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SINGULAR LIMIT SOLUTIONS FOR TWO-DIMENSIONAL ELLIPTIC PROBLEMS WITH EXPONENTIALLY DOMINATED NONLINEARITY

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Abstract

The authors consider the existence of singular limit solutions for a family of nonlinear elliptic problems with exponentially dominated nonlinearity and Dirichlet boundary condition and generalize the results of [3].

Keywords Asymptotic behaviour, Exponentially dominated nonlinearity, Singular perturbation, Weighted Hölder space

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§1. Introduction

The purpose of this paper is to consider the existence of solutions $u : \Omega \subset \mathbb{C} \to \mathbb{R}$ for the following Dirichlet problem:

$$\begin{cases} -\Delta u = \rho^2 f(u) = \rho^2(e^u + e^{\gamma u}) & \text{in } \Omega \subset \mathbb{C}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\gamma \in (0, 1)$. Our motivation is to study the existence of non-minimal solutions with singular limit as the parameter ρ tends to 0 and extend the results of [3, 2] to more general functions which are just exponentially dominated. The additional term $e^{\gamma u}$ yields the possibility of better steady state models for physical phenomena having exponential nonlinearities (see for example [1] and [6]).

The asymptotic behaviour of solutions of (1.1) is well understood thanks to the work of Nagasaki and Suzuki[7] (for $\gamma < 1/4$) and a recent work in [10]. The Green's function $G(z, z')$, defined over $\Omega \times \Omega$, is given to be the unique solution of

$$\begin{cases} -\Delta_z G(z, z') = 8\pi\delta_{z=z'} & \text{in } \Omega, \\ G(z, z') = 0 & \text{on } \partial\Omega, \end{cases}$$

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and $H(z,z') = G(z,z') + 4\log|z-z'|$ denotes the regular part of Green's function.

Theorem 1.1.[7,10] *Let Ω be a regular bounded domain of $\mathbb{C}$, $\gamma \in (0,1)$ and $\rho > 0$. Let u_ρ be a sequence of solutions of* (1.1). *Assume that, as ρ tends to 0, the sequence u_ρ converges to some non trivial function u^* in $\mathcal{D}'(\Omega)$. Then, the limit function u^* satisfies*

$$\begin{cases} -\Delta u^* = 8\pi \sum\limits_{1\le j\le k} \delta_{z_j} & \text{in } \Omega, \\ u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In addition, the point $(z_1,\cdots,z_k) \in \Omega^k$ is a critical point of the function

$$\Psi : (z_1,\cdots,z_k) \in \mathbb{C}^k \longmapsto \sum_{j=1}^{k} H(z_j,z_j) + \sum_{j\ne l} G(z_j,z_l). \tag{1.3}$$

In this paper, we deal with the converse question: given $(z_1,\cdots,z_k) \in \Omega^k$ a critical point of the function Ψ defined in (1.3) and given u^* the solution of (1.2), does there exist a family u_ρ, solutions of (1,1), which converges to the function u^* as ρ tends to 0 ? This kind of problem was considered by many authors in some special cases (see for instance [5,9] and [8]). Recently, in [3], Baraket and Pacard have constructed a family u_ρ which converges to u^* when ρ tends to 0, for $f(u) = e^u$ on a general domain Ω. Later on, in [2] this result was extended to the case $f(u) = e^u + e^{\gamma u}$ with $\gamma \in (0, 7/8)$, but the method seems not to work for $\gamma \in (7/8, 1)$. Here, we will use a general construction to solve the problem for all $3/4 \le \gamma < 1$. Our main result reads:

Theorem 1.2. *Let Ω be a regular bounded domain of $\mathbb{C}$. Let $\gamma \in (0,1)$ and $(z_1,\cdots,z_k) \in \Omega^k$ be a nondegenerate critical point of the function Ψ defined in* (1.3). *Then, there exists a one-parameter family of solutions u_ρ of* (1.1), *which converges to u^*, solution of* (1.2), *when ρ tends to* 0.

Our proof is based on some refinements of arguments in [3]. Our paper is organized as follows. We will recall some notations and results of [3] in §2, and we construct our approximate solutions in §3, where a sharp estimate on approximate solutions is established. Finally, the nonlinear problem is solved in §4. Given the fact that the proof of our result is rather technical, we shall restrict our attention to one point blow-up solutions. The case where there might be many blow up points can be treated completely similarly, though the computations should be more involved. In the following, we assume that $k = 1$, $\gamma \in [3/4, 1)$ and c denotes always a constant independent of ρ, even its value could be changed from one line to another one.

§2. Known Results in [3] and Refinements

For the sake of completeness, we recall some useful notations and results in [3]: For any $\varepsilon, \tau > 0$ and $\beta \in \mathbb{C}$, define ε to be the smallest positive solution of $\rho^2 = 8\varepsilon^2/(1+\varepsilon^2)^2$ (clearly, $\rho = O(\varepsilon)$ when $\rho \to 0$). Note

$$u_{\varepsilon,\tau}(z) = 2\log(1+\varepsilon^2) - 2\log(\varepsilon^2 + \tau^2|z|^2) + 2\log\tau,$$

$$u_{\varepsilon,\tau,\beta}(z) = 2\log(1+\varepsilon^2) - 2\log(\varepsilon^2 + \tau^2|z|^2|1+\beta z^2|^2) + 2\log\tau + 2\log|1+3\beta z^2|.$$

We know that $u_{\varepsilon,\tau}$ (resp. $u_{\varepsilon,\tau,\beta}$) are solutions of $\Delta u + \rho^2 e^u = 0$ on $\mathbb{C}$ (resp. $\mathbb{C}\setminus\{z, 1+3\beta z^2 = 0\}$). Define $L_{\varepsilon,\tau}$ and $L_{\varepsilon,\tau,\beta}$ as the following linearized operators about $u_{\varepsilon,\tau}$ and $u_{\varepsilon,\tau,\beta}$:

$$L_{\varepsilon,\tau}\omega = -\Delta\omega - \rho^2 e^{u_{\varepsilon,\tau}}\omega, \quad L_{\varepsilon,\tau,\beta}\omega = -\Delta\omega - \rho^2 e^{u_{\varepsilon,\tau,\beta}}\omega.$$

To understand the inversion of these operators, we introduce some weighted Hölder spaces as in [3]. Let $\mathcal{S} = \{z_i\}_{1\le i\le k}$ be a finite subset of Ω. We choose a positive function $d(z)$, smooth in $\Omega \setminus \mathcal{S}$ such that $d(z) = |z - z_i|$ for z sufficiently close to z_i and set

$$|u|_{k,\alpha,[\sigma,2\sigma]} = \sup_{d(z)\in[\sigma,2\sigma]} \Big(\sum_{j=0}^{k} \sigma^j |\nabla^j u(z)|\Big) + \sigma^{k+\alpha} \sup_{d(x),d(y)\in[\sigma,2\sigma]} \Big(\frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}\Big).$$

Definition 2.1. *Let Ω be a regular bounded domain of $\mathbb{C}$. For any $\nu \in \mathbb{R}$ and $\alpha \in (0,1)$, $\mathcal{S}$ a finite set of singularities in Ω, the space $C^{k,\alpha}_\nu(\Omega \setminus \mathcal{S})$ is defined to be the collection of all functions $u \in C^{k,\alpha}(\Omega \setminus \mathcal{S})$ for which the norm*

$$\|u\|_{k,\alpha,\nu} \equiv \sup_{\sigma \le \frac{1}{2}\operatorname{diam}\Omega} \sigma^{-\nu} |u|_{k,\alpha,[\sigma,2\sigma]}$$

is bounded. Moreover, define $C^{k,\alpha}_{\nu,D}(\Omega \setminus \mathcal{S}) = \{u \in C^{k,\alpha}_\nu(\Omega \setminus \mathcal{S}), u = 0 \text{ on } \partial\Omega\}$.

In all this paper, we denote by $B_r(z)$ the ball of radius r centered at z, B_r when the center is the origin 0, and $B_r^* = B_r \setminus \{0\}$. The properties of $L_{\varepsilon,\tau}$ and $L_{\varepsilon,\tau,\beta}$ are described by the following propositions:

Proposition 2.1.[3] *For all $\nu \in (1,2)$ and all $\tau > 0$, there exist two continuous linear forms $H^0_{\varepsilon,\tau}(\cdot)$ (resp. $H^1_{\varepsilon,\tau}(\cdot)$) defined from $C^{0,\alpha}_{\nu-2}(B_1^*)$ into $\mathbb{R}$ (resp. $\mathbb{C}$) such that for all $f \in C^{0,\alpha}_{\nu-2}(B_1^*)$, the solution of*

$$\begin{cases} L_{\varepsilon,\tau} w = f & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1 \end{cases} \tag{2.1}$$

can be uniquely decomposed as

$$w(z) = G_{\varepsilon,\tau}(f)(z) + H^0_{\varepsilon,\tau}(f)\phi_0\Big(\frac{\tau}{\varepsilon}z\Big) + 2H^1_{\varepsilon,\tau}(f)\cdot\frac{\tau}{\varepsilon}\phi_1\Big(\frac{\tau}{\varepsilon}z\Big),$$

where $\phi_0(z) = \frac{1-|z|^2}{1+|z|^2}$, $\phi_1(z) = \frac{2z}{1+|z|^2}$ and $z\cdot z' \equiv (z\bar{z}' + \bar{z}z')/2$, for all z, $z' \in \mathbb{C}$. In addition, the following properties hold :

- *Assume that $1 < \mu < 2$, then the linear operator $G_{\varepsilon,\tau}$ is well defined from the space $C^{0,\alpha}_{\mu-2}(B_1^*)$ into the space $C^{2,\alpha}_\mu(B_1^*)$ and stays bounded independently of $\varepsilon \in (0,1)$.*
- *Assume that $-2 < \mu < 2$, then the restriction of $G_{\varepsilon,\tau}$ to the space of functions spanned by $\{e^{\pm in\theta}h_{\pm n}(r)/n > 1\}$ is well defined from the space $C^{0,\alpha}_{\mu-2}(B_1^*)$ into the space $C^{2,\alpha}_\mu(B_1^*)$.*
- *Assume that $\mu > 0$, then the linear form $H^0_{\varepsilon,\tau}(\cdot)$ is well defined in $C^{0,\alpha}_{\mu-2}(B_1^*)$ and bounded independently of $\varepsilon \in (0,\tau/2)$.*
- *Assume that $\mu > 1$, then the linear form $H^1_{\varepsilon,\tau}(\cdot)$ is well defined in $C^{0,\alpha}_{\mu-2}(B_1^*)$ and bounded independently of $\varepsilon \in (0,1)$.*

Proposition 2.2.[3] *For all $\nu \in (1,2)$, all $\tau > 0$ and all $\beta \in \mathbb{C}$ with $|\beta| < 1/4$, there exist $\varepsilon_0 > 0$ and two continuous linear forms $H^0_{\varepsilon,\tau,\beta}(\cdot)$(resp. $H^1_{\varepsilon,\tau,\beta}(\cdot)$) defined from $C^{0,\alpha}_{\nu-2}(B_1^*)$ into $\mathbb{R}$ (resp. $\mathbb{C}$) such that for any $\varepsilon \in (0,\varepsilon_0)$ and $f \in C^{0,\alpha}_{\nu-2}(B_1^*)$, the solution of*

$$\begin{cases} L_{\varepsilon,\tau,\beta} w = f & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1 \end{cases} \tag{2.2}$$

can be uniquely decomposed as

$$w(z) = G_{\varepsilon,\tau,\beta}(f)(z) + H^0_{\varepsilon,\tau,\beta}(f)\partial_\tau u_{\varepsilon,\tau,\beta}(z) + 2H^1_{\varepsilon,\tau,\beta}(f)\cdot\partial_{\bar{z}} u_{\varepsilon,\tau,\beta}.$$

In addition, there exists $c > 0$ (*independent of* f *and* $\varepsilon < \varepsilon_0$) *such that*

$$||G_{\varepsilon,\tau,\beta}(f)||_{2,\alpha,\nu} \le c(||G_{\varepsilon,\tau}(f)||_{2,\alpha,\nu} + \varepsilon^2|H^0_{\varepsilon,\tau}(f)| + |H^1_{\varepsilon,\tau}(f)|),$$
$$|H^0_{\varepsilon,\tau,\beta}(f)| \le c(\varepsilon^2||G_{\varepsilon,\tau}(f)||_{2,\alpha,\nu} + |H^0_{\varepsilon,\tau}(f)| + \varepsilon^2|H^1_{\varepsilon,\tau}(f)|),$$
$$|H^1_{\varepsilon,\tau,\beta}(f)| \le c(\varepsilon^2||G_{\varepsilon,\tau}(f)||_{2,\alpha,\nu} + \varepsilon^2|H^0_{\varepsilon,\tau}(f)| + |H^1_{\varepsilon,\tau}(f)|).$$

Moreover

$$||\partial_r G_{\varepsilon,\tau,\beta}(f)|_{\partial B_1}||_{1,\alpha} \le c(||\partial_r G_{\varepsilon,\tau}(f)|_{\partial B_1}||_{1,\alpha} + \varepsilon^2||G_{\varepsilon,\tau}(f)||_{2,\alpha,\nu} + \varepsilon^2|H^0_{\varepsilon,\tau}(f)| + |H^1_{\varepsilon,\tau}(f)|).$$

We need however some refinements for proving our results. We define the subspace of even functions in $C^{k,\alpha}_\nu(B_1^*)$ by

$$\mathcal{E}^{k,\alpha}_\nu(B_1^*) = \{f \in C^{k,\alpha}_\nu(B_1^*);\ \text{such that } f(z) = f(-z), \forall\, z \in B_1^*\}. \tag{2.3}$$

Clearly, $\mathcal{E}^{k,\alpha}_\nu(B_1^*)$ is an algebra for any $\nu > 0$. Furthermore, we can get more precise estimations of solutions of (2.2) when f is even, in particular for the L^∞ norm $\|w\|_\infty$. The reason is that we do not have terms like $f_1(r)e^{i\theta}$ in the expansion of f, so the operators $L_{\varepsilon,\tau}$ or $L_{\varepsilon,\tau,\beta}$ are invertible on $\mathcal{E}^{0,\alpha}_{\delta-2}(B_1^*)$ for all δ in $(0,2)$, instead of $\delta \in (1,2)$ as Propositions 2.1 and 2.2 required. More precisely, we have

Proposition 2.3. *Assume that* $\delta \in (0,2)$, $\tau > 0$ *and* $|\beta| \le 1/4$ *are given. Then there exist* $\varepsilon_0 > 0$ *and a continuous linear form* $H^0_{\varepsilon,\tau,\beta}(\cdot)$, *defined from* $\mathcal{E}^{0,\alpha}_{\delta-2}(B_1^*)$ *into* $\mathbb{R}$, *such that for any* $f \in \mathcal{E}^{0,\alpha}_{\delta-2}(B_1^*)$ *and* $\varepsilon \in (0,\varepsilon_0)$, *the solution of* (2.2) *can be uniquely decomposed as* $w(z) = G_{\varepsilon,\tau,\beta}(f)(z) + H^0_{\varepsilon,\tau,\beta}(f)\partial_\tau u_{\varepsilon,\tau,\beta}(z)$. *Moreover, there exists* $c > 0$ *such that* $\|G_{\varepsilon,\tau,\beta}(f)\|_{2,\alpha,\delta} + |H^0_{\varepsilon,\tau,\beta}(f)| \le c\|f\|_{0,\alpha,\delta-2}$.

Sketch of Proof. First, we show the corresponding result for $L_{\varepsilon,\tau}$, easily obtained by Proposition 2.1. We follow the proof of Proposition 2.2 in [3]. In step 1, we get a function $\omega \in \mathcal{E}^{2,\alpha}_\delta(B_1^*)$ such that $L_{\varepsilon,\tau}\omega = f$ and $w|_{\partial B_1}$ is constant in $\mathbb{R}$. We prove then the orthogonal projection of $\mathrm{Span}\{\partial_\tau u_{\varepsilon,\tau,\beta}(e^{i\theta})\}$ on $\mathbb{R}$ in $L^2(\partial B_1)$ is one-to-one for ε sufficiently small, and finally, we complete our proof as in step 3, just by remarking that $u_{\varepsilon,\tau}$ and $u_{\varepsilon,\tau,\beta}$ are even functions.

Remark 2.1. We see that with minor changes, the results of Propositions 2.1 to 2.3 hold still if we replace B_1 by any fixed B_r $(r > 0)$. Proposition 2.3 is a key point for our proof, which will permit us to solve (1.1) in symmetric case (see §3), i.e. when Ω is B_r, $z_1 = 0$ (see Lemma 2.1), and will permit us to construct some appropriate approximate solutions.

Recall now the approximate solutions in [3]. Suppose that z_1 is a critical point of $\Psi(z) = H(z, z_1)$ and define

$$-2\log\tau_0 \equiv \Psi(z_1) = H(z_1, z_1) \text{ and } \beta_0 \equiv \frac{1}{4}\partial_z^2\Psi(z_1). \tag{2.4}$$

Let $\tau > 0$ and $a \in \mathbb{C}$ be given. We set $r_0 = \varepsilon^{2/5}$,

$$\bar{u}_\varepsilon(\tau, a, z) \equiv \chi\Big(\frac{z - z_1 - a}{r_0}\Big)u_{\varepsilon,\tau_0+\tau,\beta_0}(z - z_1 - a) + \Big[1 - \chi\Big(\frac{z - z_1 - a}{r_0}\Big)\Big]G(z, z_1 + a), \tag{2.5}$$

where χ is a C^∞, positive, radial function defined in $\mathbb{C}$ such that $\chi(z) = 1$ in $B_{5/4}$ and $\chi(z) = 0$ outside $B_{7/4}$. By the invariance of our problem under the transformation $\tau \mapsto u(\tau z) + 2\log\tau$, we may assume that $B_2(z_1) \subset\subset \Omega$ and $|\beta_0|$ can be chosen to be less than $1/4$. Define $\mathcal{L}_\varepsilon$ by $\mathcal{L}_\varepsilon\omega = -\Delta\omega - \rho^2 e^{u_{\varepsilon,\tau_0,\beta_0}(z-z_1)}\chi(z - z_1)\omega$, then the behaviours of $\mathcal{L}_\varepsilon$ as $\varepsilon \to 0$ is given by

Proposition 2.4.[3] *Assume that $1 < \nu < 2$ is given. Then, there exist $\varepsilon_0 > 0$ and two continuous linear forms $\mathcal{H}^0_\varepsilon(\cdot)$ (resp. $\mathcal{H}^1_\varepsilon(\cdot)$) (depending on ν, τ_0 and β_0), defined from $C^{0,\alpha}_{\nu-2}(\Omega\setminus\{z_1\})$ into $\mathbb{R}$ (resp. $\mathbb{C}$) such that for any $\varepsilon\in(0,\varepsilon_0)$ and $f\in C^{0,\alpha}_{\nu-2}(\Omega\setminus\{z_1\})$, the solution of*

$$\begin{cases}\mathcal{L}_\varepsilon w = f & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega\end{cases}\tag{2.6}$$

can be uniquely decomposed as

$$w(z)=\mathcal{G}_\varepsilon(f)(z)+\chi(z-z_1)\Big(\mathcal{H}^0_\varepsilon(f)\partial_\tau u_{\varepsilon,\tau^0_1,\beta_0}(z-z_1)+2\mathcal{H}^1_\varepsilon(f)\cdot\partial_{\bar z}u_{\varepsilon,\tau^0_1,\beta_0}(z-z_1)\Big).\tag{2.7}$$

In addition, there exists $c>0$ (independent of f and $\varepsilon<\varepsilon_0$) such that

$$\|\mathcal{G}_\varepsilon(f)\|_{2,\alpha,\nu}\le c\Big(\|G_{\varepsilon,\tau_0}(\bar f)\|_{2,\alpha,\nu}+|H^0_{\varepsilon,\tau_0}(\bar f)|+|H^1_{\varepsilon,\tau_0}(\bar f)|+\|f|_{\Omega\setminus B_1(z_1)}\|_{0,\alpha}\Big),$$

$$|\mathcal{H}^0_\varepsilon(f)|+|\mathcal{H}^1_\varepsilon(f)|\le c\Big(\varepsilon^2\|G_{\varepsilon,\tau_0}(\bar f)\|_{2,\alpha,\nu}+|H^0_{\varepsilon,\tau_0}(\bar f)|+|H^1_{\varepsilon,\tau_0}(\bar f)|\Big),$$

where $\bar f$ is the function $f(z+z_1)$, defined in B_1.

§3. Approximate Solutions and Their Estimates

Here, we will construct some subtle approximate solutions and obtain their convenient estimates. We know that (see [2]) the desired estimations hold with $\bar u_\varepsilon(0,0,\cdot)$ for the case $\gamma<3/4$, but for γ closer to 1, this construction seems not to be sufficient. Our idea is to deform slightly $u_{\varepsilon,\tau_0+\tau,\beta_0}$ by adding a function h to get a local solution of (1.1). In fact, let h be a solution of

$$\begin{cases}-\Delta h=\rho^2e^{u_{\varepsilon,\tau_0+\tau,\beta_0}}(e^h-1)+\rho^2e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}}e^{\gamma h} & \text{in } B_2,\\ h=0 & \text{on } \partial B_2.\end{cases}\tag{3.1}$$

Of course $u_{\varepsilon,\tau_0+\tau,\beta_0}+h$ verifies (1.1) in B_2. The existence and the sharp estimates of h are given by

Lemma 3.1. *Let δ be fixed in $(0,1-\gamma]$. Then there exist $c,\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, $|\tau|\le\varepsilon^{2\delta}$, we have a unique solution of (3.1) such that $h\in\mathcal{E}=\mathcal{E}^{2,\alpha}_\delta(B_2^*)\oplus\mathrm{Span}\{\partial_\tau u_{\varepsilon,\tau_0,\beta_0}\}$, and if $h=\bar h+\lambda\partial_\tau u_{\varepsilon,\tau_0,\beta_0}$, then $\|\bar h\|_{2,\alpha,\delta}+|\lambda|\le 2c\varepsilon^{4(1-\gamma)-\delta}$.*

Proof. Let $\mathcal{E}=\mathcal{E}^{2,\alpha}_\delta(B_2^*)\oplus\mathrm{Span}\{\partial_\tau u_{\varepsilon,\tau_0,\beta_0}\}$ be endowed with the norm $\|(\bar h,\lambda)\|_{\mathcal{E}}=\|\bar h\|_{2,\alpha,\delta}+|\lambda|$. By the regularity of $\partial_\tau u_{\varepsilon,\tau_0,\beta_0}$ (bounded), we have immediately $\|h\|_\infty\le c\|h\|_{\mathcal{E}}$ and $|\nabla h(z)|\le c\|h\|_{\mathcal{E}}/|z|$. The Proposition 2.3 and Remark 2.1 mean that $L_{\varepsilon,\tau,\beta}$ is an isomorphism from $\mathcal{E}$ into $\mathcal{E}^{0,\alpha}_{\delta-2}(B_2^*)$. We transform the equation (3.1) as follows:

$$\begin{cases}L_{\varepsilon,\tau_0,\beta_0}h=T(h)\\ \qquad=(L_{\varepsilon,\tau_0,\beta_0}-L_{\varepsilon,\tau_0+\tau,\beta_0})h+\rho^2e^{u_{\varepsilon,\tau_0+\tau,\beta_0}}(e^h-1-h)\\ \qquad\quad+\rho^2e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}}e^{\gamma h}\\ \qquad=E(h)+F(h)+G(h) \text{ in } B_2,\\ h=0 \text{ on } \partial B_2.\end{cases}\tag{3.2}$$

One result that we shall use frequently as in [3], and without comments, is that to check if a function u is an element of $C^{0,\alpha}_\nu(\Omega\setminus\{z_0\})$; it is usually sufficient to check that $|z-z_0|^{-\nu}|u(z)|\le c$ and $|z-z_0|^{1-\nu}|\nabla u(z)|\le c$. Since the two estimates are always similar in this work, we present only the calculus for $|x|^{-\nu}|u(x)|$ in general. We have then

$$\|T(0)\|_{0,\alpha,\delta-2} = \|\rho^2 e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}}\|_{0,\alpha,\delta-2} \le c\varepsilon^2 \sup_{r\le 2} \frac{r^{2-\delta}}{(\varepsilon^2+\tau_0^2 r^2)^{2\gamma}} \le c_1 \varepsilon^{4(1-\gamma)-\delta},$$

where c and c_1 are constants independent of $|\tau| \le \tau_0/2$. We get also

$$\|E(h_1)\|_{0,\alpha,\delta-2} \le c\varepsilon^{-\delta}|\tau|\|h_1\|_{\mathcal{E}}, \quad \|F(h_1)-F(h_2)\|_{0,\alpha,\delta-2} \le c\varepsilon^{4(1-\gamma)-2\delta}\|h_1-h_2\|_{\mathcal{E}},$$
$$\|G(h_1)-G(h_2)\|_{0,\alpha,\delta-2} \le c\varepsilon^{4(1-\gamma)-\delta}\|h_1-h_2\|_{\mathcal{E}} \ \text{ for any } |\tau|\le \tau_0/2,\ h_1,\ h_2\in\mathcal{E}.$$

So, for any h_1, h_2 in $B_{2c_1\varepsilon^{4(1-\gamma)-\delta}}$ of $\mathcal{E}$,

$$\|T(h_1)-T(h_2)\|_{0,\alpha,\delta-2} \le c_2(\varepsilon^{4(1-\gamma)-2\delta}+\varepsilon^{\delta})\|h_1-h_2\|_{\mathcal{E}} \text{ for any } |\tau|\le\varepsilon^{2\delta}.$$

By a classical fixed point argument, when ε is small enough, we get the existence and uniqueness of h, the solution of (3.1) with desired estimates (for $c = c_1$).

Clearly, h is a one parameter family of functions depending on τ. With the same idea, we can also estimate $\partial_\tau h$. We will denote by h_0 and $\partial_\tau h_0$ their values for $\tau = 0$.

Lemma 3.2. *Let δ, ε_0 be as in Lemma 3.1. Then for any $\varepsilon \in (0,\varepsilon_0)$, the mapping $\tau \mapsto h_\tau$, the solution of (3.2) is derivable in $[-\varepsilon^{2\delta}, \varepsilon^{2\delta}]$, and there exists $c > 0$ (independent of $\varepsilon < \varepsilon_0$) such that $\|\partial_\tau h\|_{\mathcal{E}} \le c\varepsilon^{4(1-\gamma)-\delta}$.*

Proof. It suffices to remark that $\partial_\tau h$ is the unique solution of the following equation

$$\begin{cases} L_{\varepsilon,\tau_0,\beta_0} g = T_d(g) = (L_{\varepsilon,\tau_0,\beta_0} - L_{\varepsilon,\tau_0+\tau,\beta_0})g \\ \qquad + \rho^2 e^{u_{\varepsilon,\tau_0+\tau,\beta_0}}(e^h-1)g + \gamma\rho^2 e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}} e^{\gamma h} g \\ \qquad + \left[\rho^2 e^{u_{\varepsilon,\tau_0+\tau,\beta_0}}(e^h-1-h) + \gamma\rho^2 e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}} e^{\gamma h}\right]\partial_\tau u_{\varepsilon,\tau_0,\beta_0} \text{ in } B_2, \\ \qquad g = 0 \text{ on } \partial B_2, \end{cases} \tag{3.3}$$

where h is the solution given by (3.2). As in the proof of Lemma 3.1, we get that for any $|\tau| \le \varepsilon^{2\delta}$, $\|T_d(0)\|_{0,\alpha,\delta-2} \le c_3\varepsilon^{4(1-\gamma)-\delta}$ and $\|T_d(g_1)-T_d(g_2)\|_{0,\alpha,\delta-2} \le c_4(\varepsilon^{4(1-\gamma)-2\delta} + \varepsilon^{\delta})\|g_1-g_2\|_{\mathcal{E}}$. This completes our proof.

Denote $\tilde{u}_\varepsilon(\tau,a,z) = \bar{u}_\varepsilon(\tau,a,z) + h(z-z_1)\chi(z-z_1)$. We shall get a sharp estimate on the pre-image of the error function $\Delta\tilde{u}_\varepsilon(0,0,\cdot) + \rho^2 e^{\tilde{u}_\varepsilon(0,0,\cdot)} + \rho^2 e^{\gamma\tilde{u}_\varepsilon(0,0,\cdot)}$ by the operator

$$\Lambda_\varepsilon = \Delta + \rho^2 e^{\bar{u}_\varepsilon(0,0,\cdot)}.$$

We know the following

Proposition 3.1.[3] Assume that $1 < \nu < 2$ is given. There exist c, $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$ and for all $f \in C^{0,\alpha}_{\nu-2}(\Omega\setminus\{z_1\})$, there exists a unique $w \in \mathcal{F}^{2,\alpha}_\nu$ satisfying $\Lambda_\varepsilon w = f$ in Ω and $||w||_{\mathcal{F}} \le c||f||_{0,\alpha,\nu-2}$. In addition

$$\|w\|_{\mathcal{F}} \le c\left(r_0^\nu\|\mathcal{G}_\varepsilon(f)\|_{2,\alpha,\nu} + |\mathcal{H}^0_\varepsilon(f)| + |\mathcal{H}^1_\varepsilon(f)|\right),$$

where

$$\mathcal{F}^{2,\alpha}_\nu \equiv C^{2,\alpha}_{\nu,\mathcal{D}}(\Omega\setminus\{z_1\}) \oplus \mathrm{Span}\{\partial_\tau\bar{u}_\varepsilon(0,0,\cdot)\} \oplus \mathrm{Span}\{\partial_a\bar{u}_\varepsilon(0,0,\cdot), \partial_{\bar{a}}\bar{u}_\varepsilon(0,0,\cdot)\}$$

and the norm of $w(z) = v(z) + \tau\partial_\tau\bar{u}_\varepsilon(0,0,\cdot) + 2a\cdot\partial_{\bar{a}}\bar{u}_\varepsilon(0,0,\cdot) \in \mathcal{F}^{2,\alpha}_\nu$ is defined by

$$\|w\|_{\mathcal{F}} = r_0^\nu\|v\|_{2,\alpha,\nu} + |\tau|_{\mathbb{R}} + |a|_{\mathbb{C}}.$$

Thanks to [3] (Corollary 2, page 33),

$$\|\Lambda_\varepsilon^{-1}(\Delta\bar{u}_\varepsilon(0,0,\cdot) + \rho^2 e^{\bar{u}_\varepsilon(0,0,\cdot)})\|_{\mathcal{F}} \le c\varepsilon^2 r_0^{-2}|\log\varepsilon|,$$

so it suffices to check the pre-image of the difference $\xi = \Delta(h_0\chi) + \rho^2 e^{\bar{u}_\varepsilon(0,0,\cdot)}(e^{h_0\chi} - 1) + \rho^2 e^{\gamma\bar{u}_\varepsilon(0,0,\cdot)+\gamma h_0\chi}$, where we write $h_0\chi$ as an abbreviation of $h_0(z - z_1)\chi(z - z_1)$.

In $B_1(z_1)$, $\xi = \Delta h_0 + \rho^2 e^{\bar{u}_\varepsilon(0,0,\cdot)}(e^{h_0} - 1) + \rho^2 e^{\gamma\bar{u}_\varepsilon(0,0,\cdot)+\gamma h_0}$. Using (3.1) we have

$$\xi = \rho^2\left(e^{\bar{u}_\varepsilon(0,0,\cdot)} - e^{u_{\varepsilon,\tau_0,\beta_0}}\right)\left(e^{h_0} - 1\right) + \rho^2 e^{\gamma h_0}\left(e^{\gamma\bar{u}_\varepsilon(0,0,\cdot)} - e^{\gamma u_{\varepsilon,\tau_0,\beta_0}}\right).$$

Clearly $\xi \equiv 0$ in $B_{r_0}(z_1)$. In $B_1(z_1)\backslash B_{r_0}(z_1)$, $\bar{u}_\varepsilon(0,0,\cdot) - u_{\varepsilon,\tau_0,\beta_0} = (G(z,z_1) - u_{\varepsilon,\tau_0,\beta_0})(1-\chi_1)$ where $\chi_1(z) = \chi((z - z_0)/r_0)$. By the choice of z_1, τ_0 and β_0, we have the following expansions: for any $r \in (\varepsilon, 1)$,

$$\begin{cases} u_{\varepsilon,\tau_0,\beta_0}(re^{i\theta}) - G(z_1 + re^{i\theta}, z_1) = O(\varepsilon^2 r^{-2} + r^3), \\ \partial_r u_{\varepsilon,\tau_0,\beta_0}(re^{i\theta}) - \partial_r G(z_1 + re^{i\theta}, z_1) = O(\varepsilon^2 r^{-3} + r^2). \end{cases} \tag{3.4}$$

We get then $|\xi(z)| \le c(\varepsilon^2|z - z_1|^{-1}\varepsilon^{4(1-\gamma)-\delta} + \varepsilon^2|z - z_1|^{3-4\gamma})$ in $B_1(z_1) \setminus B_{r_0}(z_1)$. Applying Proposition 2.1,

$$|H^1_{\varepsilon,\tau_0}| + |H^0_{\varepsilon,\tau_0}| + \|G_{\varepsilon,\tau_0}(\bar{\xi})\|_{2,\alpha,\nu} \le c\|\bar{\xi}\|_{0,\alpha,\nu-2} \le c\varepsilon^2 r_0^{1-\nu}, \quad \forall\, 1 < \nu < 2, \tag{3.5}$$

where $\bar{\xi} = \xi(z + z_1)|_{B_1}$. In particular,

$$\|\partial_r G_{\varepsilon,\tau_0}(\bar{\xi})|_{\partial B_1}\|_{1,\alpha} \le c\|G_{\varepsilon,\tau_0}(\bar{\xi})\|_{2,\alpha,\nu} \le c\varepsilon^2 r_0^{1-\nu}.$$

Moreover, in $\Omega \setminus B_1(z_1)$,

$$\xi = \Delta(h_0\chi) + \rho^2 e^{G(z,z_1)}(e^{h_0\chi} - 1) + \rho^2 e^{\gamma G(z,z_1)+\gamma h_0\chi}.$$

Since $\partial_\tau u_{\varepsilon,\tau_0,\beta_0}|_{\Omega\setminus B_1(z_1)}$ is bounded independently of $\varepsilon \in (0, \varepsilon_0)$ in $C^{2,\alpha}$,

$$\|\xi|_{\Omega\setminus B_1(z_1)}\|_{0,\alpha} \le c(\varepsilon^2 + \|h_0\|_{\mathcal{E}}) \le c\varepsilon^{4(1-\gamma)-\delta}. \tag{3.6}$$

Combining together (3.5), (3.6) and Proposition 2.4, we obtain: for any $1 < \nu < 2$,

$$\|\Gamma(\xi)\|_{2,\alpha,\nu} \le c\varepsilon^{4(1-\gamma)-\delta} \le c\varepsilon^{3(1-\gamma)}, \quad |\mathcal{H}^0_\varepsilon(\xi)| + |\mathcal{H}^1_\varepsilon(\xi)| \le c\varepsilon^2 r_0^{1-\nu}. \tag{3.7}$$

Using (3.7) and Proposition 3.1, recalling that $\gamma \ge 3/4$ and $\nu < 2$, so $\varepsilon^2 r_0^{1-\nu} \le \varepsilon^2 r_0^{-2}|\log\varepsilon| \le \varepsilon^{(1-\gamma)} r_0^\nu$ for ε small enough, we are led to conclude

Lemma 3.3. *Assume that $1 < \nu < 2$ and $\delta \in (0, 1-\gamma]$ are given. Then there exist $\varepsilon_0 > 0$ and $c_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\|\Lambda_\varepsilon^{-1}(\Delta\tilde{u}_\varepsilon(0,0,\cdot) + \rho^2 e^{\tilde{u}_\varepsilon(0,0,\cdot)} + \rho^2 e^{\gamma\tilde{u}_\varepsilon(0,0,\cdot)})\|_{\mathcal{F}} \le c_0\varepsilon^{(1-\gamma)} r_0^\nu.$$

§4. Nonlinear Fixed Point Argument

Now we will define some nonlinear mappings which allow us to solve (1.1) by the contraction mapping fixed point Theorem. Although the proof is similar to that in [3], some new technical difficulties appear. As in [3], for any $|a| < 1/4$ in $\mathbb{C}$, we define a family of diffeomorphisms $\Xi_a : \Omega \to \Omega$ depending smoothly on a, satisfying $\Xi_0 = Id$ and

$$\Xi_a(z) = z - a, \quad \forall\, z \in B_2(z_1).$$

We look for a solution of (1.1) in the form $\bar{u}_\varepsilon(\tau, a, z) + (h_\tau(z - z_1)\chi(z - z_1) + v) \circ \Xi_a$, with $w = (v, \tau, a) \in \mathcal{F}^{2,\alpha}_\nu$. Using $h_\tau\chi$ as an abbreviation of $h_\tau(z - z_1)\chi(z - z_1)$, we denote

$$\begin{aligned} \mathcal{M}(v,\tau,a) &= \Delta[\bar{u}_\varepsilon(\tau,a,z) + (h_\tau\chi)\circ\Xi_a + v\circ\Xi_a]\circ\Xi_a^{-1} \\ &\quad + \rho^2 e^{\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + h_\tau\chi + v} + \rho^2 e^{\gamma[\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + h_\tau\chi + v]}, \\ \mathcal{N}(v,\tau,a) &= \Delta(\bar{u}_\varepsilon(\tau,a,z) + v\circ\Xi_a)\circ\Xi_a^{-1} + \rho^2 e^{\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + v}, \end{aligned}$$

and $\mathcal{R}(v,\tau,a) = \mathcal{M}(v,\tau,a) - \mathcal{N}(v,\tau,a)$. By [3, Lemma 4, page 34], we can say that $D\mathcal{N}|_{(0,0,0)} = \Lambda_\varepsilon + \mathcal{S}$, with $\mathcal{S}$ a small perturbation satisfying $\|\mathcal{S}w\|_{0,\alpha,\nu-2} \le cr_0^{2-\nu}\|w\|_{\mathcal{F}}$. Notice also that $\mathrm{supp}(h_\tau\chi) \subset B_{7/4}$, hence $\Delta[(h_\tau\chi)\circ\Xi_a]\circ\Xi_a^{-1} \equiv \Delta(h_\tau\chi)$ for any $|a| < 1/4$.

We study now the behaviour of $D\mathcal{R}|_{(0,0,0)}$, when ε tends to 0. First, we check $D\mathcal{R}|_{(0,0,0)}(0,\tau,a)$. Define $\mathcal{R}_1(\tau,a) = \mathcal{R}(0,\tau,a)$, we have

$$\mathcal{R}_1(\tau,a) = \Delta(h_\tau\chi) + \rho^2 e^{\bar u_\varepsilon(\tau,a,z)\circ\Xi_a^{-1}}(e^{h_\tau\chi}-1) + \rho^2 e^{\gamma\bar u_\varepsilon(\tau,a,z)\circ\Xi_a^{-1}+\gamma h_\tau\chi}.$$

In $B_1(z_1)$, $\chi \equiv 1$,

$$\mathcal{R}_1(\tau,a) = \rho^2(e^{h_\tau}-1)\left(e^{\bar u_\varepsilon(\tau,a,z)\circ\Xi_a^{-1}} - e^{u_{\varepsilon,\tau_0+\tau,\beta_0}}\right) + \rho^2 e^{\gamma h_\tau}\left(e^{\gamma\bar u_\varepsilon(\tau,a,z)\circ\Xi_a^{-1}} - e^{\gamma u_{\varepsilon,\tau_0+\tau,\beta_0}}\right),$$

so $\mathcal{R}_1 \equiv 0$ in $B_{r_0}(z_1)$. In $B_1(z_1)\setminus B_{r_0}(z_1)$, since $\bar u_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} = \chi_1 u_{\varepsilon,\tau_0+\tau,\beta_0} + (1-\chi_1)G(z+a,z_1+a)$ with $\chi_1(z) = \chi((z-z_1)/r_0)$, we obtain

$$\begin{aligned} D\mathcal{R}_1|_{(0,0)}(\tau,a) &= \rho^2 e^{h_0}(e^{\bar u_\varepsilon(0,0,\cdot)} - e^{u_{\varepsilon,\tau_0,\beta_0}})\tau\partial_\tau h_0 \\ &+ \rho^2(e^{h_0}-1)(\chi_1 e^{\bar u_\varepsilon(0,0,\cdot)} - e^{u_{\varepsilon,\tau_0,\beta_0}})\tau\partial_\tau u_{\varepsilon,\tau_0,\beta_0} \\ &+ \gamma\rho^2 e^{\gamma h_0}(e^{\gamma\bar u_\varepsilon(0,0,\cdot)} - e^{\gamma u_{\varepsilon,\tau_0,\beta_0}})\tau\partial_\tau h_0 \\ &+ \gamma\rho^2 e^{\gamma h_0}(\chi_1 e^{\gamma\bar u_\varepsilon(0,0,\cdot)} - e^{\gamma u_{\varepsilon,\tau_0,\beta_0}})\tau\partial_\tau u_{\varepsilon,\tau_0,\beta_0} \\ &+ \rho^2(e^{h_0}-1)e^{\bar u_\varepsilon(0,0,\cdot)}(1-\chi_1)\partial_a H(z+a,z_1+a)|_{a=0}\cdot a \\ &+ \gamma\rho^2 e^{\gamma\bar u_\varepsilon(0,0,\cdot)+\gamma h_0}(1-\chi_1)\partial_a H(z+a,z_1+a)|_{a=0}\cdot a, \end{aligned}$$

where h_0 reads as $h_0(z-z_1)$. Straightforwardly, by Lemmas 3.1, 3.2, the choice of z_1 and the expansion (3.4), we get

$$\|D\mathcal{R}_1|_{(0,0)}(\tau,a)\|_{0,\alpha,\nu-2} \le c(\varepsilon^2 r_0^{2-\nu-4\gamma} + \varepsilon^2 r_0^{-2-\nu}\varepsilon^{4(1-\gamma)-\delta})(|\tau|+|a|)$$

in $B_1(z_1)\setminus B_{r_0}(z_1)$. Finally, in $\Omega\setminus B_1(z_1)$,

$$\mathcal{R}_1(\tau,a) = \Delta(h_\tau\chi) + \rho^2 e^{G(z,z_1+a)\circ\Xi_a^{-1}}(e^{h_\tau\chi}-1) + \rho^2 e^{\gamma G(z,z_1+a)\circ\Xi_a^{-1}+\gamma h_\tau\chi}.$$

By the regularity of h_τ and G, we obtain

$$\|D\mathcal{R}_1|_{(0,0)}(\tau,a)\|_{0,\alpha,\nu-2} \le c(\varepsilon^{4(1-\gamma)-\delta} + \varepsilon^2)(|\tau|+|a|).$$

Combining all these estimates, we have proved the following result:

Lemma 4.1. *Assume that $\nu\in(1,2)$ and $\delta\in(0,1-\gamma]$ are given, then there exists $c>0$ independent of $\varepsilon\in(0,\varepsilon_0)$, such that*

$$\|D\mathcal{R}_1|_{(0,0)}(\tau,a)\|_{0,\alpha,\nu-2} \le c\varepsilon^{(1-\gamma)}(|\tau|+|a|).$$

Otherwise, we can repeat the previous proof for any $|\tau|\le\varepsilon^{2\delta}$ and $|a|\le 1/4$, since Lemmas 3.1, 3.2 hold still, and Ξ_a is still a translation in $B_2(z_1)$, which means

Lemma 4.2. *Let $\nu\in(1,2)$ and $\delta\in(0,1-\gamma]$ be given. There exists $\varepsilon_0, c>0$ such that $\mathcal{R}_1$ is a Lipschitz function with Lipschitz constant less than $c\varepsilon^{1-\gamma}$ over $B_{\varepsilon^{2\delta}}\times B_{1/4}\subset\mathbb{R}\times\mathbb{C}$ and for any $\varepsilon\in(0,\varepsilon_0)$.*

Now, we shall estimate $D\mathcal{R}|_{(0,0,0)}(v,0,0)$. Denote $\zeta(v) = D\mathcal{R}|_{(0,0,0)}(v,0,0)$. We are not able to prove directly that the operator ζ goes to zero with respect to the norm $\mathcal{F}$ when ε tends to 0; however, we can prove that the pre-image by Λ_ε of this operator goes to 0, which will be sufficient for our proof. More precisely, we shall prove

Lemma 4.3. *Assume that ν is a constant in $(5/3, 2)$ and $\delta \in (0, 1-\gamma]$. Then there exist $\varepsilon_0, c > 0$ such that*

$$\left\|\Lambda_\varepsilon^{-1}\left(D\mathcal{R}|_{(0,0,0)}(v,0,0)\right)\right\|_{\mathcal{F}} \le c\varepsilon^{3(1-\gamma)} r_0^\nu \|v\|_{2,\alpha,\nu}$$

for any $\varepsilon \in (0, \varepsilon_0)$ and $v \in C^{2,\alpha}_{\nu,D}(\Omega \setminus \{z_1\})$.

Proof. Denote $\zeta(v) = \mathcal{R}(v,0,0) = \rho^2(e^{h_0\chi}-1)e^{\bar{u}_\varepsilon(0,0,\cdot)}v + \gamma\rho^2 e^{\gamma h_0\chi + \gamma\bar{u}_\varepsilon(0,0,\cdot)}v$. Fix also $\mu > 1$ satisfying $(\nu-\mu) > 2\nu/5$, which is possible since $\nu > 5/3$. Applying Proposition 2.1 and denoting $\bar{\zeta} = \zeta(z+z_1)|_{B_1}$, we get

$$\|G_{\varepsilon,\tau_0}(\bar{\zeta})\|_{2,\alpha,\mu} \le c\|\bar{\zeta}\|_{0,\alpha,\mu-2} \le c\varepsilon^{4(1-\gamma)-\delta+\nu-\mu}\|v\|_{2,\alpha,\nu},$$

hence

$$\begin{aligned}
&\|G_{\varepsilon,\tau_0}(\bar{\zeta})\|_{2,\alpha,\nu} \le c\|\bar{\zeta}\|_{0,\alpha,\nu-2} \le c\varepsilon^{3(1-\gamma)}\|v\|_{2,\alpha,\nu},\\
&\quad |H^0_{\varepsilon,\tau_0}(\bar{\zeta})| \le c\|\bar{\zeta}\|_{0,\alpha,\delta-2} \le c\varepsilon^{4(1-\gamma)+\nu-2\delta}\|v\|_{2,\alpha,\nu} \le c\varepsilon^{2(1-\gamma)}\varepsilon^\nu\|v\|_{2,\alpha,\nu},\\
&\quad |H^1_{\varepsilon,\tau_0}(\bar{\zeta})| \le c\|\bar{\zeta}\|_{0,\alpha,\mu-2} \le c\varepsilon^{4(1-\gamma)-\delta+\nu-\mu}\|v\|_{2,\alpha,\nu} \le c\varepsilon^{3(1-\gamma)}r_0^\nu\|v\|_{2,\alpha,\nu}.
\end{aligned}$$

By Proposition 2.4, we have

$$\|\mathcal{G}_\varepsilon(\zeta)\|_{2,\alpha,\nu} \le c\varepsilon^{3(1-\gamma)}\|v\|_{2,\alpha,\nu} \text{ and } |\mathcal{H}^0_\varepsilon(\zeta)| + |\mathcal{H}^1_\varepsilon(\zeta)| \le c\varepsilon^{3(1-\gamma)}r_0^\nu\|v\|_{2,\alpha,\nu}. \tag{3.8}$$

The proof is completed by Proposition 3.1.

Now we can tackle the fixed point problem, which is the last step of our proof. For all $w = (v,\tau,a) \in \mathcal{F}^{2,\alpha}_\nu$, decompose

$$\begin{aligned}
\mathcal{M}(v,\tau,a) = {} & \mathcal{M}(0,0,0) + D\mathcal{M}_{|(0,0,0)}(v,\tau,a) + [\mathcal{M}(v,\tau,a) - \mathcal{M}(0,\tau,a)\\
& - D\mathcal{M}|_{(0,\tau,a)}(v,0,0)] + (D\mathcal{M}_{|(0,\tau,a)} - D\mathcal{M}_{|(0,0,0)})(v,0,0)\\
& + [\mathcal{M}(0,\tau,a) - \mathcal{M}(0,0,0) - D\mathcal{M}_{|(0,0,0)}(0,\tau,a)],
\end{aligned}$$

where $D\mathcal{M}|_{(0,0,0)} = \Lambda_\varepsilon + \mathcal{S} + D\mathcal{R}|_{(0,0,0)}$. The proof preceeds with the following steps.

Step 1. Let $f_0 = \mathcal{M}(0,0,0)$. By Lemma 3.3, we know that there exists $c_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, $\|\Lambda_\varepsilon^{-1}\mathcal{M}(0,0,0)\|_{\mathcal{F}} \le c_0\varepsilon^{1-\gamma}r_0^\nu$.

From now on, we fix $\nu \in (5/3, 2)$, $\delta = 1-\gamma$. Let $w_1, w_2 \in \mathcal{F}$ satisfying $\|w_i\|_{\mathcal{F}} \le 2c_0\varepsilon^{1-\gamma}r_0^\nu$.

Step 2. Define

$$\begin{aligned}
f_1(w) &= \mathcal{M}(v,\tau,a) - \mathcal{M}(0,\tau,a) - D\mathcal{M}|_{(0,\tau,a)}(v,0,0)\\
&= \rho^2 e^{\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + h_\tau\chi}(e^v - 1 - v) + \rho^2 e^{\gamma\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + \gamma h_\tau\chi}(e^{\gamma v} - 1 - \gamma v).
\end{aligned}$$

Remarking that f_1 is independent of a in $B_{r_0}(z_1)$, and using the estimates on h and $\partial_\tau h$ (notice that $|\tau| \le 2c_0\varepsilon^{1-\gamma}r_0^\nu$ implies that $|\tau| \le \varepsilon^{2\delta}$ for ε small enough since $\gamma \ge 3/4$), we get

$$\|f_1(w_1) - f_1(w_2)\|_{0,\alpha,\nu-2} \le c\varepsilon^{1-\gamma+\nu}r_0^{-\nu}\|w_1 - w_2\|_{\mathcal{F}}.$$

Step 3. Define

$$\begin{aligned}
f_2(w) &= (D\mathcal{M}_{|(0,\tau,a)} - D\mathcal{M}_{|(0,0,0)})(v,0,0)\\
&= \Delta(v\circ\Xi_a)\circ\Xi_a^{-1} - \Delta v + \rho^2\left(e^{\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1}) + h_\tau\chi} - e^{\bar{u}_\varepsilon(0,0,\cdot) + h_0\chi}\right)v\\
&\quad + \gamma\rho^2\left(e^{\gamma\bar{u}_\varepsilon(\tau,a,z)\circ\Xi_a^{-1} + \gamma h_\tau\chi} - e^{\gamma\bar{u}_\varepsilon(0,0,\cdot) + \gamma h_0\chi}\right)v.
\end{aligned}$$

Denoting ${f_2}^1(w) = \Delta(v\circ\Xi_a)\circ\Xi_a^{-1} - \Delta v$, we remark that this quantity does not depend on

τ, and ${f_2}^1 \equiv 0$ in $B_1(z_1)$; we get

$$\|{f_2}^1(w_1) - {f_2}^1(w_2)\|_{0,\alpha,\nu-2} \le c\varepsilon^{1-\gamma}(r_0^\nu\|v_1 - v_2\|_{2,\alpha,\nu} + |a_1 - a_2|).$$

Denote ${f_2}^2 = f_2 - {f_2}^1$, it is easy to see that ${f_2}^2$ does not depend on a in $B_{r_0}(z_1)$. By some direct computations, we then get

$$||{f_2}^2(w_1) - {f_2}^2(w_2)||_{0,\alpha,\nu-2} \le c\varepsilon^{1-\gamma}(r_0^\nu||v_2 - v_1||_{2,\alpha,\nu} + |\tau_2 - \tau_1| + r_0^3|a_2 - a_1|).$$

Step 4. Let

$$f_3(\tau, a) = \mathcal{M}(0, \tau, a) - \mathcal{M}(0, 0, 0) - D\mathcal{M}|_{(0,0,0)}(0, \tau, a).$$

We notice that

$$\mathcal{M}(0, \tau, a) = \mathcal{N}(0, \tau, a) + \mathcal{R}(0, \tau, a).$$

Combining the estimation about $\mathcal{N}(0, \tau, a)$ as in [3, Lemma 5], we get that f_3 is a Lipschitz mapping on $B_{2c_0\varepsilon^{1-\gamma}r_0^\nu}$ in $\mathcal{F}_\nu^{2,\alpha}$, with Lipschitz constant tending to 0, as $\varepsilon \to 0$.

Finally, the nonlinear operator $\mathcal{K}_\varepsilon$ we deal with is just defined by

$$\mathcal{K}_\varepsilon(w) = -\Lambda_\varepsilon^{-1}(\mathcal{S}w + D\mathcal{R}|_{(0,0,0)}w + f_0 + f_1(w) + f_2(w) + f_3(\tau, a)).$$

Combining the results of the above 4 steps, Lemmas 4.1, 4.3 and Proposition 3.1, we conclude that there exists $\varepsilon_0 > 0$, such that $\mathcal{K}_\varepsilon$ is well defined and $\mathcal{K}_\varepsilon$ is a contraction from the ball of radius $2c_0\varepsilon^{1-\gamma}r_0^\nu$ in $(\mathcal{F}_\nu^{2,\alpha}, \|\cdot\|_\mathcal{F})$ into itself for any $\varepsilon \in (0, \varepsilon_0)$, so there exists a unique $w \in B_{2c_0\varepsilon^{1-\gamma}r_0^\nu}$ of $(\mathcal{F}_\nu^{2,\alpha}, \|\cdot\|_\mathcal{F})$, such that

$$\mathcal{K}_\varepsilon(w) = w, \quad \text{i.e. } \mathcal{M}(w) = 0.$$

This completes the proof of our Theorem 1.2.

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