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SOME REMARKS ON SINGULAR DARBOUX TRANSFORMATIONS FOR UNITONS**

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Abstract

In the paper [2] the authors defined the singular Darboux transformations and established an explicit formula for constructing unitons from a simply connected Riemann surface M to the group $U(N)$. The formula is obtained as a limit of an infinite consequence of Darboux transformations through some renormalization procedure. In the present paper the authors give a complete proof of the fact that the formula gives a global solution of harmonic maps without singularity.

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§1. Introduction

Harmonic maps from Riemann surfaces to Lie groups and symmetric spaces have received much attention recently[1−5]. An important class of harmonic maps called unitons is introduced by Uhlenbeck[1]. In [2,3], Darboux transformation method is introduced, and the explicit expressions for Darboux matrices are used to obtain new harmonic maps from a known one. By introducing the singular Darboux transformations and making use of a kind of renormalization procedure, the authors present a purely algebraic explicit method to construct harmonic maps and unitons. Particularly, the singular Darboux transformation realizes the singular Bäcklund transformation proposed in [1] by an explicit algebraic algorithm. The singular Bäcklund transformation is also called flag transformation by some authors. This method is used in [5] to obtain many flag transformations which led to some new factorizations for $U(N)$-unitons.

In [3], the procedure of renormalization is proved to be convergent pointwisely. In the present note we prove that the limiting procedure converges to harmonic maps without singularities.

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§2. Harmonic Maps and Extended Harmonic Maps

Consider a $C^\infty-$map $\phi : M^2 \to U_n$ from a simply connected compact Riemann surface. Define

$$A = \frac{\partial \phi}{\partial \overline{z}}\phi^{-1} = \phi_{\overline{z}}\phi^{-1}, \quad B = \phi_z\phi^{-1}, \tag{2.1}$$

where z is local complex isothermal coordinate. Then A and B satisfies

$$A_z - B_{\overline{z}} + [A, B] = 0 \tag{2.2}$$

and

$$A^* = -B. \tag{2.3}$$

Moreover if A and B satisfy

$$A_z + B_{\overline{z}} = 0 \tag{2.4}$$

then ϕ is called harmonic.

The Lax pair of harmonic maps from M^2 to U_n is

$$\Phi_{\overline{z}} = \frac{1-\mu}{2}A\Phi, \quad \Phi_z = \frac{1-\mu^{-1}}{2}B\Phi, \quad \mu \in C^* = C\setminus\{0\}. \tag{2.5}$$

The integrability conditions of (2.5) are (2.2) and (2.4). A non-degenerated $n \times n$ matrix solution Φ of (2.5) is called an extended harmonic map. It can be normalized so that

$$\Phi(1) = I, \quad \Phi^*(\sigma(\mu))\Phi(\mu) = I, \tag{2.6}$$

where $\sigma(\mu) = \overline{\mu}^{-1}$. It follows that an extended harmonic map Φ implies a map from M^2 to the loop group ΩU_n. Hence, if Φ is an extended harmonic map, then $\Phi(-1) : M \to U_n$ is harmonic. Conversely, if $\phi : M \to U_n$ is harmonic, and if M is simply connected, then there exists an extended harmonic map Φ such that $\phi = \Phi(-1)$. This Φ is unique up to multiplication on the right by an element $\gamma \in \Omega U_n$ such that $\gamma(-1) = I$.

Darboux transformation is a constructive method for obtaining new extended harmonic maps (and hence new harmonic maps) from a known extended harmonic map via purely algebraic algorithm.

Let ε be a complex number satisfying $\varepsilon \neq 0, |\varepsilon| \neq 1$, L_1 and L_2 be $N\times k$ and $N\times(N-k)$ constant matrices respectively such that $L_2^*L_1 = 0$. Let π_0 be the Hermitian projection on the constant subspace $V := \text{span}\{L_2\} \subset \mathbb{C}^n$. Set

$$\gamma_\varepsilon(\mu) = \frac{(\overline{\varepsilon}-1)(\mu-\varepsilon)}{(1-\varepsilon)(\overline{\varepsilon}\mu-1)}. \tag{2.7}$$

It has been proved in [2,3] that

$$\Phi^{(\varepsilon)}_{(\mu)} = (\pi_\varepsilon + \gamma_\varepsilon\pi_\varepsilon^\perp)\Phi(\mu)(\pi_0 + \gamma_\varepsilon^{-1}\pi_0^\perp) \tag{2.8}$$

is a new extended solution, where π_ε and $\pi_\varepsilon^\perp$ are Hermitian projections on span$\{\ \Phi(\sigma(\varepsilon))L_2\}$ and span$\{\Phi(\varepsilon)L_1\}$ respectively. We can calculate π_ε and $\pi_\varepsilon^\perp$ explicitly

$$\begin{aligned}\pi_\varepsilon &= \Phi(\sigma(\varepsilon))L_2(L_2^*\Phi^*(\sigma(\varepsilon))\Phi(\sigma(\varepsilon))L_2)^{-1}L_2^*\Phi^*(\sigma(\varepsilon)),\\ \pi_\varepsilon^\perp &= \Phi(\varepsilon)L_1(L_1^*\Phi^*(\varepsilon)\Phi(\varepsilon)L_1)^{-1}L_1^*\Phi^*(\varepsilon).\end{aligned} \tag{2.9}$$

The transformation $\Phi(\mu) \to \Phi^{(\varepsilon)}(\mu)$ (or $\phi \to \phi^{(\varepsilon)} = \Phi^{(\varepsilon)}_{-1}$) is called a Darboux transformation.

A uniton is a harmonic map $\phi : M \to U_n$ which has an algebraic extended harmonic map $\Phi : M \to \Omega_{\rm alg}U_n$. In this case, Φ is called an extended uniton. An extended uniton Φ can be expressed as the following type up to multiplication on the right by a constant loop $\rho \in \Omega_{\rm alg}U_n$:

$$\Phi = \sum_{\alpha=0}^{m} \mu^\alpha T_\alpha. \tag{2.10}$$

§3. Regularity of Singular Darboux Transformation

Let $\Phi(\mu)$ be an extended uniton. We construct its Darboux transformation (2.8). Thus we have a family of extended untions $\Phi^{(\varepsilon)}(\mu)$. Let us recall the renormalization process of [3] in treating the limit $\Phi^{(\varepsilon)}(\mu)$ as $\varepsilon \to 0$. The expansion of $\Phi(\varepsilon)L_1$ as a polynomial of ε is denoted by

$$\Phi(\varepsilon)L_1 = X^0 = X^0_0 + X^0_1\varepsilon + \cdots + X^0_n\varepsilon^n. \tag{3.1}$$

Here $X^0_0, \cdots, X^0_n$ are $N\times k$ matrices. Choose some columns of X^0 to constitute a submatrix of X^0

$$\widetilde{X}^1 = \widetilde{X}^1_0 + \widetilde{X}^1_1\varepsilon \cdots + \widetilde{X}^n_1\varepsilon \tag{3.2}$$

such that $\widetilde{X}^1_0$ consists of linearly independent columns which span the matrix X^0_0. The other columns, after substracting suitable linear combinations of the columns of $\widetilde{X}^1$ from them, constitute a matrix

$$\varepsilon\widetilde{\widetilde{X}}^1 = \varepsilon(\widetilde{\widetilde{X}}^1_0 + \widetilde{\widetilde{X}}^1_1\varepsilon + \cdots). \tag{3.3}$$

Then, the $N \times k$ matrix

$$X^1 = [\widetilde{X}^1, \widetilde{\widetilde{X}}^1] \tag{3.4}$$

is of rank k and its columns span the subspace span$\{\Phi(\varepsilon)L_1\}$ too. Thus, we have done an operation on X^0 to obtain X^1. We can apply the same operation on X^1 to obtain X^2, on X^2 to obtain X^3 and so on. Finally, we obtain an $N \times k$ matrix

$$X = X_0 + X_1\varepsilon + \cdots + X_n\varepsilon^n \tag{3.5}$$

such that X_0 is of rank k and the columns of X span the subspace span$\{\Phi(\varepsilon)L_1\}$. Evidently, the columns of X_0 span a subspace which is the limit of span$\{\Phi(\varepsilon)L_1\}$ as $\varepsilon \to 0$. Hence $\pi^\perp_\varepsilon$ approaches a limit $\pi^\perp$ as $\varepsilon \to 0$. For generic points, the convergence of the derivatives of $\Phi^{(\varepsilon)}(\mu)$ is also clear, but it should be proved that the limit is regular everywhere without exceptional points.

It is well-known that the energy of a uniton is $4\pi N$ for some integer N (see [6, 7]). On the other hand, $\phi^{(\varepsilon)} = \Phi^{(\varepsilon)}_{-1}$ is a system of harmonic maps, depending on ε continuously. So, we have

$$E(\phi^{(\varepsilon)}) = 4\pi N.$$

By the well-known weak compactness of harmonic maps from the Riemann surface M^2(see [8]), there exists a subsequence $\phi^{(\varepsilon_i)}$ of $\phi^{(\varepsilon)}$ such that $\lim\limits_{\varepsilon_i\to 0}\phi^{(\varepsilon_i)} = \widehat{\phi}$ in $C^2(\Omega)$ for any $\Omega \subset\subset M^2\backslash\{p_1, \cdots, p_l\}$ and $\widehat{\phi}$ extends to a smooth harmonic map $M^2 \to U_n$.

From the above discussion of the renormalization process, we know that the limit $\pi_\varepsilon(p)$ $(\varepsilon \to 0)$ exists at each point $p \in M^2$, which is independent of the choice of subsequences. Moreover, the limit π (or $\pi^\perp$) is unique at each point and $\lim \Phi^{(\varepsilon)}(-1) = \hat{\phi}$. Hence we have

Theorem 3.1. *The sequence of the Darboux transformations $\Phi^{(\varepsilon)}(\mu)$ of $\Phi(\mu)$ converges to an extended uniton*

$$\widehat{\Phi}(\mu) = (\pi + \mu\pi^\perp)\Phi(\mu)(\pi_0 + \mu^{-1}\pi_0^\perp) \tag{3.6}$$

globally and $\widehat{\Phi}(-1)$ is, exactly, a uniton solution.

Remark 3.1.

(i) The transformation of extended solution of harmonic maps in the form (3.6) is also called a flag transformation by some authors. The Hermitian projection π is called flag factor of the harmonic map $\phi = \Phi(-1)$. Such kind of limit is called Darboux limit in [5]. Since

$$\text{rank}(\pi) = \text{rank}(\pi_\varepsilon) = \text{rank}(L_2),$$

we can construct flag factors with any ranks by Darboux limit.

(ii) The uniton number of $\widehat{\Phi}_\mu$ is less than or equal to that of Φ_μ. But it can not be added by the singular Darboux transformations. The problem of adding uniton numbers has been considered by J. C. Wood[7]. However, the problem of adding uniton numbers by algebraic algorithm is still open.

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