary 2.1. *Let G be a nonabelian group. If $|\text{Ob}(G)| = 1$, then one of the following assertions is true*:

(1) *$G = PL$, where P is a normal Sylow p-subgroup of G for some prime p and L is*

an abelian p-complement in G. Moreover, P is abelian, $P = G' \times C_P(L)$ *(hence* $G = G'L \times C_P(L)$*),* G' *is an elementary abelian minimal normal subgroup of G and* $G'L/C_L(G')$ *is a doubly transitive Frobenius group with the Frobenius kernel* $G'C_L(G')/C_L(G') \cong G'$ *and a Frobenius complement* $L/C_L(G')$.

In addition, $|NL(G)| = |Z(G)| = |C_P(L)C_L(G')|$ *and* $\mathrm{c.d.}(G) = \{1, |G'| - 1\}$.

(2) $G = P \times A$, *where A is an abelian group of odd order and P is a 2-group with* $|P'| = 2$. *In particular, every* $\chi \in NL(G)$ *vanishes on* $G - Z(G)$ *and G is isoclinic to an extra-special 2-group.*

In addition, $|NL(G)| = |Z(G)|/|G'| = |Z(P)||A|/2$ *and* $\mathrm{c.d.}(G) = \{1, |P/Z(P)|^{1/2}\}$.

Proof. Since $|\mathrm{Ob}(G)| = 1$, $NL(G)$ itself is an $L(G)$-orbit. Hence, by virtue of Lemma 1.1 the kernel of the unique $L(G)$-orbit is 1. In addition, by Theorem 1.1 and [1, (12.6)], G is solvable with $m(G') = 2$. So, using Theorem 2.1 completes the proof.

By virtue of Theorem 1.1 and [4, Proposition 3.1(e), p.115], we obtain the following

Lemma 2.2. *Let* $1 < G' < G$. *Then* (G, G') *is a Camina pair with* $m(G') = n + 1$ *if and only if* $|NL(G)| = n$ *and the action of* $L(G)$ *on* $NL(G)$ *is trivial, that is,* $\lambda\chi = \chi$ *for every* $\lambda \in L(G)$ *and every* $\chi \in NL(G)$.

Using [5, Theorem (2.1), p.69], we get the following Theorem 2.2 and Theorem 2.3, the proofs of which are omitted in order to save space.

Theorem 2.2. *Let* $1 < G' < G$. *Suppose that* (G, G') *is a Camina pair with* $m(G') = 3$, *that is,* $|NL(G)| = 2$ *and the action of* $L(G)$ *on* $NL(G)$ *is trivial. Then G is one of the following types*:

(1) $G = G' \rtimes C$ *is a Frobenius group with the Frobenius kernel* G' *and a cyclic Froubenius complement C. Furthermore,* G' *is a minimal normal subgroup of G and is an elementary abelian p-group, where p is an odd prime, and* $|G'| - 1 = 2|C|$.

(2) *G is an extra-special* 3*-group.*

(3) $G = (C_3 \times C_3) \rtimes Q_8$ *is a Frobenius group with the Frobenius kernel* $C_3 \times C_3$ *(the elementary abelian group of order* 9*) and a Frobenius complement* Q_8 *(the qauternion group of order* 8*).*

Theorem 2.3. *Let* $1<G'<G$. *Suppose that* (G, G') *is a Camina pair with* $m(G') = 4$, *that is,* $|NL(G)| = 3$ *and the action of* $L(G)$ *on* $NL(G)$ *is trivial. Then G is one of the following types*:

(1) *G is a Frobenius group with the Frobenius kernel* G' *and a Frobenius complement L,* G' *is an elementary abelian minimal normal subgroup of G and* $|G'| - 1 = 3|L|$.

(2) *G is isomorphic to the normalizer of a Sylow* 2*-subgroup of the Suzuki simple group* $Sz(2^k)$, *where* $k > 1$ *is odd.*

(3) *G is a semi-extraspecial* 2*-group with* $|G'| = 4$. *(See* [6] *for the definition of semi-extraspecial p-groups.)*

Remark. From Corollary 2.1 and Theorem 2.2 we immediately obtain the classification of groups G with $|NL(G)| = 1$ or $|NL(G)| = 2$ (see [7] and [8]).

§3. The Case Where $|\mathrm{Ob}(G)|=2$

In this section we classify the groups G with $|\mathrm{Ob}(G)| = 2$. In order to save space, we omit the proofs of Lemma 3.1 and Lemma 3.2.

Lemma 3.1. *Let G be a non-nilpotent solvable group. Assume that the commutative subgroup G' of G is a p-group, G' is not a minimal normal subgroup of G and G' contains only one minimal normal subgroup M of G. Then*:

(1) $G = PL$, *where P is a normal Sylow p-subgroup of G and L is an abelian p-complement in G.*

(2) $P = G'C_P(L), [G', C_P(L)] \trianglelefteq G$ *and* $[G', C_P(L)] = 1$ *or* M.

(3) $[P, L] = G'$, $M \leq Z(P)$, $P' = 1$ *or* M.

(4) *If* $M \leq C_P(L)$, *then*:

(i) $C_P(L) \trianglelefteq G$ and $[G', C_P(L)] = 1$. In particular, $M \leq Z(G)$.

(ii) G' *is a special p-group with* $(G')' = M$.

(5) *If* $M \nleq Z(G)$, *then*:

(i) $Z(G) \cap G' = 1$ *and* $G' \cap C_P(L) = 1$. *In particular,* $C_P(L)$ *is abelian.*

(ii) $C_L(M) = C_L(G') = C_L(P) = C_L(G'/M) \leq Z(G)$.

(6) *If* $M \nleq Z(G)$ *and* $m(G') = 3$, *then*:

(i) $G'L/MC_L(G')$ *is a doubly transitive Frobenius group with the Frobenius kernel*
$$G'C_L(G')/MC_L(G') \cong G'/M$$
and a Frobrnius complement $LM/MC_L(G') \cong L/C_L(G')$.

(ii) $ML/C_L(M)$ *is a doubly transitive Frobenius group with the Frobenius kernel*
$$MC_L(M)/C_L(M) \cong M$$
and a Frobenius complement $L/C_L(M)$.

(iii) $|M| = p^n|$, $|G'| = p^{2n}$, *and* $|L/C_L(G')| = p^n - 1$.

(iv) $G'L/C_L(G')$ *is a Frobenius group with the Frobenius kernel* $G'C_L(G')/C_L(G') \cong G'$.

Lemma 3.2. *Let $x, y \in G - G'$, and let $\chi \in NL(G)$. Assume that* $xG' = yG' = Cl_G(x) \cup Cl_G(y)$ *and χ does not vanish on xG'. If χ is rational valued and $\chi(g) \geq -1$ for every $g \in G$, then either $\chi(x) = -1$ or $\chi(y) = -1$. Furthermore, if $\chi(y) = -1$, then* $|Cl_G(y)| = |G||G'|/(|G| + |G'||\mathrm{Ob}(\chi)|)$ *and* $|Cl_G(x)| = |G'|^2|\mathrm{Ob}(\chi)|/(|G| + |G'||\mathrm{Ob}(\chi)|)$.

Theorem 3.1. *Assume that G has no non-trivial abelian direct factors . If $|\mathrm{Ob}(G)| = 2$, then G is one of the following types*:

(1) *G' is an elementary abelian minimal normal subgroup of G of order p^n, $G = G'L$, where L is an abelian p-complement in G, $C_L(G') = Z(G)$, and $G/Z(G)$ is a Frobenius group with the Frobenius kernel $G'Z(G)/Z(G) \cong G'$ and a Frobenius complement $L/Z(G)$. Furthermore, $|G'| - 1 = 2|L/Z(G)|$.*

$$\text{c.d.}(G) = \{1, (|G'| - 1)/2\} = \{1, (p^n - 1)/2\},$$
$$\mathrm{Obl}(G) = (|Z(G)|, |Z(G)|) = (|C_L(G')|, |C_L(G')|).$$

(2) *G is a 3-group and $|G'| = 3$. In particular, G is isoclinic to an extra-special 3-group and every $\chi \in NL(G)$ vanishes on $G - Z(G)$,*

$$\text{c.d.}(G) = \{1, (|G/Z(G)|)^{1/2}\},$$
$$\mathrm{Obl}(G) = (|Z(G)/G'|, |Z(G)/G'|) = (|Z(G)|/3, |Z(G)|/3).$$

(3) *G is a 2-group with class 3, and $|G'| = 4$. (In this case, there are no general expressions for* c.d.(G) *and* Obl(G).)

(4) *$G' \cong Q_8$, the quaternion group of order 8, $G = PL$, where P is a normal Sylow 2-subgroup of G and L is a non-normal Sylow 3-subgroup of G. Moreover, $P = G'C_P(L)$,*

$[G', C_P(L)] = 1$, $G' \cap C_P(L) = G'' = (Q_8)'$ *and* $G/C_P(L)C_L(P) \cong A_4$, *where* A_4 *is the alternative group on 4 letters.*

$$\text{c.d.}(G) = \{1, 3, |P/Z(P)|^{1/2}\} = \{1, 3, 2|C_P(L)/Z(C_P(L))|)^{1/2}\},$$
$$\text{Obl}(G) = (|C_P(L)C_L(P)|/2, 3|Z(C_P(L))C_L(P)|/2) = (|C_P(L)C_L(P)|/2, 3|Z(G)|/2).$$

(5) G' *is an abelian* p*-group,* $G = PL$, *where* P *is a normal Sylow* p*-subgroup of* G *and* L *is an abelian* p*-complement in* G. *Moreover,* $P = G'C_P(L)$, $G' \cap C_P(L) = 1$, $1 < P' < G'$,

$$[G', C_P(L)] = P' \leq Z(P), \quad C_L(P') = C_L(G') = C_L(P) = C_L(G'/P') \leq Z(G),$$

$P'L/C_L(P')$ *is a doubly transitive Frobenius group with the Frobenius kernel*

$$P'C_L(P')/C_L(P') \cong P',$$

and $G'L/P'C_L(G')$ *is a doubly trnsitive Frobenius group with the Frobenius kernel*

$$G'C_L(G')/P'C_L(G') \cong G'/P'$$

and a Frobenius complement $P'L/P'C_L(G') \cong L/C_L(G')$.

The further information:

$|P'| = p^n$, $|G'| = p^{2n}$, $G'L/C_L(G')$ *is a Frobenius group of order* $p^{2n}(p^n - 1)$,

$$C_P(L)/(Z(P) \cap C_P(L)) = C_P(L)/(Z(G) \cap C_P(L))$$

is an elementary abelian group of order p^n,

$$Z(G) = (Z(P) \cap C_P(L)) \times C_L(P),$$

and $PZ(G)/Z(G)$ *is a special* p*-group of order* p^{3n}.

$$\text{c.d.}(G) = \{1, p^{n-1}, p^n(p^n - 1)\}, \quad \text{Obl}(G) = (p^n|Z(G)|, |Z(G)|).$$

(6) G'' *is elementary abelian with order* p^n, G' *is a Frobenius group of order* p^nq *with the Frobenius kernel* G'', $G = G''H$, $G'' \cap H = 1, G' \cap H = H'$, $|H'| = q$, *the action of* H *on the set* $G'' - \{1\}$ *is transitive, and* $H = H'L \times A$, *where* A *is an abelian* q*-group and* L *is a* q'*-group. Moreover,* $H'L/C_L(H')$ *is a doubly transitive Frobenius group of order* $q(q-1)$. *(Hence,* $Z(G) = \text{Cor}_G(H)$, *and* $|H| = |Z(H)|q(q-1) = |C_L(H')||A|q(q-1)$).

$$\text{c.d.}(G) = \{1, q-1, p^n - 1\},$$
$$\text{Obl}(G) = (|Z(H)|, |H|/(|G''| - 1)) = (|Z(H)|, |H|/(p^n - 1)).$$

The Further information:

(i) *If* $|H|/(p^n - 1) = |Z(G)|$, *then* $G/Z(G) = G''H/Z(G)$ *is a doubly transitive Frobenius group with the Frobenius kernel* $G''Z(G)/Z(G) \cong G''$ *and a Frobenius complement* $H/Z(G)$, *and* $\pi(LZ(G)/Z(G)) = \pi(Z(H)/Z(G))$.

(ii) *If* $|H|/(p^n - 1) > |Z(G)|$, *then there exists a positive integer* k *such that* $k < n$, $k|n$, *and* $q - 1 = \sum\limits_{i=0}^{n/k-1} p^{ik}$.

(7) G'' *is elementary abelian with order* p^n, G' *is a Frobenius group of order* $2p^n$ *with the Frobenius kernel* G'', $G = G''H$, $G'' \cap H = 1$, $G' \cap H = H'$, *and* $H = Q \times A$, *where* Q *is a* 2*-group and* A *is an abelian* $2'$*-group. Moreover, the action of* H *on the set* $G'' - \{1\}$ *is transitive.*

The further information:

$Z(G) = \mathrm{Cor}_G(H)$, $G' \cap Z(G) = 1$, $|H'| = |Q'| = 2$. *In addition, for the group* $\overline{G} = G/Z(G)$, $\overline{G} = \overline{G''}\overline{H}$ *is a doubly transitive Frobenius group with the Frobenius kernel* $\overline{G''} \cong G''$ *and a Frobenius complement* $\overline{H} = H/Z(G)$, *and* $\overline{H} = Q_8 \times \overline{A}$, *where* $\overline{A}$ *is a cyclic* $2'$*-group.*

$$\mathrm{c.d.}(G) = \{1, |G''| - 1, (|Q/Z(Q)|)^{1/2}\} = \{p^n - 1, (|Q/Z(Q)|)^{1/2}\},$$
$$\mathrm{Obl}(G) = (|H|/(|G''| - 1), |Z(H)|/2) = (|H|/(p^n - 1), |Z(H)|/2).$$

Proof. By virtue of Theorem 1.1 and [1, (12.6)], G is solvable and $m(G') = 3$. If G' is a minimal normal subgroup of G, then by Lemma 2.1 and Theorem 2.1 we get that G is either the type (1) or the type (2) since G has no non-trivial abelian direct factors by hypothesis. So, in what follows we assume that G' is not a minimal normal subgroup of G. Then, G' contains only one minimal normal subgroup M of G. Furthermore, G' is an abelian p-group, or a special p-group with $(G')' = M$, or else a Frobenius group with the Frobenius kernel $M = G''$. Clearly, $m(M) = 2$ and $m_{G/M}(G'/M) = 2$. Also, we note that when G is not nilpotent and G' is a p-group, we can apply Lemma 3.1 to G.

Case I G' is a p-group.

In this case, $G = PL$, where P is a normal Sylow p-subgroup of G and L is an abelian p-complement in G.

We first assume that $M \leq Z(G)$. Then, $|M| = 2$, so that $p = 2$ and P is a 2-group. It follows that G' is either an abelian 2-group or an extra-special 2-group with $(G')' = M$. We need to distinguish two subcases, depending on whether or not $L = 1$.

Subcase (i) $L = 1$, that is, $G = P$ is a 2-group.

Clearly, G'/M is a central subgroup of G/M of order 2, so that $|G'| = 4$. $[G', G] \neq 1$; otherwise $G' \leq Z(G)$ and thus $m(G') = 4$, a contradiction. Hence, $[G', G] = M$ and $[G, G, G, G] = [G', G, G] = [M, G] = 1$. So, G has class 3 and G is the type (3).

Subcase (ii) $L \neq 1$.

Since L is abelian and G has no non-trivial abelian direct factors by hypothesis, $G = PL$ is not nilpotent, and so we can apply Lemma 3.1 to G. Hence, G' is an extra-special 2-group with $(G')' = M$ (see Lemma 3.1(4)). In particular, $\exp(G') = 4$. Thus, G' has only one subgroup of order 2. It follows that $G' \cong Q_8$ (see [2, 8.2, p.310]), L is an abelian 3-group and $G/C_P(L)C_L(P) \cong A_4$ (see Lemma 3.1). Then, using Lemma 3.1, we conclude that G is the type (4). It remains to determine c.d.(G) and Obl(G).

Since $G/C_L(P) \sim G/C_P(L)C_L(P) \cong A_4$, $G/C_L(P)$ has an irreducible character χ_1 of degree 3. On the other hand, $|G/C_L(P)| = |A_4||C_P(L)| = 12|C_P(L)|$, and so $G/C_L(P)$ has a non-linear irreducible character χ_2 such that $3 \nmid \chi_2(1)$. Then, by [1, (6.19)] we have

$$\chi_2(1) \in \mathrm{c.d.}(G'C_P(L)C_L(P)/C_L(P)) = \mathrm{c.d.}(G'C_P(L)) = \mathrm{c.d.}(P).$$

$P' = M$ (see Lemma 3.1(3), (4)) is a minimal normal subgroup of P of order 2, so that by Lemma 2.1 and Theorem 2.1 we have $\chi_2(1) = (|P|/|Z(P)|)^{1/2}$. Hence

$$\mathrm{c.d.}(G) = \{1, 3, (|P/Z(P)|)^{1/2}\}.$$

Since $Z(P) = Z(C_P(L))$ and $G' \cap C_P(L) = M$ (see Lemma 3.1 (2), (4)), we have

$$\begin{aligned}|P/Z(P)| &= |G'C_P(L)/Z(C_P(L))| = |G'||C_P(L)|/|G' \cap C_P(L)||Z(C_P(L))| \\ &= |Q_8||C_P(L)|/|M||Z(C_P(L))| = 4|C_P(L)|/|Z(C_P(L))|.\end{aligned}$$

Thus

$$\mathrm{c.d.}(G) = \{1, 3, 2(|C_P(L)/Z(C_P(L)|)^{1/2}\}.$$

Since $G'\cap(C_P(L)C_L(P))=M$ and $G/C_P(L)C_L(P)\cong A_4$, noticing that $|\mathrm{Ob}_{A_4}(\chi_1)|=1$, by Theorem 1.3(2) we get $|\mathrm{Ob}_{G/M}(\chi_1)|=|C_P(L)C_L(P)/M|=|C_P(L)C_L(P)|/2$. Then, since $(G/M)/(G/M)'\cong G/G'$, $|\mathrm{Ob}(\chi_1)|=|\mathrm{Ob}_{G/M}(\chi_1)|=|C_P(L)C_L(P)|/2$. Let $|\mathrm{Ob}(\chi_2)|=s$. Then

$$\begin{aligned}|G|&=|G/G'|+(|C_P(L)C_L(P)|/2)\chi_1(1)^2+s\chi_2(1)^2\\&=|G/G'|+9|C_P(L)C_L(P)|/2+4s|C_P(L)/Z(C_P(L))|.\end{aligned}$$

It follows from this that $|\mathrm{Ob}(\chi_2)|=s=3|C_L(P)Z(C_P(L))|/2$. So, we have

$$\mathrm{Obl}(G)=(|\mathrm{Ob}(\chi_1)|,|\mathrm{Ob}(\chi_2)|)=(|C_P(L)C_L(P)|/2,3|C_L(P)Z(C_P(L))|/2).$$

Now we assume that $M\not\leq Z(G)$ and set $|M|=p^n$. Then, $Z(G)\cap G'=1$. Hence, G is not nilpotent and we can apply Lemma 3.1 to G. We have to consider the following two subcases for G'.

Subcase (i) G' is a special p-group with $(G')'=M$.

We claim that $C_P(L)\leq Z(G)$. On the contrary, we suppose that $C_P(L)\not\leq Z(G)$. Take $x\in C_P(L)-Z(G)$. For $g\in G'$, $x^g=x[x,g]\in xM\subseteq xZ(P)$ and $(C_P(x))^g=C_P(x^g)=C_P(x)$ (see Lemma 3.1(2),(3)). Thus, $C_P(x)\trianglelefteq G'C_P(L)L=G$ (see Lemma 3.1 (2), (5)). $G'\not\leq C_P(x)$; otherwise, $G=G'C_P(L)L=C_P(x)C_P(L)L$ and hence $x\in Z(G)$ (see Lemma 3.1(5)), contradicting the choice of x. It follows that $G'\cap C_P(x)=M$. Hence, we have $[g,x]\neq 1$ for every $g\in G'-M$. Then, since $P'=M\leq Z(P)$ and $|G'/M|=|M|=p^n$ (see Lemma 3.1(3), (6)), we have

$$(*)\qquad G'/M\cong M,\quad gM\to[g,x]\in M.$$

Note that $L/C_L(M)$ is a cyclic group of order p^n-1 and acts irreducibly on both $G'/M=G'/G''$ and $M=G''$ (see Lemma 3.1(6)). Thus, by virtue of the above isomorphism $(*)$ and [9, Exercises 19, p.215], $|L/C_L(M)|<p^n-1$, a contradiction. So, our asertion has been proved.

Since $G'\cap C_P(L)=1$ (see Lemma 3.1(5)) and $C_P(L)\leq Z(G)$, $G=G'C_P(L)L=G'L\times C_P(L)$, forcing $C_P(L)=1$ because G has no non-trivial abelian direct factors by hypothesis. Thus, $G=G'L$, and $G/C_L(G')$ is a Frobenius group with the Frobenius kernel $G'C_L(G')/C_L(G')$ (see Lemma 3.1(6)). In particular, $(G/C_L(G'),G'C_L(G')/C_L(G'))$ is a Camina pair. Then, since $m_{G/C_L(G')}(G'C_L(G')/C_L(G'))=m(G')=3$, G' is a minimal normal subgroup of G (see Theorem 2.2), a contradiction. Therefore, this subcase can not occur.

Subcase (ii) G' is an abelian p-group.

First, we note that the arguments in the above paragraph (Subcase (i)) show that P can not be abelian and $[G',C_P(L)]\neq 1$. Hence, $P'=M$ and $[G',C_P(L)]=M=P'$ (see Lemma 3.1(2), (3)).

Next, we prove the following statement: $|C_P(L)/(C_P(L)\cap Z(G))|=|C_P(L)/(C_P(L)\cap Z(P))|=|M|=p^n$, and $C_P(L)/(C_P(L)\cap Z(P))$ is elementary abelian.

Let $y\in C_P(L)L$ such that $x^y=x$ for some $x\in G'-M$. Since $C_P(L)L$ is abelian and the action of $C_P(L)L$ on the set $G'-M$ is transitive, $z^y=z$ for every $z\in G'-M$. Then, for every $t\in M$, since $tx\in G'-M$, we have that $t^yx=(tx)^y=tx$ and so $t^y=t$. Therefore $y\in C_{C_P(L)L}(G')=C_{C_P(L)}(G')\times C_L(G')$. This implies that

$$|G'-M|=|C_P(L)L|/|C_{C_P(L)}(G')\times C_L(G')|=|C_P(L)/C_{C_P(L)}(G')||L/C_L(G')|,$$

and thus $p^n(p^n-1) = |G'-M| = |C_P(L)/C_{C_P(L)}(G')|(p^n-1)$ (see Lemma 3.1(6)). It follows that $|C_P(L)/C_{C_P(L)}(G')| = p^n = |M|$. Note that $C_{C_P(L)}(G') = C_P(L)\cap Z(G) = C_P(L)\cap Z(P)$ (see Lemma 3.1(2), (5)). Thus, $|C_P(L)/(C_P(L)\cap Z(G))| = |C_P(L)/(C_P(L)\cap Z(P))| = p^n = |M|$. Since $[C_P(L),P] \le P' \le M \le Z(P)$ (see Lemma 3.1(3)), $[C_P(L)^p,P] = [C_P(L),P]^p \le M^p = 1$, that is, $C_P(L)^p \le Z(P)$. Then, since $C_P(L)$ is abelian (see Lemma 3.1(5)), $\Phi(C_P(L)) \le Z(P)$ and $C_P(L)/(C_P(L)\cap Z(P))$ is elementary abelian. This completes the proof of the above-mentioned statement.

Hence, using Lemma 3.1, we conclude that G is the type (5). (It is easy to see that $Z(G) = (Z(P)\cap C_P(L))\times C_L(P)$ and $PZ(G)/Z(G)$ is a special p-group of order p^{3n}.)

Below, we will determine c.d.(G) and $\mathrm{Obl}(G)$. For this, we will freely use the information about the structure of G, which is given in (5).

G has an irreducible character χ_1 of degree p^n-1 and $M \le \ker\chi_1$. Let χ_2 be an irreducible chracter of G such that $M \not\le \ker\chi_2$ (such a character χ_2 exisits, according to Lemma 1.1). Then, for any irreducible chracter θ of G' such that $M \not\le \ker\theta$, by [1, Problems (6.2), p.95] we have $\theta^G = e\sum\{\chi|\chi\in \mathrm{Ob}(\chi_2)\}$, where e is a positive integer. So, by virtue of the Clifford's theorem, $(\chi_2)_{G'} = e\sum\{\theta|\theta\in \mathrm{Irr}(G')-\mathrm{Irr}(G'/M)\}$. It follows from this that $\chi_2(1) = e(|G'|-|G'/M|) = ep^n(p^n-1)$. On the other hand, according to [1, (6.15)], $\chi_2(1)$ divides $|G/(G'(Z(P)\cap C_P(L))C_L(P))| = p^n(p^n-1)$. So, $\chi_2(1) = p^n(p^n-1)$, $e=1$ and $|\mathrm{Ob}(\chi_2)| = |G/G'|/(p^n(p^n-1)) = |C_P(L)L|/(p^n(p^n-1)) = |(Z(P)\cap C_P(L))C_L(P)|$. Hence $|G| = |G/G'| + |\mathrm{Ob}(\chi_1)|(p^n-1)^2 + |(Z(P)\cap C_P(L))C_P(L)|(p^n(p^n-1))^2$. It follows from this that $|\mathrm{Ob}(\chi_1)| = p^n|(Z(P)\cap C_P(L))C_L(P)|$. We therefore have

$$\text{c.d.}(G) = \{1, p^n-1, p^n(p^n-1)\},$$
$$\mathrm{Obl}(G) = \{p^n|(Z(P)\cap C_P(L))C_L(P)|, |(Z(P)\cap C_P(L))C_L(P)|\} = \{p^n|Z(G)|, |Z(G)|\}.$$

Case (II) G' is a Frobenius group with the Frobenius kernel $M = G''$.

In this case, $|G'| = p^nq$, where $p^n = |M|$. Let Q_1 be a Frobenius complement of G'. Then $G' = MQ_1$. By the Frattini argument, $G = MN_G(Q_1)$. Clearly, $N_G(Q_1) < G$. Set $H = N_G(Q_1)$. Then, $G = MH$, H is a maximal subgroup of G, $M\cap H = 1$, $H \cong G/M$, and $Z(G) = \mathrm{Cor}_G(H) := \cap\{H^x|x\in G\} = \cap\{H^x|x\in M\}$.

Note that the action of H on the set $M-\{1\}$ is transitive. Hence, for Hx and Hy, where $x,y\in M-\{1\}$, there exists an element $z\in H$ such that $Hxz = Hy$. This implies that G is doubly transitive on the set $\{Hx|x\in M\} = \{Hx|x\in G\}$. Thus, by [1, (5.17)] we have $(1_H)^G = 1_G + \chi_1$, where $\chi_1 \in \mathrm{Irr}(G)$. Clearly, χ_1 is a rational valued character, $\chi_1(g) \ge -1$ for every $g\in G$, and $\chi_1(1) = |M|-1 = p^n-1$. Note that $(1_H)^G(x) = 0$ for every $x\in M-\{1\}$. Hence, $\chi_1(x) = -1$ for every $x\in M-\{1\}$. Let $\chi_2 \in NL(H) = NL(G/M)$. Then, $\mathrm{Ob}(G) = \{\mathrm{Ob}(\chi_1), \mathrm{Ob}(\chi_2)\}$.

Since $H\cong G/M$ and $H/H' \cong (G/M)/(G/M)' \cong G/G', \mathrm{Ob}_H(\chi_2) = \mathrm{Ob}_{G/M}(\chi_2) = \mathrm{Ob}(\chi_2)$. Note that $m_H(H') = m_{G/M}(G'/M) = 2$. So, by Theorem 1.1, $|\mathrm{Ob}(H)| = |\mathrm{Ob}(G/M)| = 1$. Therefore, in view of Corollary 2.1 we have to consider the following two subcases.

Subcase (i) $H = QL$, where Q is a normal Sylow q-subgroups and L is an abelian q-complement in H. Moreover, $Q_1 = H' \le Q$, Q is abelian, $Q = Q_1\times C_Q(L)$, $Z(H) = C_Q(L)\times C_L(Q)$, $H'L/C_L(Q) = Q_1L/C_L(Q_1)$ is a doubly transitive Frobenius group of order $q(q-1)$, and $|H| = |Z(H)|q(q-1)$.

In this subcase, G is the type (6). It remains to determine c.d.(G) and $\mathrm{Obl}(G)$ and prove the statements (i) and (ii) given in (6).

We have that $|\mathrm{Ob}_H(\chi_2)| = |NL(H)| = |Z(H)|$ and $\chi_2(1) = q-1$ (see Corollary 2.1). Thus, $|\mathrm{Ob}(\chi_2)| = |\mathrm{Ob}_H(\chi_2)| = |Z(H)|$. Let $|\mathrm{Ob}(\chi_1)| = s$. Then

$$|G| = |G:G'| + |Z(H)|\chi_2(1)^2 + s\chi_1(1)^2 = |G:G'| + |Z(H)|(q-1)^2 + s(p^n-1)^2.$$

It follows that $|\mathrm{Ob}(\chi_1)| = s = |H|/(|M|-1) = |H|/(p^n-1)$. So, we have that

$$\text{c.d.}(G) = \{1, q-1, p^n-1\},$$
$$\mathrm{Obl}(G) = \{|Z(H)|, |H|/(|M|-1)\} = \{|Z(H)|, |H|/(p^n-1)\}.$$

Below, we will prove statements (i) and (ii) in (6). Note that $Z(G) = \mathrm{Cor}_G(H)$, $Z(G)\cap G' = 1$, $Z(H/Z(G)) = Z(H)/Z(G)$, and $Z(G/Z(G)) = 1$. So, by Theorem 1.3 we may assume that $Z(G) = 1$.

If $|H|/(p^n-1) = |Z(G)| = 1$, then $|H| = p^n - 1 = |M| - 1$ and hence $G = MH$ is a doubly transitive Frobenius group with a Frobenius complement $H = Q_1L \times C_Q(L)$. Hence, $C_Q(L) = 1$ and $Z(H) = C_Q(L) \times C_L(Q) = C_L(Q) = C_L(Q_1) > 1$ (see [2, 8.15, p.505; 8.18, p.506]). Also, we note that H is a Frobenius complement, while $H/Z(H) = Q_1L/C_L(Q_1)$ is a Frobenius group and $|Q_1| = q$. We therefore have that $\pi(L) = \pi(C_L(Q_1)) = \pi(Z(H))$ (see [2, 8.15(b), p.505]), completing the proof of the statement (i) in (6).

Now, we assume that $|H|/(p^n-1) > 1 = |Z(G)|$. For $x \in Z(H) - \{1\}$, $C_M(x) = 1$; otherwise, $C_M(x) = M$ and $x \in Z(G) = 1$, a contradiction. There exists an element $h \in H - Z(H)$ such that $C_M(h) \neq 1$; or else $G = MH$ would be a Frobenius group and $|H|/(p^n-1) = |H|/(|M|-1) \leq 1$, a contradiction. Hence, $\mathrm{Ob}(\chi_1)$ does not vanish on $G'h$, $\chi_1(1) \neq -1$, $|Cl_G(h)| = p^{n-k}q$, where k is a positive integer. Clearly, we may assume without loss of generality that $h \in L - Z(H)$. Then, since $\mathrm{Ob}(\chi_2)$ vanishes on $G - Z(H)G'$ (see Corollary 2.1), $\mathrm{Ob}(\chi_2)$ vanishes on $G'h$. Hence, $m(G'h) = 2$ (see Theorem 1.4), and thus by Lemma 3.2 we get that $p^{n-k}q = |Cl_G(h)| = |G'|^2|\mathrm{Ob}(\chi_1)|/(|G| + |G'||\mathrm{Ob}(\chi_1)|)$. It follows that $q = (p^n-1)/(p^k-1)$, $k < n$, $k|n$, and $q - 1 = \sum\limits_{i=0}^{n/k-1} p^{ik}$. This proves the statement (ii) in (6).

Subcase (ii) $q = 2$ and $H = Q \times A$, where A is an abelian $2'$-group and Q is a 2-group with $|Q'| = |H'| = |Q_1| = 2$. (In particular, $H' = Q_1 \leq Z(H)$.)

In this subcase, $\chi_2(1) = (|H/Z(H)|)^{1/2} = (|Q/Z(Q)|)^{1/2}$ and $|\mathrm{Ob}(\chi_2)| = |\mathrm{Ob}_H(\chi_2)| = |Z(H)|/2$ (see Corollary 2.1). Then, since $\chi_1(1) = p^n - 1 = |M| - 1$, $|\mathrm{Ob}(\chi_1)| = |H|/(|M| - 1) = |H|/(p^n-1)$.

Now, we determine the structure of $G/Z(G) = G/\mathrm{Cor}_G(H)$. As mentioned in Subcase (i), we may assume that $Z(G) = \mathrm{Cor}_G(H) = 1$. Then, $C_G(M) = M$ (see [2, 3.2, p.159]), $C_H(M) = 1$.

First of all, we show that χ_1 vanishes on $G - M$. Clearly, $MZ(H)$ is a Frobenius group with the Frobnenius kernel M and a Frobenius complement $Z(H)$. Then, since $\chi_1(m) = -1$ for every $m \in M - \{1\}$, the irreducible constituents of $(\chi_1)_{MZ(H)}$ are not linear, and so χ_1 vanishes on $MZ(H) - M$ (see [1, (6.34)]). Therefore, in order to show that χ_1 vanishes on $G - M$, we need only to show that χ_1 vanishes on $G'h = MH'h$ for every $h \in H - Z(H)$,

Suppose that χ_1 does not vanish on $G'h$ for some $h \in H - Z(H)$. Then, since $\mathrm{Ob}(\chi_2)$

vanishes on $G'h$ (see Corollary 2.1(2)), there exists a G-conjugacy class C in $G'h$ such that

$$|C| = |G'|^2|\mathrm{Ob}(\chi_1)|/(|G| + |G'||\mathrm{Ob}(\chi_1)|) = \frac{4p^{2n}}{p^{2n}+1}$$

(see Lemma 3.2 and Theorem 1.4). It follows from the above equality that $p^n + 1 = 4$, so that $n = 1$ and $p = 3$. Thus, M is a cyclic group of order 3 and this implies that $|H| = |H/C_H(M)| = 2$ and $|H'| = |Q'| = 1$, a contradiction. Hence, χ_1 vanishes on $G'h$ for every $h \in H - Z(H)$. So, as mentioned in the previous paragraph, we have proved that χ_1 vanishes on $G - M$.

Then, since $\chi_1(1) = p^n - 1$ and $\chi_1(m) = -1$ for every $m \in M - \{1\}$, by [1, (2.29)] we have that

$$|H| = [(\chi_1)_M, (\chi_1)_M] = 1/|M| \sum_{m\in M} \chi_1(m)^2 = ((p^n - 1)^2 + (p^n - 1))/p^n = p^n - 1 = |M| - 1.$$

Hence, $G = MH$ is a doubly transitive Frobenius group with the Frobenius kernel M and a Frobenius complement H. It follows that $Q \cong Q_8$ and A is a cyclic $2'$-group. Thus, G is the type (7). This completes the proof of the theorem.

Corollary 3.1. *If G has exactly three non-linear irreducible characters, then G is one of the following types*:

(1) *$G = G'L$ is a Frobenius group with the Frobenius kernel G' and a cyclic Frobenius complement L, G' is an elementary abelian minimal normal subgroup of G and $|G'| - 1 = 3|L|$.*

(2) *G is isomorphic to the normalizer of a Sylow 2-subgroup of the Suzuki simple group $Sz(2^k)$, where $k > 1$ is odd.*

(3) *G is a semi-extrspecial 2-group with $|G'| = 4$.*

(4) *$G = G' \rtimes L$, $|G'| = p^n$, L is a cyclic group of order $3(p^n - 1)$, and $G/C_L(G')$ is a doubly transitive Frobenius group with the Frobenius kernel $G'C_L(G')/C_L(G') \cong G'$ and a Frobenius complement $L/C_L(G')$.*

(5) *$G = G'L \times C_3$, where C_3 is a cyclic group of order 3 and $G'L$ is a doubly transitive Frobenius group with the Frobenius kernel G' and a Frobenius complement L.*

(6) *$G = P \times C_3$, where P is an extra-special 2-group and C_3 is a cyclic group of order 3.*

(7) *G is a dihedral group of order 16.*

(8) *G is a semidihedral group of order 16.*

(9) *G is a generalized quaternion group of order 16.*

(10) *$|G| = 2^5$. G is the 44th group in the 51 groups of order 2^5 given in [10].*

(11) *$|G| = 2^5$. G is the 45th group in the 51 groups of order 2^5 given in [10].*

(12) *$|G| = 2^5$. G is the 46th group in the 51 groups of order 2^5 given in [10].*

(13) *$|G| = 2^5$. G is the 47th group in the 51 groups of order 2^5 given in [10].*

(14) *$|G| = 2^5$. G is the 48th group in the 51 groups of order 2^5 given in [10].*

(15) *$G = G''H$ is a doubly transitive Frobenius group with the Frobenius kernel G'' and a Frobenius complement H, $H = H'L$, $H' \cap L = 1$, $|H'| = q$, $|Z(H)| = 2$, and L is a 2-group. Moreover, $H/Z(H)$ is a doubly transitive Frobenius group with the Frobenisu kernel $Z(H)H'/Z(H) \cong H'$.*

(16) *$G \cong S_4$.*

Proof. Set $NL(G) = \{\chi_1, \chi_2, \chi_3\}$. Wwe have $|\mathrm{Ob}(G)| \leq 3$. If $|\mathrm{Ob}(G)| = 3$, then G is one of the types (1)–(3) (see Theorem 2.3). If $|\mathrm{Ob}(G)| = 1$, then G is one of the types

(4)–(6) (see Corollary 2.1).

Now, let us assume that $\mathrm{Ob}(G) = 2$. Then, $\mathrm{Obl}(G) = (1, 2)$ and G has no non-trivial abelian direct fators (see Lemma 1.2). So, we can apply Theorem 3.1 to G. We may assume without loss of generality that $\mathrm{Ob}(\chi_1) = \{\chi_1, \chi_2\}$ and $\mathrm{Ob}(\chi_3) = \{\chi_3\}$. Then, χ_3 vanishes on $G - G'$. Since $\sum \mathrm{Obl}(G) = 3$, by using the information about lengths of orbits given in Theorem 3.1, we see that G can not be (1), (2), (4) and (5) of the types given in Theorem 3.1.

Further, we claim that G is not the type (7) in Theorem 3.1. In fact, if G is the type (7) in Theorem 3.1, then, using the notation in Theorem 3.1(7), H is a 2-group and $p^n - 1 = 2^t$ for some positive integer t. It follows that $n = 1$ and G'' is of order p. This implies $H' \leq C_H(G'')$, a contradiction.

Now, we assume that G is the type (3) in Theorem 3.1. Then, noticing that χ_3 vanishes on $G - G'$, we have $Z(G) < G'$ and $|Z(G)| = 2$. Furthermore, by [11, Theorem 1.3, p.1452], $|G| = 2^4$ or 2^5. If $|G| = 2^4$, by [2, 11.9(a), p.339] G is one of the types (7)–(9). If $|G| = 2^5$, then, using the notaion of [10], G is a stem group of one of the families Γ_6 and Γ_7 (see the proof of Theorem 4.2 of [10, pp.18–19]). Hence, by [10, p.55 and p.59] G is one of the types (10)–(14).

Finally, we assume that G is the type (6) in Theorem 3.1. Then, using the notation in Theorem 3.1(6), one of the following two cases occurs:

(i) $|Z(H)| = 2$ and $|H|/(p^n - 1) = 1$. (ii) $Z(H) = 1$ and $|H|/(p^n - 1) = 2$.

Note that we have $Z(G) = 1$ because $Z(G) < Z(H)$ and $|Z(H)| \leq 2$. Thus, for the case (i), G is the type (15) by the statement (i) in Theorem 3.1(6). For the case (ii), by the statement (ii) in Theorem 3.1(6) we get that $q - 1 = \sum\limits_{i=1}^{n/k-1} p^{ik}$, where k is a positive integer such that $k < m$ and $k|m$. Hence

$$2 = |H|/(P^n - 1) = q(q-1)/(p^n - 1) = q\Big(\sum_{i=1}^{n/k-1} p^{ik}\Big)/(p^n - 1) = qp^k\Big(\sum_{i=0}^{n/k-2} p^{ik}\Big)/(p^n - 1),$$

forcing that $p = 2$, $n = 2$, and $q = 3$. It follows that $|G| = p^n q(q-1) = 24$ and $G \cong S_4$, the type (16). This completes the proof.

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