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BLOW UP OF SOLUTIONS TO THE CAUCHY PROBLEM FOR NONLINEAR WAVE EQUATIONS**

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Abstract

The author proves blow up of solutions to the Cauchy problem of certain nonlinear wave equations and, also, estimates the time when the blow up occurs.

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§1. Introduction

This paper deals with solutions of certain nonlinear wave equations of the form

$$\Box u = |u_t|^p, \tag{1.1}$$

corresponding to initial conditions

$$u(0,x) = f(x) \quad u_t(0,x) = g(x) \quad x \in R^n, \tag{1.2}$$

where

$$\Box = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2 \tag{1.3}$$

is the wave operator.

We are interested in showing the "blow up" of solutions to (1.1)–(1.2). For that, we require

$$(p-1)(n-1) \le 2. \tag{1.4}$$

If $(p-1)(n-1) > 2$, global solutions of wave equation subject to very general perturbations of order p exist provided the initial data are sufficiently small (see [6] and references therein). We are also interested in estimating the time when "blow up" occurs. For initial data of the form

$$u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x) \tag{1.5}$$

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with constant $0 < \varepsilon \le 1$, smallness can be measured conveniently by the size of ε for fixed f, g. We define "life span" $T(\varepsilon)$ of the solutions of (1.1),(1.5) to be the largest value such that solutions exist for $x \in R^n, \quad 0 \le t < T(\varepsilon)$.

Theorem 1.1. *Suppose that f, g are smooth functions with compact support:*

$$\mathrm{supp} f, g \subset \{x : |x| \le \rho\}. \tag{1.6}$$

If g satisfies

$$\int g(x)dx > 0, \tag{1.7}$$

then we have the following estimates for the life span $T(\varepsilon)$ of solutions of (1.1),(1.5)*:*

(i) *If $(n-1)(p-1) < 2$, then there exists a positive constant A which is independant of ε such that*

$$T(\varepsilon) \le A\varepsilon^{-\frac{p-1}{1-(n-1)(p-1)/2}}. \tag{1.8}$$

(ii) *If $(n-1)(p-1) = 2$, then there exist a positive constant B which is independent of ε such that*

$$T(\varepsilon) \le \exp(B\varepsilon^{-(p-1)}). \tag{1.9}$$

In Theorem 1.1, we prove the "blow up" of solutions of (1.1)–(1.2) under the condition (1.6). However, we can prove that solutions of (1.1)–(1.2) must blow up for any nontrivial initial data.

Theorem 1.2. *Suppose that f, g are smooth functions with compact support property* (1.6)*. If $(n-1)(p-1) \le 2$, then the solution of* (1.1)–(1.2) *always blow up in finite time provided f, g are not both identically zero.*

The previous study on the blow-up of solutions of nonlinear wave equations can be found in [1, 2, 4].

§2. Proof of Theorem 1.1

Let

$$x = (y, z), \quad y \in R, \quad z \in R^{n-1},$$

and define a new function

$$U(y,t) = \int_{R^{n-1}} u(y,z,t)dz. \tag{2.1}$$

Then U satisfies the wave equation in one space dimension of the form

$$(\partial_t^2 - \partial_y^2)U(y,t) = \int |u_t(y,z,t)|^p dz \tag{2.2}$$

with respect to to initial conditions

$$U(y,0) = U_0(y), \quad U_t(y,0) = U_1(y). \tag{2.3}$$

Here

$$U_0(y) = \int_{R^{n-1}} f(y,z)dz, \quad U_1(y) = \int_{R^{n-1}} g(y,z)dz. \tag{2.4}$$

By D'Alembert's formula, we get

$$U(y,t)=\frac{\varepsilon}{2}(U_0(y+t)+U_0(y-t)+\int_{y-t}^{y+t}U_1(\xi)d\xi)$$
$$+\frac{1}{2}\int_0^t d\tau\int_{y-t+\tau}^{y+t-\tau}d\xi\int|u_t(\xi,z,\tau)|^p dz. \tag{2.5}$$

By (1.6), we get

$$\operatorname{supp} f,g\subset\{y:|y|\le\rho\}. \tag{2.6}$$

Thus, (2.5) implies that

$$U(y,t)=M\varepsilon+\frac{1}{2}\int_0^t d\tau\int_{y-t+\tau}^{y+t-\tau}d\xi\int|u_\tau(\xi,y,\tau)|^p dz \tag{2.7}$$

for $-t+\rho\le y\le t-\rho, t\ge\rho$, where

$$M=\frac{1}{2}\int g(x)dx>0.$$

Making use of Holder's inequality in the z-integral of the right-hand side of (2.6), we get by support property (1.6) that

$$U(y,t)\ge M\varepsilon+C\int_0^t d\tau\int_{y-t+\tau}^{y+t-\tau}\frac{|U_t(\xi,\tau)|^p}{((\tau+\rho)^2-\xi^2)^{(n-1)(p-1)/2}}d\xi. \tag{2.8}$$

Here and hereafter, C will be a constant independent of ε and it may chang from line to line.

Now we shall fix a characteristic line $t-y=\rho$. Define a new function

$$p(y)=U(y,y+\rho). \tag{2.9}$$

Then, cutting the domain of the interal (2.8) and inverting the order of (ξ,τ), we get

$$p(y)\ge M\varepsilon+C\int_\rho^y d\xi\int_{\xi-\rho}^{\xi+\rho}\frac{|U_\tau(\xi,\tau)|^p}{((\tau+\rho)^2-\xi^2)^{(n-1)(p-1)/2}}d\tau. \tag{2.10}$$

It follows from (2.10) that, for $y\ge\rho$,

$$v(y)\ge M\varepsilon+C\int_\rho^y\frac{d\xi}{(\xi+\rho)^{(n-1)(p-1)/2}}\int_{\xi-\rho}^{\xi+\rho}|U_\tau(\xi,\tau)|^p d\tau. \tag{2.11}$$

In the τ-integral of (2.11), Holder's inequality yields that

$$v(y)\ge M\varepsilon+C\int_\rho^y\frac{d\xi}{(\xi+\rho)^{(n-1)(p-1)/2}}\Big|\int_{\xi-\rho}^{\xi+\rho}U_\tau(\xi,\tau)d\tau\Big|^p. \tag{2.12}$$

Thus, noting (2.9), we get

$$v(y)\ge M\varepsilon+C\int_\rho^y\frac{|v(\xi)|^p}{(\xi+\rho)^{(n-1)(p-1)/2}}d\xi,\quad y\ge\rho. \tag{2.13}$$

Now we introduce a function w satisfying the integral equation

$$w(y)=M\varepsilon+C\int_\rho^y\frac{|w(\xi)|^p}{(\xi+\rho)^{(n-1)(p-1)/2}}d\xi,\quad y\ge\rho. \tag{2.14}$$

Then it follows that

$$v(y)\ge w(y).$$

So, the life span of v is less than the one of w which will be the upper bound of $T(\varepsilon)$.

It is easy to check that w is a solution to the o.d.e

$$w'(y) = \frac{C|w(y)|^p}{(y+\rho)^{(n-1)(p-1)/2}} \tag{2.15}$$

with initial condition

$$w(\rho) = M\varepsilon. \tag{2.16}$$

Thus, in the case $(n-1)(p-1) < 2$, we get

$$w(y) = ((\varepsilon M)^{-(p-1)} + c'(2\rho)^{1-(n-1)(p-1)/2} - c'(y+\rho)^{1-(n-1)(p-1)/2})^{-\frac{1}{p-1}}, \tag{2.17}$$

where

$$c' = \frac{c(p-1)}{1-(n-1)(p-1)/2}. \tag{2.18}$$

Thus

$$T(\varepsilon) \le A\varepsilon^{\frac{-(p-1)}{1-(n-1)(p-1)/2}}. \tag{2.19}$$

When $(n-1)(p-1) = 2$, integrating (2.15)–(2.16), we get

$$w(y) = ((\varepsilon M)^{-(p-1)} - c'' \log(\frac{y+\rho}{2\rho}))^{-\frac{1}{p-1}}, \tag{2.20}$$

where $c'' = C(p-1)$. Therefore

$$T(\varepsilon) \le \exp(B\varepsilon^{-(p-1)}).$$

This ends the proof of Theorem 1.1.

§3. Proof of Theorem 1.2

We prove the theorem by contradiction. If u is the global solution of (1.1)–(1.2), then by Theorem 1.1 (in which we take $t = \tau$ for any $\tau \ge 0$ to be the initial time), we get

$$\int g(x) \le 0, \tag{3.1}$$

$$\int u_\tau(\tau, x)dx \le 0, \quad \forall \tau > 0. \tag{3.2}$$

Integrating (1.1) with respect to x, we obtain

$$\partial_t \int u_t(t,x)dx = \int |u_t(t,x)|^p dx.$$

Theorefore

$$\int u_t(t,x)dx = \int g(x)dx + \int_0^t \int |u_\tau(\tau,x)|^p dx d\tau. \tag{3.3}$$

Let

$$D = -\int g(x)dx.$$

Then we get from (3.2)–(3.3) that

$$D \ge \int_0^\infty \int |u_\tau(\tau,x)|^p dx dt. \tag{3.4}$$

Let

$$x = (y,z) \quad y \in R, z \in R^{n-1},$$

$$E = \int_{-\infty}^{-\rho} \int_0^{+\infty} \int |u_t(t,y,z)|^p dz dt dy. \tag{3.5}$$

We shall prove that $E > 0$, otherwise

$$u_t(t, y, z) = 0, \quad \forall y \le -\rho.$$

Thus, u is independent of t when $y \le -\rho$. Noting that $\mathrm{supp}\, f$ lies in the set $(y \ge -\rho)$, we conclude that

$$u(t, y, z) \equiv 0, \quad y \le -\rho. \tag{3.6}$$

By integrating (1.1)–(1.2) with respect to z and applying D'Alembert's formula, we get

$$\int u(t, y, z) = \frac{-D}{2} + \frac{1}{2}\int_0^t \int_{y-t+\tau}^{y+t-\tau} \int |u_\tau(\tau, \xi, z)|^p dz d\xi d\tau \tag{3.7}$$

for $-t + \rho \le y \le t - \rho$. Taking $(t, y) = (2\rho - \rho)$ in (3.7), we get

$$D = \int_0^{2\rho} \int_{y+2\rho-\tau}^{y-2\rho+\tau} \int |u_\tau(\tau, \xi, z)|^p dz d\xi d\tau. \tag{3.8}$$

Then it follows from (3.4) that

$$u_t(t, x) \equiv 0, \quad \forall t \ge 2\rho.$$

Thus, u is independent of t when $t \ge 2\rho$. By (1.1), u is then a harmonic function of x which has compact support when $t \ge 2\rho$; this would imply that u is identically 0 when $t \ge 2\rho$. Reversing the time interval and solving (1.1) with $t = 2\rho$ as initial time, we concluded that u is identically 0, which contradicts the fact that f, g are both identically 0. Therefore $E > 0$. Let

$$p(y) = \int u(y + \rho, y, z) dz. \tag{3.9}$$

Then by (3.4),(3.5) and (3.7)

$$p(y) < -E/2, \quad \forall y \ge \rho. \tag{3.10}$$

We get from (3.4) that

$$D \ge \int_\rho^{+\infty} \int_{\xi-\rho}^{\xi+\rho} \int |u_\tau(\tau, \xi, z)|^p dz d\tau d\xi. \tag{3.11}$$

A similar argument as in the proof of Theorem 1.1 proves

$$\int_{\xi-\rho}^{\xi+\rho} \int |u_\tau(\tau, \xi, z)|^p dz d\tau \ge c\frac{|p(\xi)|^p}{(\xi + \rho)^{(n-1)(p-1)/2}}. \tag{3.12}$$

Thus

$$D \ge CE^p \int_\rho^{+\infty} \frac{d\xi}{(\xi + \rho)^{(n-1)(p-1)/2}}. \tag{3.13}$$

The right hand side is equal to $+\infty$ when $(n - 1)(p - 1) \le 2$, which is a contradiction, so the theorem is proved.

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