Chin. Ann. of Math.
22B: 3(2001),281-286.

AUTOMORPHISM GROUPS OF 4-VALENT CONNECTED CAYLEY GRAPHS OF p-GROUPS****

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Abstract

Let G be a p-group (p odd prime) and let $X = \mathrm{Cay}(G, S)$ be a 4-valent connected Cayley graph. It is shown that if G has nilpotent class 2, then the automorphism group $\mathrm{Aut}(X)$ of X is isomorphic to the semidirect product $G_R \rtimes \mathrm{Aut}(G, S)$, where G_R is the right regular representation of G and $\mathrm{Aut}(G, S)$ is the subgroup of the automorphism group $\mathrm{Aut}(G)$ of G which fixes S setwise. However the result is not true if G has nilpotent class 3 and this paper provides a counterexample.

Keywords Cayley graphs, Normal Cayley graphs, Automorphism groups
2000 MR Subject Classification 05C25, 20B25
Chinese Library Classification O157.5, O152.9 **Document Code** A
Article ID 0252-9599(2001)03-0281-06

§1. Introduction

Let G be a finite group and S a subset of G such that $1 \notin S$ and $S = S^{-1}$. The Cayley graph $X = \mathrm{Cay}(G, S)$ of G with respect to S is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) \mid g \in G, s \in S\}$. From the definition the following two facts are obvious: (1) the automorphism group $\mathrm{Aut}(X)$ of X contains G_R, the right regular representation of G, as a subgroup, and (2) X is connected if and only if S generates the group G.

For a Cayley graph $X = \mathrm{Cay}(G, S)$ of a finite group G with respect to S, X is said to be normal if G_R is a normal subgroup of the automorphism group $\mathrm{Aut}(X)$ of X. The study of the normality of Cayley graphs is important in many cases, for example CI-subsets (for the concept, see [11]), symmetric graphs and half-transitive graphs (see [5,12]), etc. For abelian groups (i.e., groups with nilpotent class 1), Baik, et al.[2,3] classified the Cayley graphs of valency 3, 4, or 5, which are not normal. For nonabelian p-groups of order p^3 (p odd prime), Feng, et al.[5] proved that all connected Cayley graphs of valency 4 are normal. In this paper, we obtain the same result for a p-group of nilpotent class 2.

Manuscript received April 21, 2000. Revised August 29, 2000.

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****Project supported by the National Natural Science Foundation of China (No.10071002) and Com2MaC-KOSEF.

Theorem 1.1. *All* 4*-valent connected Cayley graph of a* p*-group* (p *odd prime*) *of nilpotent class* 2 *are normal.*

In general, it is known to be difficult to determine the normality of Cayley graphs. The only groups, for which the complete information about the normality of Cayley graphs is available, are the cyclic groups of prime order[1] and the groups of order twice a prime[4]. Since Wang, Wang and Xu[10] obtained all normal disconnected Cayley graphs, we only consider the connected ones in Theorem 1.1. For a p-group of nilpotent class 3, Theorem 1.1 is not true and in the third section we provide a 4-valent connected Cayley graph which is not normal.

Let u be a vertex of a Cayley graph $X = \mathrm{Cay}(G, S)$. We denote $N_i(u) = \{v \in V(X) \mid d(u, v) = i\}$ (i positive integer), where $d(u, v)$ denotes the distance between u and v in X. Write $A = \mathrm{Aut}(X)$ and by A_1 we denote the stabilizer of 1 in A. For other group- and graph-theoretic notions used later, the reader can refer to [7] and [8].

Before the end of this section, we list some preliminary results.

Proposition 1.1.[11, Proposition 1.5] *Let* $X = \mathrm{Cay}(G, S)$ *be a Cayley graph of* G *with respect to* S *and let* $A = \mathrm{Aut}(X)$*. Then* X *is normal if and only if* $A_1 = \mathrm{Aut}(G, S)$*.*

In view of Theorem 1.2 of [2] we have

Proposition 1.2. *Let* G *be an abelian* p*-group and let* $X = \mathrm{Cay}(G, S)$ *be a* 4*-valent connected Cayley graph. If* X *is not normal, then* X *is the complete graph* K_5*,* G *is the cyclic group* $\mathbb{Z}_5$ *and* $S = G\backslash\{1\}$*.*

Proposition 1.3.[5, Theorem 3.1] *All* 4*-valent connected Cayley graphs of a nonabelian group of order* p^3 (p *odd prime*) *are normal.*

Let G be a finite group and let $X = \mathrm{Cay}(G, S)$ be a 4-valent connected Cayley graph of G. Denote by $(g, z_1g, z_2z_1g, \ldots, z_{n-1}z_{n-2}\cdots z_2z_1g, z_nz_{n-1}\cdots z_2z_1g = g)$ a cycle of length n in X where $z_i \in S$ $(1 \le i \le n)$. Obviously $z_{i+\ell}\cdots z_{i+1}z_i \ne 1$ $(1 \le i \le i+\ell \le n)$ except $z_nz_{n-1}\cdots z_2z_1 = 1$. For simplicity we use $C_g(z_nz_{n-1}\cdots z_2z_1)$ to denote this cycle. Feng et al.[5, Lemma 2.6 and Remark 1] obtained the following Proposition.

Proposition 1.4. *Let* $X = \mathrm{Cay}(G, S)$ *be a* 4*-valent connected Cayley graph where* G *is a* p*-group of nilpotent class less than or equal to* 2 *and* $S = \{x, y, x^{-1}, y^{-1}\}$*. If either* $p > 3$*, or* $p = 3$ *and the order* $o([x, y])$ *of the communicator* $[x, y]$ *is greater than* 3*, then for any* $g \in G$ *we have*

(1) *there are exactly two cycles of length* 8 *in* X *which have* xg*,* g *and* $x^{-1}g$ *as three consecutive vertices on them*: $C_g(xyx^{-1}y^{-1}y^{-1}x^{-1}yx)$, $C_g(xy^{-1}x^{-1}yyx^{-1}y^{-1}x)$;

(2) *there are exactly seven cycles of length* 8 *in* X *which have* xg*,* g *and* yg *as three consecutive vertices on them*:

$$C_g(y^{-1}x^{-1}yyx^{-1}y^{-1}xx),\ C_g(y^{-1}x^{-1}x^{-1}y^{-1}xyyx),\ C_g(y^{-1}y^{-1}xyx^{-1}x^{-1}yx),$$
$$C_g(y^{-1}xyx^{-1}y^{-1}x^{-1}yx),\ C_g(y^{-1}x^{-1}yxyx^{-1}y^{-1}x),$$
$$C_g(y^{-1}x^{-1}yx^{-1}y^{-1}xyx),\ C_g(y^{-1}x^{-1}y^{-1}xyx^{-1}yx).$$

§2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by introducing the following three lemmas.

Lemma 2.1. *Let* $G = \langle a, b, c \mid a^9 = b^3 = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$*. Then all* 4*-valent connected Cayley graphs of* G *are normal.*

Proof. Let $X = \mathrm{Cay}(G, S)$ be a 4-valent connected Cayley graph and let $S = \{x, y, x^{-1}, y^{-1}\}$. Since all the elements of order 3 in G are commutative, by $G = \langle S\rangle$, we may assume

$o(x) = 9$ and $x = a^i b^j c^k$ $(3 \not| i)$. Thus a, b and c satisfy the same relations as do x, b and c^i and so we can assume that $S = \{a, a^{-1}, y, y^{-1}\}$. Now we consider these two cases $o(y) = 3$ or 9 separately.

Case 1: $o(y) = 3$.

In this case we have $y = a^{3i} b^j c^k$. Since $G = \langle S \rangle$, we have $3 \not| j$. Then a, b and c satisfy the same relations as do a, y and c^j. We can assume that $S = \{a, a^{-1}, b, b^{-1}\}$.

Let $\alpha \in A_1$ and let α fix S pointwise. We shall prove $\alpha = 1$. Since there is a unique triangle through each vertex of X, if α fixes g, ag $(g \in G)$, it also fix $a^i g$ (i integer) and if α fixes g and transposes ag and $a^{-1}g$, it also transposes $a^i g$ and $a^{-i} g$ (i integer). Since α fixes 1 and a, it fixes a^j for each integer j. Suppose that α transposes ab and $a^{-1}b$. Then α transposes $a^i b$ and $a^{-i} b$ and specially transposes $a^3 b$ and $a^{-3} b$. However $(a^3 b)^\alpha = (ba^3)^\alpha \in \{ba^3, b^{-1}a^3\}$ since α fixes a^3. Consequently $a^{-3}b = ba^3$ or $b^{-1}a^3$, which are impossible. Thus α fixes $N_1(b)$ pointwise and similarly $N_1(b^{-1})$ pointwise.

Let $C_1(baz_1z_2z_3z_4ab)$ $(z_1, z_2, z_3, z_4 \in S)$ be a cycle of length 8 which have ab, b, 1, b^{-1}, $a^{-1}b^{-1}$ as consecutive vertices on it. Then $baz_1z_2z_3z_4ab = 1$. Clearly there are two a^{-1}s and two b^{-1}s in $\{z_1, z_2, z_3, z_4\}$ and $z_4 \neq a^{-1}$. With these we can easily prove that there is only one such cycle, that is, $C_1(bab^{-1}a^{-2}b^{-1}ab)$. Since α fixes ab, b, 1, b^{-1}, $a^{-1}b^{-1}$, it fixes each vertex of the cycle and specially $a^{-1}b^{-1}ab$. Let $d(x, y)$ denote the distance between x and y in X $(x, y \in G)$. It is easy to check that $d(b^{-1}a, a^{-1}b^{-1}ab) = 2$ and $d(ba, a^{-1}b^{-1}ab) \neq 2$ which shows that α fixes $N_1(a)$ pointwise and similarly $N_1(a^{-1})$ pointwise.

By the connectivity of X, $\alpha = 1$. This implies that A_1 acts on S faithfully. The fact that there is only one triangle passing 1 gives rise to $|A_1| \leq 4$. On the other hand, it is easy to prove $|\mathrm{Aut}(G, S)| \geq 4$ and hence $A_1 = \mathrm{Aut}(G, S)$. By Proposition 1.1, X is normal.

Case 2: $o(y) = 9$.

In this case we have $y = a^{3i\pm1} b^j c^k$. Thus $3 \not| j$ because $G = \langle S \rangle$. Since a, $a^{3i} b^j c^k$ and c^j satisfy the same relations as do a, b and c, we can assume that

$$S = \{a, a^{-1}, ab, (ab)^{-1}\} \text{ or } \{a, a^{-1}, a^{-1}b, (a^{-1}b)^{-1}\}.$$

Since the automorphism of G, induced by $a \mapsto a^{-1}$, $b \mapsto b$, $c \mapsto c^{-1}$, maps $\{a$, a^{-1}, ab, $(ab)^{-1}\}$ to $\{a$, a^{-1}, $a^{-1}b$, $(a^{-1}b)^{-1}\}$, we can assume that $S = \{a, a^{-1}, ab, (ab)^{-1}\}$.

Now we state several facts and leave the simple proofs to the reader: (1) through 1 there are exactly 8 cycles of length 6 and each of them passes exactly one element of $T = \{a^8bc^2$, ab^2c, a^3, ab^2, a^6, a^8bc, ab^2c^2, $a^8b\}$ which is a subset of $N_3(1)$; (2) of the above cycles of length 6, two pass a and a^{-1}, two a and ab, two a^{-1} and $(ab)^{-1}$, two ab and $(ab)^{-1}$, and none of them passes a and $(ab)^{-1}$, or a^{-1} and ab; (3) none of the above cycles of length 6 passes a^2bc^2, a^7b^2c, a^2b or $a^7b^2c^2$ which are the elements of $N_1(a)$, $N_1(a^{-1})$, $N_1(ab)$, and $N_1((ab)^{-1})$ respectively, and there is at least one such cycle passing the other elements of $N_1(a)$, $N_1(a^{-1})$, $N_1(ab)$, and $N_1((ab)^{-1})$; (4) all the cycles of length 8, which contain 1 and two elements of T, pass bc^2 or b^2c which belong to $N_4(1)$.

Let $\alpha \in A_1$ and α fixes a. By the above fact (2) α fixes $(ab)^{-1}$, by (1) and (4) α fixes T and $\{b^2c, bc^2\}$ setwise, and by (3) α fixes a^2bc^2. Since $|N_1(a^2bc^2) \cap N_1(bc^2)| = |\{abc^2\}| = 1$ and $|N_1(a^2bc^2) \cap N_1(b^2c)| = 0$, α fixes b^2c and bc^2, and since $N_1(bc^2) \cap T = \{a^8bc^2$, $ab^2c^2\}$, $N_1(a^8bc^2) \cap N_1(a) = \{b^2\}$, and $N_1(ab^2c^2) \cap N_1(a) = \phi$, α fixes a^8bc^2 and ab^2c^2. Then α fixes a^{-1} by $N_1(a^8bc^2) \cap N_1(a^{-1}) = \{a^7\}$ and $N_1(a^8bc^2) \cap N_1(ab) = \phi$. Thus, α fixes S pointwise. We also have

$$N_1(a^8bc^2) \cap N_1(a) = \{b^2\}, \qquad N_1(a^8bc^2) \cap N_1(a^{-1}) = \{a^7\},$$
$$N_1(ab^2c^2) \cap N_1(ab) = \{a^2b^2c^2\}, \quad N_1(ab^2c^2) \cap N_1((ab)^{-1}) = \{b^2c^2\}.$$

These results together with α fixing a^2bc^2, a^7b^2c, a^2b and $a^7b^2c^2$ (the above fact (3)) show that α fixes $N_1(a)$, $N_1(a^{-1})$, $N_1(ab)$ and $N_1((ab)^{-1})$ pointwise. By the connectivity of X, $\alpha = 1$. Thus $|A_1| \leq 4$.

On the other hand, the automorphisms of G induced by $a \mapsto a^{-1}$, $b \mapsto b^2c^2$, $c \mapsto c$ and $a \mapsto ab$, $b \mapsto b^{-1}$, $c \mapsto c^{-1}$ belong to $\mathrm{Aut}(G,S)$ and hence $|\mathrm{Aut}(G,S)| \geq 4$. Consequently $A_1 = \mathrm{Aut}(G,S)$ and X is normal.

Lemma 2.2. *Let G be a p-group of nilpotent class 2 and let $X = \mathrm{Cay}(G,S)$ be a 4-valent connected Cayley graph where $S = \{x, x^{-1}, y, y^{-1}\}$. If either $p > 3$, or $p = 3$ and $o([x,y]) \neq 3$, then X is normal.*

Proof. Set $A = \mathrm{Aut}(X)$ and let $\alpha \in A_1$. It is enough to show that $\alpha \in \mathrm{Aut}(G,S)$ by Proposition 1.1, that is, for any positive integer m,

$$(s_1s_2\cdots s_m)^\alpha = s_1^\alpha s_2^\alpha \cdots s_m^\alpha, \tag{2.1}$$

where $s_i \in S$ for $1 \leq i \leq m$.

Let $n(g_1, g_2, \ldots, g_\ell)$ denote the number of cycles of length 8 containing $g_1, g_2, \ldots, g_\ell$ as consecutive vertices on them. By Proposition 1.4 we have $n(x,1,x^{-1}) = 2$ and $n(x,1,y) = 7$. Similarly, $n(y,1,y^{-1}) = 2$ and $n(x^{-1},1,y) = n(x^{-1},1,y^{-1}) = n(x,1,y^{-1}) = 7$. Thus $(s^{-1})^\alpha = (s^\alpha)^{-1}$ for any $s \in S$.

Clearly (2.1) holds for $m = 1$. Assume $m \geq 2$. Set $t = s_3s_4\cdots s_m$ ($t = 1$ if $m = 2$). By inductive hypothesis, we have $t^\alpha = s_3^\alpha s_4^\alpha \cdots s_m^\alpha$, $(s_1t)^\alpha = s_1^\alpha t^\alpha$, $(s_1^{-1}t)^\alpha = (s_1^{-1})^\alpha t^\alpha = (s_1^\alpha)^{-1}t^\alpha$, $(s_2t)^\alpha = s_2^\alpha t^\alpha$, and $(s_2^{-1}t)^\alpha = (s_2^\alpha)^{-1}t^\alpha$.

If $s_1 = s_2^{-1}$, (2.1) holds by $s_1^\alpha = (s_2^{-1})^\alpha = (s_2^\alpha)^{-1}$. By Proposition 1.4(1), we have $n(x^{-1}g, g, xg, x^2g) = 0$ and $n(x^{-1}g, g, xg, yxg) = n(x^{-1}g, g, xg, y^{-1}xg) = 1$, where $g \in G$. Similarly, for $s, s' \in S$ and $g \in G$, $n(s^{-1}g, g, sg, s^2g) = 0$ and $n(s^{-1}g, g, sg, s'sg) = 1$, where $s' \neq s^{\pm 1}$. Thus $(s_2^2t)^\alpha = (s_2^\alpha)^2t^\alpha$ by $n(s_2^{-1}t,\ t,\ s_2t,\ s_2^2t) = n((s_2^\alpha)^{-1}t^\alpha,\ t^\alpha, s_2^\alpha t^\alpha, (s_2^2t)^\alpha) = 0$ and $n((s_2^\alpha)^{-1}t^\alpha, t^\alpha, s_2^\alpha t^\alpha, (s_2^\alpha)^2t^\alpha) = 0$. This implies that (2.1) holds for $s_1 = s_2$. Hence we let $s_1 \neq s_2^{\pm 1}$. Then $s_1^\alpha \neq (s_2^\alpha)^{\pm 1}$. By Proposition 1.4(2), $n(yg, g, xg, x^2g) = n(yg, g, xg, y^{-1}xg) = 1$ and $n(yg, g, xg, yxg) = 5$ where $g \in G$. Similarly, for $s, s' \in S$ and $g \in G$, $n(s'g, g, sg, s^2g) = n(s'g, g, sg, (s')^{-1}sg) = 1$ and $n(s'g, g, sg, s'sg) = 5$ where $s' \neq s^{\pm 1}$. Noting that $n(s_1t, t, s_2t, s_1s_2t) = n(s_1^\alpha t^\alpha, t^\alpha, s_2^\alpha t^\alpha, (s_1s_2t)^\alpha) = 5$ and $n(s_1^\alpha t^\alpha, t^\alpha, s_2^\alpha t^\alpha, s_1^\alpha s_2^\alpha t^\alpha) = 5$, we have $(s_1s_2t)^\alpha = s_1^\alpha s_2^\alpha t^\alpha$, as required.

Lemma 2.3. *Let G be a 3-group of nilpotent class 2 and let $X = \mathrm{Cay}(G,S)$ be a 4-valent connected Cayley graph where $S = \{x, y, x^{-1}, y^{-1}\}$. If $o([x,y]) = 3$, then X is normal.*

Proof. Set $A = \mathrm{Aut}(X)$ and let G_R be the right regular representation of G. We shall prove $G_R \lhd A$.

First we claim that G_R is a Sylow 3-subgroup of A. Supposing the contrary, we have 3 divides $|N_A(G_R) : G_R|$. By [11, Proposition 1.3], $N_A(G_R) = G_R \cdot \mathrm{Aut}(G,S)$ and consequently 3 divides $|\mathrm{Aut}(G,S)|$. Let $\alpha \in \mathrm{Aut}(G,S)$ be an element of order 3. Then α fixes exactly one element in S. However, this is impossible since if $\mathrm{Aut}(G,S)$ fixes s ($s \in S$), it also fixes s^{-1}.

Obviously, A is a $\{2,3\}$-group and hence solvable. Let N be a minimal normal subgroup of A. Since $A = A_1 \cdot G_R$ and G_R is a Sylow 3-subgroup of A, A_1 is a Sylow 2-subgroup of A. Thus $O_2(A) = 1$ and N is an elementary abelian 3-group. Consequently $N \leq G_R$. If $N = G_R$, of course $G_R \lhd A$. Thus we assume that $N \neq G_R$.

Denote by $\Sigma = \{B_0, B_1, \cdots, B_{3^\ell-1}\}$ the set of orbits of N on $V(X)$. Then Σ is a complete block system of A. Consider the corresponding quotient graph $\overline{X}$ of X defined by $V(\overline{X}) = \Sigma$ and $(B_i, B_j) \in E(\overline{X})$ if and only if there exist $v_i \in B_i$, $v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. Let

K be the kernel of the action of A on Σ. Clearly N acts regularly on each B_i ($0 \le i \le 3^\ell - 1$). Since $|\Sigma| = 3^\ell$ is odd, $\overline{X}$ has valency 2 or 4. Now we need two sublemmas.

Sublemma 1. *K acts faithfully on each B_i.*

Proof. It is efficient to prove that K acts faithfully on B_0. We can assume $1 \in B_0$. If $\overline{X}$ has valency 4, then $K_1 = 1$ since the elements of $N_1(1)$ belong to different blocks. Clearly K is faithful on B_0. Thus we let $\overline{X}$ be of valency 2 and let $\alpha \in K$. Then α fixes B_0 pointwise. We only need to prove $\alpha = 1$.

Without loss of generality, let (B_0, B_1) be an edge of $\overline{X}$. If there are some edges of X in B_0, then the induced subgraph $\langle B_0 \rangle$ of X has valency 2 because $|V(\langle B_0 \rangle)|$ is odd. Hence each vertex in B_0 is adjacent to a unique vertex in B_1 that implies α fixes B_1 pointwise. By the connectivity of X, α fixes each B_i pointwise ($0 \le i \le 3^\ell - 1$), that is, $\alpha = 1$. If there is no edge of X in B_0, then each vertex in B_0 is adjacent to exactly two vertices in B_1 because $|\Sigma| = 3^\ell$ is odd. Conversely, each vertex in B_1 is adjacent to exactly two vertices in B_0 and consequently α fixes B_1 pointwise. Again by the connectivity of X, $\alpha = 1$.

Sublemma 2. *If $\overline{X}$ has valency 2, then the Sylow 3-subgroup of $C_A(N)$ is normal in A.*

Proof. Set $C = C_A(N)$ and let C_2 be a Sylow 2-subgroup of C. Let $\alpha \in C_2 \cap K$. By Frattini argument, $K = K_1 N$ where K_1 is the stabilizer of 1 in K. It follows that K_1 is a Sylow 2-subgroup of K and hence there exists a $\beta \in K$ such that $C_2 \cap K \le K_1^\beta = K_{1^\beta}$. Set $g_0 = 1^\beta$. Then $\alpha \in K_{1^\beta} = K_{g_0}$. From $\alpha \in C_A(N)$ we have $\alpha\gamma = \gamma\alpha$ for any $\gamma \in N$, which implies that $(g_0^\gamma)^\alpha = g_0^{\alpha\gamma} = g_0^\gamma$, that is, α fixes g_0^γ for any $\gamma \in N$. If $g_0 \in B_i$, then α fixes B_i pointwise by the transitivity of N on B_i. By Sublemma 1, we have $\alpha = 1$. Thus $C_2 \cap K = 1$ and $|C_2 K| = |C_2| \cdot |K|$.

Since $A/K \le \mathrm{Aut}(\overline{X}) \cong D_{2\cdot 3^\ell}$ where $D_{2\cdot 3^\ell}$ is the dihedral group of order $2 \cdot 3^\ell$, we obtain that $|A|$ divides $2 \cdot 3^\ell \cdot |K|$. Noting that A_1 and K_1 are Sylow 2-subgroups of A and K respectively, we have $|A_1| = |K_1|$ or $2|K_1|$. Since the order of a Sylow 2-subgroup of $C_2 K$ is no more than $|A_1|$, we have $|C_2| \le 2$ by $|C_2 K| = |C_2| \cdot |K|$. Now C is a $\{2,3\}$-group and $|C_2| \le 2$. By Sylow's Theorem, the Sylow 3-subgroup of C is normal in C and hence normal in A since $C \lhd A$.

Now we return to the proof of Lemma 2.3. For the simplicity of statement, we identify G with G_R.

To use induction on $|G|$, let $|G| = 3^3$. Then Lemma 2.2 is true by Proposition 1.3. Assume that $|G| \ge 3^4$. If $\overline{X}$ has valency 4, then $K = N \cdot K_1 = N$. In this case, $\overline{X} = \mathrm{Cay}(\overline{G}, \overline{S})$ with $\overline{G} = G/N$ and $\overline{S} = \{xN, x^{-1}N, yN, y^{-1}N\}$. If $\overline{G}$ has nilpotent class 2, the inductive hypothesis shows $\overline{G} \lhd \mathrm{Aut}(\overline{X})$ and thus $G \lhd A$. If $\overline{G}$ has nilpotent class less than 2, we have the same result by Proposition 1.2. Thus we let $\overline{X}$ be of valency 2.

If $N \le Z(G)$, Lemma 2.2 holds by Sublemma 2. Let $N \not\le Z(G)$. Since $N \cap Z(G) \ne 1$, we have $|N| \ne 3$. By the arbitrarity of N, we may assume that A has no minimal normal subgroup of order 3.

Set $z = [x, y]$. Since $G' \le Z(G)$ (G has nilpotent class 2), we have that for any integers i, j, $x^i y^j = y^j x^i z^{ij}$. Thus by $G = \langle S \rangle$, $G' = \langle z \rangle$ and by $o(z) = 3$, $(x^i y^j)^3 = x^{3i} y^{3j}$. Furthermore we can obtain that $g^3 \in Z(G)$ for any $g \in G$.

Note that N is an elementary abelian 3-group and $N \not\le Z(G)$. There exists an $x^i y^j z^k \in N$ such that $o(x^i y^j z^k) = 3$ and $x^i y^j z^k \notin Z(G)$. This implies $o(x^i y^j) = 3$ and $x^i y^j \notin Z(G)$ which forces $3 \nmid i$ or $3 \nmid j$. Without loss of generality, we can assume that $3 \nmid i$. Thus $o([x^i y^j, y]) = 3$. With these results and $G = \langle x, y \rangle = \langle x^i y^j, y \rangle$, y has order at least $|G|/3^2$.

If G has an element of order $|G|/3$, by [9, Theorem 5.3.4],

$$G = \langle a, b \mid a^{3^{n-1}} = b^3 = 1, [a, b] = a^{3^{n-2}} \rangle. \tag{2.2}$$

If G has no element of order $|G|/3$, set $a = y$, $b = x^i y^j$ and $c = [y, x^i y^j]$. Then

$$G = \langle a, b, c \mid a^{3^{n-2}} = b^3 = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle. \tag{2.3}$$

Given a finite 3-group H, we define $\Omega_1(H) = \langle h^3 \mid h \in H \rangle$. Clearly $\Omega_1(H) \lhd\lhd H$, that is, $\Omega_1(H)$ is a character subgroup of H. In both cases (2.2) and (2.3), $\Omega_1(G) = \langle a^3 \rangle$ and $\Omega_1(\Omega_1(G)) = \langle a^9 \rangle$. Since $a^3 \in Z(G)$, $\Omega_1(G) \le C = C_A(N)$. Let C_3 be a Sylow 3-subgroup of C. By Sublemma 2, we have $C_3 \lhd C$ and so $\Omega_1(G) \le C_3$. Then $\Omega_1(\Omega_1(G)) \le \Omega_1(C_3)$. Since $\Omega_1(G)$ is cyclic, $\Omega_1(C_3)$ is cyclic. Suppose $\Omega_1(\Omega_1(G)) \ne 1$. Then $\Omega_1(C_3)$ has a character subgroup of order 3. By $\Omega_1(C_3) \lhd\lhd C_3$ and $C_3 \lhd A$, A has a normal subgroup of order 3, contrary to our assumption. Thus we have $\Omega_1(\Omega_1(G)) = \langle a^9 \rangle = 1$. It implies that $n = 3$ for the group G given in (2.2), and $n = 3$ or 4 for the group given in (2.3). When $n = 3$, we have $G_R \lhd A$ by Proposition 1.3 and when $n = 4$, we have the same result by Lemma 2.1.

§3. A Cayley Graph Which Is Not Normal

In this section we give an example of 4-valent connected Cayley graph of a 3-group of order 81 with nilpotent class 3, which is not normal.

Example 3.1. Let $G = \langle a, b, c \mid a^9 = b^3 = c^3 = 1, [a, b] = a^3, [a, c] = b, [b, c] = 1 \rangle$ and $S = \{a, ac, a^{-1}, (ac)^{-1}\}$. Then G has nilpotent class 3 and $X = \mathrm{Cay}(G, S)$ is not normal. Moreover if we set $A = \mathrm{Aut}(X)$ and $A_1^* = \{\alpha \in A_1 \mid s^\alpha = s, \ \ \forall s \in S\}$, then $A_1^* \cong \mathrm{Aut}(G, S) \cong \mathbb{Z}_2$ and $A_1/A_1^* \cong D_8$.

The proof of this example is simple, but tedious. We omit it. As a complement, we can easily obtain that X is not normal with the help of a computer software "Nauty" or "Megma". In fact it is not hard to prove $\mathrm{Aut}(G, S) \cong \mathbb{Z}_2$ and these softwares can give us $|A_1| = 16$. This implies $A_1 \ne \mathrm{Aut}(G, S)$ and hence X is not normal by Proposition 1.1.

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