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ASYMPTOTIC ANALYSIS OF DYNAMIC PROBLEMS FOR LINEARLY ELASTIC SHELLS—JUSTIFICATION OF EQUATIONS FOR DYNAMIC KOITER SHELLS**

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Abstract

Under certain conditions, the dynamic equatioins of membrane shells and the dynamic equations of flexural shells are obtained from dynamic equations of Koiter shells by the method of asymptotic analysis.

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§1. Introduction

In this paper, $\alpha,\beta,\sigma,\tau,\cdots$ take their values in the set $\{1,2\}, i,j,k,l,\cdots$ take their values in the set $\{1,2,3\}$.

In [1] and [2] under certain conditions, starting from the three-dimensional dynamic equations of elastic shells we have given the justifications of dynamic equations of membrane shells and flexural shells respectively. In this paper, we shall show that, starting from the dynamic equations of Koiter shells, we can also get the dynamic equations of membrane shells and flexural shells by the asymptotic analysis. In this way we give the justification of dynamic equations of Koiter shells, too. The main results of this paper are Theorem 2.2 and Theorem 3.1.

Consider a family of linearly elastic shells with the same middle surface $S=\vec{\varphi}(\overline{\omega})$ in $\mathbf{R}^3$ and thickness 2ε, where $\omega\subset\mathbf{R}^2$ is a bounded domain with Lipschitz-continuous boundary $\gamma=\partial\omega$, and $\vec{\varphi}\in C^3(\overline{\omega};\mathbf{R}^3)$ such that two vectors $\vec{a}_\alpha(y)=\partial_\alpha\vec{\varphi}(y)(\alpha=1,2)$ are linearly independent at any fixed point $y\in\omega$. $\vec{a}_\alpha(y)(\alpha=1,2)$ form a covariant basis of the tangent plane to the surface $S=\vec{\varphi}(\overline{\omega})$ at the point $\vec{\varphi}(y)$, and two vectors $\vec{a}^\alpha(y)(\alpha=1,2)$ defined by

$$\vec{a}^\alpha(y)\cdot\vec{a}_\beta(y)=\delta^\alpha_\beta$$

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constitute a contravariant basis of the same tangent plane. Let

$$\vec{a}_3(y)=\vec{a}^3(y)=\frac{\vec{a}_1(y)\times\vec{a}_2(y)}{|\vec{a}_1(y)\times\vec{a}_2(y)|}.$$

All the elastic shells are clamped along a part of their lateral faces, the middle line of which is $\vec{\varphi}(\gamma_o)$, where γ_o is a nonempty part of γ with positive length. For each $\varepsilon>0$, let $u_i^\varepsilon(i=1,2,3)$ denote the covariant components of displacement field $u_i^\varepsilon\vec{a}^i$ on the middle surface $S=\vec{\varphi}(\overline{\omega})$. $\vec{u}^\varepsilon=(u_i^\varepsilon)$ solves the following two-dimensional equations of W. T. Koiter shells[3,4]: find $\vec{u}^\varepsilon=(u_i^\varepsilon)\in V_k(\omega)$ such that

$$\begin{aligned}&\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy+\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&=\int_\omega p^{i,\varepsilon}v_i\sqrt{a}dy,\quad\forall\vec{v}=(v_i)\in V_k(\omega)\ ,\end{aligned}\tag{1.1}$$

where

$$\begin{aligned}V_k(\omega)&=\{\vec{v}=(v_i)\in H^1(\omega)\times H^1(\omega)\times H^2(\omega);v_i=\partial_\nu v_3=0\ \text{ on }\ \gamma_o\},\\a^{\alpha\beta\sigma\tau}&=\frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau}+2\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}),\\a&=\det(a^{\alpha\beta}),\end{aligned}$$

$a^{\alpha\beta}$ being the first fundamental form of the middle surface,

$$\begin{aligned}\gamma_{\alpha\beta}&=\frac{1}{2}(\partial_\alpha v_\beta+\partial_\beta v_\alpha)-\Gamma^\sigma_{\alpha\beta}v_\sigma-b_{\alpha\beta}v_3,\\\rho_{\alpha\beta}(\vec{v})&=\partial_{\alpha\beta}v_3-\Gamma^\sigma_{\alpha\beta}\partial_\sigma v_3+b^\sigma_\beta(\partial_\alpha v_\sigma-\Gamma^\tau_{\alpha\sigma}v_\tau)\\&\quad+b^\sigma_\alpha(\partial_\beta v_\sigma-\Gamma^\tau_{\beta\sigma}v_\tau)+b^\sigma_{\alpha|\beta}v_\sigma-c_{\alpha\beta}v_3,\\p^{i,\varepsilon}&=\frac{1}{2}\int_{-\varepsilon}^{\varepsilon}f^{i,\varepsilon}dx_3^\varepsilon,\end{aligned}$$

$\lambda>0$ and $\mu>0$ are the Lamé constants of elastic materials, which is independent of ε. $a^{\alpha\beta\sigma\tau}$ is the contravariant components of the two-dimensional elastic tensor. The functions $\gamma_{\alpha\beta}(\cdot)$ represent the covariant components of the linearized change of metric tensor of the surface S. $\rho_{\alpha\beta}(\cdot)$ are the covariant components of the linearized change of curvature tensor. The functions $f^{i,\varepsilon}\in L_2(\Omega^\varepsilon)(\Omega^\varepsilon=\omega\times(-\varepsilon,\varepsilon))$ express the body force density acting on the elastic shells. The meaning of other notations such as $\Gamma^\sigma_{\alpha\beta}$, b^σ_α, $b^\sigma_{\alpha|\beta}$, $c_{\alpha\beta}\in C^o(\overline{\omega})$ can be found in [1], and ∂_ν represents the outward normal derivative operator on boundary $\gamma=\partial\omega$.

§2. Relationship Between the Dynamic Equations of Koiter Shells and the Dynamic Equations of Membrane Shells

Consider the following dynamic problems: $\forall\ T\geq 0$,

$$\begin{aligned}&\varepsilon\int_\omega\vec{u}^\varepsilon_{tt}\cdot\vec{v}\sqrt{a}dy+\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&+\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&=\int_\omega p^{i,\varepsilon}v_i\sqrt{a}dy,\quad\forall\vec{v}=(v_i)\in V_k(\omega)\quad(0\leq t\leq T),\end{aligned}\tag{2.1}$$

$$\vec{u}^\varepsilon(y,0)=\vec{\varphi}(y),\ \vec{u}^\varepsilon_t(y,0)=\vec{\psi}(y)\ .\tag{2.2}$$

Remark. Instead of (2.2), we can take

$$\vec{u}^\varepsilon(y,0)=\vec{\varphi}(\varepsilon)(y),\quad \vec{u}_t^\varepsilon(y,0)=\vec{\psi}(\varepsilon)(y)\ ,$$

but we should require that there exist $\vec{\varphi}(y)\in V_k(\omega)$ and $\vec{\psi}(y)\in L_2(\omega)$ such that when $\varepsilon\to 0$,

$$\|\vec{\varphi}(\varepsilon)(y)-\vec{\varphi}(y)\|_{V_k(\omega)}\to 0\ ,\quad \|\vec{\psi}(\varepsilon)(y)-\vec{\psi}(y)\|_{L_2(\omega)}\to 0\ .$$

In this section we make the following assumptions: The elastic shells are clamped on the whole lateral surface. The middle surface S is elliptic, $\gamma\in C^4$, $\vec{\varphi}\in C^5(\overline{\omega};R^3)$ (these three assumptions ensure that $\{\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|_{0,\omega}^2\}^{\frac{1}{2}}$ is an equivalent norm in the space $V_M(\omega)=H_o^1(\omega)\times H_o^1(\omega)\times L_2(\omega)$ (see [7])). There exist functioins $f^i(x)$ independent of ε such that $f^{i,\varepsilon}(x)=f^i(x),\quad \forall x\in\Omega$. By the transform $x_3^\varepsilon=\varepsilon x_3$, (2.1) can be changed into the following form:

$$\int_\omega \vec{u}_{tt}^\varepsilon\cdot\vec{v}\sqrt{a}dy+\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy+\frac{\varepsilon^2}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy$$
$$=\frac{1}{2}\int_\omega\Big(\int_{-1}^1 f^i dx_3\Big)v_i\sqrt{a}dy,\qquad \forall\vec{v}=(v_i)\in V_k(\omega)\ \ (0\le t\le T).\tag{2.3}$$

Denote $\vec{q}=(q_i)$, where $q_i=\frac{1}{2}\int_{-1}^1 f^i dx_3 (i=1,2,3)$.

By Galerkin method we have the following

Theorem 2.1. *If $\vec{q}(x,t),\vec{q}_t(x,t)\in L_\infty(0,T;V_k^*(\omega))$, where $V_k^*(\omega)$ is the daul space of $V_k(\omega)$, $\vec{\varphi}(y)\in V_k(\omega)$ and $\vec{\psi}(y)\in L_2(\omega)$, then there exists a unique solution $\vec{u}^\varepsilon(x,t)$ to the problem* (2.2), (2.3) *such that*

$$\vec{u}^\varepsilon(x,t)\in L_\infty(0,T;V_k(\omega)),\quad \vec{u}_t^\varepsilon(x,t)\in L_\infty(0,T;L_2(\omega)),\quad \vec{u}_{tt}^\varepsilon(x,t)\in L_\infty(0,T;V_k^*(\omega)).$$

In what follows we will establish a priori estimate for the solution to the problem (2.2), (2.3).

Specially taking $\vec{v}=\vec{u}_t^\varepsilon$ in (2.3), we get

$$\frac{1}{2}\frac{d}{dt}\int_\omega(\vec{u}_t^\varepsilon)^2\sqrt{a}dy+\frac{1}{2}\frac{d}{dt}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$+\frac{\varepsilon^2}{6}\frac{d}{dt}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy=\int_\omega\vec{q}\cdot\vec{u}_t^\varepsilon\sqrt{a}dy\quad (0\le t\le T).$$

Then, integrating it from 0 to t, we get

$$\frac{1}{2}\int_\omega(\vec{u}_t^\varepsilon)^2\sqrt{a}dy+\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$+\frac{\varepsilon^2}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$=\frac{1}{2}\int_\omega(\vec{\psi}(y))^2\sqrt{a}dy+\frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{\varphi}(y))\gamma_{\alpha\beta}(\vec{\varphi}(y))\sqrt{a}dy$$
$$+\frac{\varepsilon^2}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{\varphi}(y))\rho_{\alpha\beta}(\vec{\varphi}(y))\sqrt{a}dy+\int_o^t\int_\omega\vec{q}\cdot\vec{u}_t^\varepsilon\sqrt{a}dydt\quad (0\le t\le T).\tag{$*$}$$

Obviously,

$$\frac{1}{2}\int_\omega (\vec\psi(y))^2\sqrt{a}dy + \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec\varphi(y))\gamma_{\alpha\beta}(\vec\varphi(y))\sqrt{a}dy$$
$$+\frac{\varepsilon^2}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec\varphi(y))\rho_{\alpha\beta}(\vec\varphi(y))\sqrt{a}dy \le C.$$

Here and hereafter C stands for a positive constant independent of ε. Substituting the above inequaity into equality $(*)$ yields

$$\frac{1}{2}\int_\omega (\vec u^\varepsilon_t)^2dy + \frac{1}{2}\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec u^\varepsilon)\|^2_{o,\omega} + \frac{1}{6}\Sigma_{\alpha,\beta}\|\varepsilon\rho_{\alpha\beta}(\vec u^\varepsilon)\|^2_{o,\omega}$$
$$\le C\Big[1+\int_o^t\int_\omega(\vec q)^2dydt + \int_o^t\int_\omega(\vec u^\varepsilon_t)^2dydt\Big] \quad (0\le t\le T).$$

Then, by Gronwall inequality we obtain

Lemma 2.1. *If $\vec q(y,t)\in L_\infty(0,T;L_2(\omega))$, $\vec\varphi(y)\in V_k(\omega),\vec\psi(y)\in L_2(\omega)$, then we have*

$$\int_\omega (\vec u^\varepsilon_t)^2dy + \Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec u^\varepsilon)\|^2_{o,\omega} + \Sigma_{\alpha,\beta}\|\varepsilon\rho_{\alpha\beta}(\vec u^\varepsilon)\|^2_{o,\omega}\le C \quad (0\le t\le T).$$

Since the middle surface S is elliptic , $\gamma\in C^4$ and $\vec\varphi\in C^5(\overline\omega;R^3)$, from the conclusion in [7] (also see [8, 9]) we know that $\{\Sigma_{\alpha,\beta}\|\gamma_{\alpha,\beta}(\vec u^\varepsilon)\|^2_{o,\omega}\}^{\frac12}$ is an equivalent norm in the space $V_M(\omega) = H^1_o(\omega)\times H^1_o(\omega)\times L_2(\omega)$. By Lemma 2.1, $\vec u^\varepsilon$ is uniformly bounded with respect to ε in $L_\infty(0,T;V_M(\omega))$, $\{\varepsilon\rho_{\alpha\beta}(\vec u^\varepsilon)\}_{\varepsilon\ge0}$ is uniformly bounded with respect to ε in $L_\infty(0,T;L_2(\omega))$. Hence, there exists a subsequence (still denoted by $(\vec u^\varepsilon)_{\varepsilon>0}$) and functions $\vec u\in L_\infty(0,T;V_M(\omega))$, $\rho^{-1}_{\alpha\beta}\in L_\infty(0,T;L_2(\omega))$ such that, when $\varepsilon\to0$,

$$\vec u^\varepsilon \overset{*}{\rightharpoonup} \vec u \text{ weak-star in } L_\infty(0,T;V_M(\omega)), \tag{2.4}$$

$$\varepsilon\rho_{\alpha\beta}(\vec u^\varepsilon) \overset{*}{\rightharpoonup} \rho^{-1}_{\alpha\beta} \text{ weak-star in } L_\infty(0,T;L_2(\omega)), \tag{2.5}$$

$$\vec u^\varepsilon_t \overset{*}{\rightharpoonup} \vec u_t \text{ weak-star in } L_\infty(0,T;L_2(\omega)). \tag{2.6}$$

By (2.6), for any fixed $\vec v\in V_k(\omega)$, when $\varepsilon\to0$, $\int_\omega \vec u^\varepsilon_t\vec v\sqrt{a}dy$ converges to $\int_\omega \vec u_t\vec v\sqrt{a}dy$ weak-star in $L_\infty(0,T)$, then $\int_\omega \vec u^\varepsilon_t\vec v\sqrt{a}dy$ converges to $\int_\omega \vec u_t\vec v\sqrt{a}dy$ in $\mathcal{D}'(0,T)$. Therefore, when $\varepsilon\to0$,

$$\frac{d}{dt}\int_\omega \vec u^\varepsilon_t\vec v\sqrt{a}dy \to \frac{d}{dt}\int_\omega \vec u_t\vec v\ \sqrt{a}dy \text{ in } \mathcal{D}'(0,T). \tag{2.7}$$

Since $\vec u^\varepsilon_t$, $\vec u_t\in L_\infty(0,T;L_2(\omega))$, we infer $\vec u^\varepsilon_t,\vec u_t\in\mathcal{D}'(0,T;L_2(\omega))$, hence $\vec u^\varepsilon_{tt}$, $\vec u_{tt}\in\mathcal{D}'(0,T;L_2(\omega))$.

It follows from (2.7) that when $\varepsilon\to0$,

$$\int_\omega \vec u^\varepsilon_{tt}\vec v\sqrt{a}dy \to \int_\omega \vec u_{tt}\vec v\sqrt{a}dy \text{ in } \mathcal{D}'(\text{ 0,T}). \tag{2.8}$$

Taking $\varepsilon\to0$ in (2.3), we get

$$\int_\omega \vec u_{tt}\vec v\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec u)\gamma_{\alpha\beta}(\vec v)\sqrt{a}dy$$
$$=\int_\omega\Big(\int_{-1}^1 f^idx_3\Big)v_i\sqrt{a}dy, \quad \forall\vec v=(v_i)\in V_k(\omega). \tag{2.9}$$

From (2.3) and (2.9) we know that the convergence (2.8) in $\mathcal{D}'(0,T)$ is actually the weak-star convergence in $L_\infty(0,T)$. It is easy to verify that the initial condition (2.2) for $\vec u$ takes the following form:

$$\vec{u}(y,0)=\vec{\varphi}(y),\quad \vec{u}_t(y,0)=\vec{\psi}(y). \tag{2.10}$$

Since the solution to the problem (2.9)–(2.10) is unique , the convergence (2.4)–(2.6) holds for the whole family $(\vec{u}^\varepsilon)_{\varepsilon>0}$. Summing up the results of this section, we get the following main theorem.

Theorem 2.2. *Suppose that* $\vec{q}(y,t)\in L_\infty(0,T;L_2(\omega)),\vec{q}_t(y,t)\in L_\infty(0,T;V_k^*(\omega))$*, and* $\vec{u}^\varepsilon$ *is the solution to the following problem*:

$$\begin{aligned}&\int_\omega \vec{u}^\varepsilon_{tt}\cdot\vec{v}\sqrt{a}dy+\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy+\frac{\varepsilon^2}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&=\frac{1}{2}\int_\omega\Big(\int_{-1}^1 f^i dx_3\Big)v_i\sqrt{a}dy,\quad \forall\vec{v}=(v_i)\in V_k(\omega),\\&\vec{u}^\varepsilon(y,0)=\vec{\varphi}(y),\quad \vec{u}^\varepsilon_t(y,0)=\vec{\psi}(y),\end{aligned}$$

where $\vec{\varphi}(y)\in V_k(\omega),\vec{\psi}(y)\in L_2(\omega)$*. Then there exists function* $\vec{u}(y,t)\in L_\infty(0,T;V_M(\omega))$ *such that, when* $\varepsilon\to 0$, $\vec{u}^\varepsilon$ *converges to* $\vec{u}$ *weak-star in* $L_\infty(0,T;V_M(\omega))$, $\vec{u}^\varepsilon_t$ *converges to* $\vec{u}_t$ *weak-star in* $L_\infty(0,T;L_2(\omega))$*, and* $\vec{u}$ *is the solution to the following two-dimensional dynamic equations of membrane shells*:

$$\begin{aligned}&\int_\omega \vec{u}_{tt}\vec{v}\sqrt{a}dy+\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u})\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&=\frac{1}{2}\int_\omega\Big(\int_{-1}^1 f^i dx_3\Big)v_i\sqrt{a}dy,\quad \forall\vec{v}=(v_i)\in V_k(\omega),\\&\vec{u}(y,0)=\vec{\varphi}(y),\quad \vec{u}_t(y,0)=\vec{\psi}(y).\end{aligned}$$

§3. Relationship Between Dynamic Equations of Koiter Shells and Dynamic Equations of Flexural Shells

In this section we make the following assumptions about the body force density:

$$f^{i,\varepsilon}(x^\varepsilon)=\varepsilon^2 f^i(x),\quad \forall x\in\Omega=\omega\times(-1,1), \tag{3.1}$$

where $x^\varepsilon=(x_1,x_2,x_3^\varepsilon)$ corresponds to $x=(x_1,x_2,x_3)$ by the transform $x_3^\varepsilon=\varepsilon x_3$ $(-1\le x_3\le 1)$.

Consider the following dynamic problem: $\forall\ T>0$,

$$\begin{aligned}&\varepsilon^3\int_\omega \vec{u}^\varepsilon_{tt}\cdot\vec{v}\sqrt{a}dy+\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&+\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy\\&=\int_\omega p^{i,\varepsilon}v_i\sqrt{a}dy,\quad \forall\vec{v}=(v_i)\in V_k(\omega)\ \ (0\le t\le T),\end{aligned} \tag{3.2}$$

$$\vec{u}^\varepsilon(y,0)=\varepsilon\vec{\varphi}(y)\ ,\ \vec{u}^\varepsilon_t(y,0)=\vec{\psi}(y). \tag{3.3}$$

Remark. Instead of (3.3) we can take

$$\vec{u}^\varepsilon(y,0)=\varepsilon\vec{\varphi}(\varepsilon)(y),\ \vec{u}^\varepsilon_t(y,0)=\vec{\psi}(\varepsilon)(y),$$

but we should require that there exist functions $\vec{\varphi}(y)\in V_k(\omega)$ and $\vec{\psi}(y)\in L_2(\omega)$ such that, when $\varepsilon \to 0$,

$$\|\vec{\varphi}(\varepsilon)(y) - \vec{\varphi}(y)\|_{V_k(\omega)} \to 0, \quad \|\vec{\psi}(\varepsilon)(y) - \vec{\psi}(y)\|_{L_2(\omega)} \to 0.$$

Denote $\vec{q} = (q_i)(i = 1, 2, 3)$, where $q_i = \frac{1}{2}\int_{-1}^{1} f^i(x)dx_3$.

Equation (3.2) can be changed into

$$\int_\omega \vec{u}^\varepsilon_{tt} \cdot \vec{v}\sqrt{a}dy + \frac{1}{\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy + \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy$$
$$= \frac{1}{2}\int_\omega \Big(\int_{-1}^{1} f^i dx_3\Big)v_i\sqrt{a}dy, \quad \forall \vec{v} = (v_i) \in V_k(\omega) \quad (0 \le t \le T). \tag{3.4}$$

In what follows we will establish a priori estimate for the solution to the problem (3.3)–(3.4). For this purpose, specially taking $\vec{v} = \vec{u}^\varepsilon_t$ in (3.4), we get

$$\frac{1}{2}\frac{d}{dt}\int_\omega (\vec{u}^\varepsilon_t)^2\sqrt{a}dy + \frac{1}{2\varepsilon^2}\frac{d}{dt}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$+ \frac{1}{6}\frac{d}{dt}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$= \int_\omega \vec{q} \cdot \vec{u}^\varepsilon_t\sqrt{a}dy \quad (0 \le t \le T).$$

Then, integrating it from 0 to t yields

$$\frac{1}{2}\int_\omega (\vec{u}^\varepsilon_t)^2\sqrt{a}dy + \frac{1}{2\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$+ \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$= \frac{1}{2}\int_\omega (\vec{u}^\varepsilon_t(y,0))^2\sqrt{a}dy + \frac{1}{2\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon(y,0))\gamma_{\alpha\beta}(\vec{u}^\varepsilon(y,0))\sqrt{a}dy$$
$$+ \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon(y,0))\rho_{\alpha\beta}(\vec{u}^\varepsilon(y,0))\sqrt{a}dy + \int_o^t\int_\omega \vec{q} \cdot \vec{u}^\varepsilon_t\sqrt{a}dydt \quad (0 \le t \le T).$$

From (3.3) we know that

$$\frac{1}{2}\int_\omega (\vec{u}^\varepsilon_t(y,0))^2\sqrt{a}dy + \frac{1}{2\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon(y,0))\gamma_{\alpha\beta}(\vec{u}^\varepsilon(y,0))\sqrt{a}dy$$
$$+ \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon(y,0))\rho_{\alpha\beta}(\vec{u}^\varepsilon(y,0))\sqrt{a}dy \le C. \tag{3.5}$$

Hence, we have

$$\frac{1}{2}\int_\omega (\vec{u}^\varepsilon_t)^2\sqrt{a}dy + \frac{1}{2\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$+ \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$
$$\le C\Big[1 + \int_o^t\int_\omega (\vec{q})^2\sqrt{a}dydt + \int_o^t\int_\omega (\vec{u}^\varepsilon_t)^2\sqrt{a}dydt\Big] \quad (0 \le t \le T).$$

Noting that from the conclusion in [4], $\{\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{L_2(\omega)} + \Sigma_{\alpha\beta}\|\rho_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{L_2(\omega)}\}^{\frac{1}{2}}$ is an equivalent norm in the space $V_k(\omega)$, we have

$$\frac{1}{\varepsilon^2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy + \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{u}^\varepsilon)\sqrt{a}dy$$

$$\geq C\Big(\Sigma_{\alpha,\beta}\|\frac{1}{\varepsilon}\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}+\Sigma_{\alpha,\beta}\|\rho_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}\Big)$$
$$\geq C\Big(\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}+\Sigma_{\alpha,\beta}\|\rho_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}\Big)\geq C\|\vec{u}^\varepsilon\|_{V_k(\omega)}. \tag{3.6}$$

Hence, by Gronwall inequality we get

Lemma 3.1. *If $\vec{q}(y,t)\in l_\infty(0,T;L_2(\omega))$, $\vec{\varphi}(y)\in V_k(\omega)$ and $\vec{\psi}(y)\in L_2(\omega)$, then we have*

$$\int_\omega(\vec{u}^\varepsilon_t)^2\sqrt{a}dy+\Sigma_{\alpha,\beta}\|\frac{1}{\varepsilon}\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}+\Sigma_{\alpha,\beta}\|\rho_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{o,\omega}\leq C\quad(0\leq t\leq T).$$

By Lemma 3.1 we have

$$n\|\vec{u}^\varepsilon\|_{V_k(\omega)}\leq C(\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{L_2(\omega)}+\Sigma_{\alpha\beta}\|\rho_{\alpha\beta}(\vec{u}^\varepsilon)\|^2_{L_2(\omega)})\leq C\quad(0\leq t\leq T). \tag{3.7}$$

Therefore we can select a subsequence (still denoted by $(\vec{u}^\varepsilon)_{\varepsilon>0}$) and there exist functions $\vec{u}\in L_\infty(0,T;V_k(\omega))$ and $\chi_{\alpha\beta}\in L_\infty(0,T;L_2(\omega))$ such that, when $\varepsilon\to 0$,

$$\vec{u}^\varepsilon \overset{*}{\rightharpoonup} \vec{u}\quad\text{weak-star in } L_\infty(0,T;V_k(\omega)), \tag{3.8}$$
$$\frac{1}{\varepsilon}\gamma_{\alpha\beta}(\vec{u}^\varepsilon) \overset{*}{\rightharpoonup} \chi_{\alpha\beta}\quad\text{weak-star in } L_\infty(0,T;L_2(\omega)), \tag{3.9}$$
$$\vec{u}^\varepsilon_t \overset{*}{\rightharpoonup} \vec{u}_t\quad\text{weak-star in } L_\infty(0,T;L_2(\omega)). \tag{3.10}$$

By (3.9), when $\varepsilon\to 0$, $\gamma_{\alpha\beta}(\vec{u}^\varepsilon)$ converges to 0 weak-star in $L_\infty(0,T;L_2(\omega))$. Moreover, by (3.8) it is easy to see that, when $\varepsilon\to 0$, $\gamma_{\alpha\beta}(\vec{u}^\varepsilon)$ converges to $\gamma_{\alpha\beta}(\vec{u})$ weak-star in $L_\infty(0,T;L_2(\omega))$. Hence $\gamma_{\alpha\beta}(\vec{u})=0$, then $\vec{u}\in L_\infty(0,T;V_F(\omega))$, where

$$V_F(\omega)=\{\vec{\eta}\in V_k(\omega);\gamma_{\alpha\beta}(\vec{\eta})=0\}$$

is the space of inextensional displacement. By (3.10), when $\varepsilon\to 0$,

$$\int_\omega\vec{u}^\varepsilon_t\vec{v}\sqrt{a}dy \overset{*}{\rightharpoonup} \int_\omega\vec{u}_t\vec{v}\sqrt{a}dy\quad\text{weak-star in } L_\infty(0,T),\ \forall\vec{v}\in V_F(\omega). \tag{3.11}$$

Since $\vec{u}^\varepsilon_t,\vec{u}_t\in L_\infty(0,T;L_2(\omega))$, we can infer $\vec{u}^\varepsilon_t,\vec{u}_t\in\mathcal{D}'(0,T;L_2(\omega))$, therefore $\vec{u}^\varepsilon_{tt}$, $\vec{u}_{tt}\in\mathcal{D}'(0,T;L_2(\omega))$, where $\mathcal{D}'(0,T;L_2(\omega))$ is the generalized function space from [0,T] to $L_2(\omega)$. It follows from (3.11) that, when $\varepsilon\to 0$,

$$\int_\omega\vec{u}^\varepsilon_t\vec{v}\sqrt{a}dy \ \to\ \int_\omega\vec{u}_t\vec{v}\sqrt{a}dy \text{ in } \mathcal{D}'(0,T),$$

then

$$\frac{d}{dt}\int_\omega\vec{u}^\varepsilon_t\vec{v}\sqrt{a}dy \ \to\ \frac{d}{dt}\int_\omega\vec{u}_t\vec{v}\sqrt{a}dy \text{ in } \mathcal{D}'(0,T),$$

that is,

$$\int_\omega\vec{u}^\varepsilon_{tt}\vec{v}\sqrt{a}dy \ \to\ \int_\omega\vec{u}_{tt}\vec{v}\sqrt{a}dy\quad\text{in } \mathcal{D}'(0,T). \tag{3.12}$$

For any fixed $\vec{v}\in V_F(\omega)$, taking $\varepsilon\to 0$ in (3.4) gives

$$\int_\omega\vec{u}_{tt}\vec{v}\sqrt{a}dy+\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy$$
$$=\frac{1}{2}\int_\omega\Big(\int_{-1}^1 f^i dx_3\Big)v_i\sqrt{a}dy,\ \forall\vec{v}\in V_F(\omega). \tag{3.13}$$

From (3.4) and (3.13), the convergence (3.12) in $\mathcal{D}'($ 0,T) is actually the weak-star convergence in $L_\infty(0,T)$.

When $\varepsilon \to 0$, it is easy to verify that the initial condition (3.3) for $\vec{u}$ takes the following form:

$$\vec{u}(y,0) = 0\ ,\ \vec{u}_t(y,0) = \vec{\psi}(y). \tag{3.14}$$

Since the solution to the problem (3.13)–(3.14) is unique, the convergence (3.8) holds for the whole family $(\vec{u}^\varepsilon)_{\varepsilon>0}$.

Summing up the above result, we have the following main theorem.

Throrem 3.1. *Suppose that* $V_F(\omega) \neq \{0\}$ *and* $\vec{u}^\varepsilon \in L_\infty(0,T;V_k(\omega))$ *is the soluton to the following problem*:

$$\varepsilon^3 \int_\omega \vec{u}^\varepsilon_{tt} \cdot \vec{v}\sqrt{a}dy + \varepsilon \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{u}^\varepsilon)\gamma_{\alpha\beta}(\vec{v})\sqrt{a}dy$$
$$+ \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u}^\varepsilon)\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy = \varepsilon^3\int_\omega q_i v_i\sqrt{a}dy, \quad \forall\vec{v} = (v_i) \in V_k(\omega) \quad (0 \le t \le T),$$
$$\vec{u}^\varepsilon(y,0) = \varepsilon\vec{\varphi}(y),\ \vec{u}^\varepsilon_t(y,0) = \vec{\psi}(y),$$

where $\vec{\varphi}(y) \in V_k(\omega)$, $\vec{\psi}(y) \in L_2(\omega)$, $\vec{q} = (q_i)$ $(q_i = \frac{1}{2}\int_{-1}^1 f^i dx_3)$ *satisfies* $\vec{q}(y,t) \in L_\infty(0,T;L_2(\omega))$, $\vec{q}_t(y,t) \in L_\infty(0,T;V_k^*(\omega))$. *Then, when* $\varepsilon \to 0$,

$$\vec{u}^\varepsilon \overset{*}{\rightharpoonup} \vec{u} \text{ weak-star in } L_\infty(0,T;V_k(\omega))$$

and $\vec{u} \in L_\infty(0,T;V_F(\omega))$ *is the solution to the two-dimensional dynamic equations of flexural shells*:

$$\int_\omega \vec{u}_{tt}\vec{v}\sqrt{a}dy + \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{u})\rho_{\alpha\beta}(\vec{v})\sqrt{a}dy$$
$$= \frac{1}{2}\int_\omega \Big(\int_{-1}^1 f^i dx_3\Big) v_i\sqrt{a}dy, \quad \forall\vec{v} \in V_F(\omega) \quad (0 \le t \le T),$$
$$\vec{u}(y,0) = 0,\ \vec{u}_t(y,0) = \vec{\psi}(y).$$

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