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ON THE LIMIT CYCLES OF PLANAR AUTONOMOUS SYSTEMS***

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Abstract

The authors investigate the global properties of general autonomous systems on the plane and establish criteria for the nonexistence, existence and uniqueness of limit cycles. As application examples, the limit cycles for some polynomial systems are studied.

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§1. Introduction

The theory of limit cycles is a very active research field of qualitative theory of ordinary differential equations. There have been many mathematicians studying the nonexistence, existence and uniqueness of limit cycles for plane systems, and most attenttions were paid to some special forms (see [2-4, 6-10] and the references cited therein). As we know, for the general system on the plane

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P, Q : R^2 \to R$ are continuously differentiable, there are some well-known results. The most beautiful result is the Poincare-Bendixson theorem which concerns the existence of a limit cycle on an annular region. For the nonexistence of limit cycles of (1.1) we can use Poincare or Dulac function method respectively (see [6, 7] for example). We note that in many cases these methods are only valid on a bounded region and a limit cycle may exist on a larger area. Recently, Giacomin, Llibre and Vian[1] studied the limit cycles of general system (1.1), and obtained two theorems concerning the nonexistence and uniqueness of a limit cycle in a bounded region. Some applications to quadratic and cubic systems were also presented in the same paper. Up to now, there have been very few global results concerning the nonexistence and uniqueness of a limit cycle of general form (1.1) on the whole plane. Comparatively, most results in this direction were obtained for some special systems of Lienard type.

In this paper we also study the limit cycles of general system (1.1), and give some sufficient conditions for the existence and noexistence of limit cycles (Theorems 2.1–2.3). Results on

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the maximal number (including the uniqueness) of limit cycles are also given (Theorem 2.4). Comparing with the known results mentioned above, our conditions are concrete, global and valid on the whole plane. We also completely solve a problem of the uniquness of a limit cycle for a cubic system studied in [1, 5] (Propsition 2.1).

The paper is organized as follows. The statements of the main results are presented in Section 2, and their proofs are given in Section 3. Application examples of the main theorems to specific systems are provided in Section 4.

§1. Statement of the Main Results

We first give the following definition.

Definition 2.1. *A basic cycle of* (1.1) *is a simple closed invariant curve of* (1.1) *such that the Poincare map of* (1.1) *is well-defined on at least one side of the curve.*

Obviously, according to this definition a periodic orbit of (1.1) is a basic cycle. A homoclinic or heteroclinic loop is also a basic cycle.

It is easy to see that in any neighborhood of a basic cycle there is a smooth simple closed curve whose index with respect to the vector field defined by (1.1) is equal to 1. For the notation of index of a cloed curve with repect to a vector field, the reader can consult [4, 7].

Consider the C^1 system (1.1). For the nonexistence of limit cycles we have the following

Theorem 2.1. *Suppose the following conditions hold.*

(i) *There exist $y_0 \geq 0$, $x_0 \leq 0 \leq x_1$, and continuous functions $S(x) \geq 0$ and $R(y) > 0$ defined for $x_0 \leq x \leq x_1$ and $y \geq y_0$ respectively such that*

(a) $P(x,y) \neq 0$, $\frac{Q(x,y)}{P(x,y)} \geq R(y)S(x)$ *for $x_0 < x < x_1, y > y_0$*;

(b) $M \equiv \int_{y_0}^{+\infty} \frac{dy}{R(y)} \leq \int_{x_0}^{x_1} S(x)dx$.

(ii) *There exists C^1 function $F(x,y)$ such that the equation $F(x,y) = F(x_0,y_0)$ defines a simple closed curve C_0 surrounding the origin, and that any singular point of* (1.1) *outside C_0, if exists, has nonpositive index.*

(iii) *$\mu(PF_x + QF_y) \geq 0$ for $\mu = P((x_0+x_1)/2, y_0+1)$ and all points in the interier of the curve C_0, and $PF_x + QF_y \neq 0$ along any basic cycle of* (1.1) *in the closed region surrounded by C_0.*

Then the system (1.1) *has no basic cycles on the plane.*

Theorem 2.2. *Suppose the following conditions are satisfied.*

(i) *There exist $x_0 \leq 0$ and a continuous function $p(x)$ which is defined for $x \leq x_0$ and is differentiable for $x \notin E$, where E is a finite subset of the interval $(-\infty, x_0]$, such that $P(x_0,y)$ keeps constant sign for $y \geq p(x_0) \equiv y_0$, $\mu = -P(x_0,y^*) \neq 0$ for some $y^* > y_0$, and*

$$\mu[Q(x,p(x)) - p'(x)P(x,p(x))] \geq 0 \quad \textit{for} \quad x \leq x_0, x \notin E,$$
$$P^2(x,p(x)) + Q^2(x,p(x)) \neq 0 \quad \textit{for} \quad x \leq x_0.$$

(ii) *The condition* (ii) *of Theorem* 2.1 *holds with $y_0 = p(x_0)$.*

(iii) *The condition* (iii) *of Theorem* 2.1 *holds with $\mu = -P(x_0,y^*)$.*

Then the system (1.1) *has no basic cycles on the plane.*

The proof of Theorem 2.2 implies the following corollary which is not direct from the theorem.

Corollary 2.1. *Let the condition* (i) *of either Theorem* 2.1 *or* 2.2 *hold for $(x_0,y_0) = (0,0)$. If any singular point of* (1.1) *not at the origin has nonpositive index, then* (1.1) *has no basic cycles on the plane.*

For the existence of a limit cycle we have the following theorem.

Theorem 2.3. *Suppose the following conditions are satisfied.*

(i) *There exist constants* $a_1 < 0 < a_2$ *and a* C^1 *function* $F(x,y)$ *with* $\lim\limits_{x^2+y^2\to\infty} F(x,y) = +\infty$ *such that*

$$PF_x + QF_y \le 0 \quad \text{for} \quad x \notin [a_1,a_2], |y| < \infty.$$

(ii) *There exist* $N > 0$ *and a continuous function* $R(y) > 0$ *for* $|y| \ge N$ *such that*

$$\int_N^{+\infty} \frac{dy}{R(y)} = +\infty, \qquad \int_{-N}^{-\infty} \frac{dy}{R(y)} = -\infty,$$

$$yPx,y) > 0, \quad \frac{Q(x,y)}{P(x,y)} \le R(y) \quad \text{for} \quad x \in [a_1,a_2], |y| \ge N.$$

(iii) *There exists a continuous function* $p(x)$ *defined for* $x \le a_1$ *(resp.,* $x \ge a_2$*) and being differentiable for* $x \le a_1, x \notin E$ *(resp.,* $x \ge a_2, x \notin E$*) where* E *is a finite subset of the interval* $(-\infty, a_1]$ *(resp.,* $[a_2,+\infty)$*) such that*

$$\begin{aligned} &y_1 \equiv p(a_1) \le N,\ P(a_1,y) > 0 \quad \text{for} \quad y > y_1, \\ &Q(x,p(x)) \le p'(x)P(x,p(x)) \quad \text{for} \quad x \le a_1, x \notin E \end{aligned} \tag{2.1}$$

(*resp.,*

$$\begin{aligned} &y_2 \equiv p(a_2) \ge -N,\ P(a_2,y) < 0 \quad \text{for} \quad y < y_2, \\ &Q(x,p(x)) \ge p'(x)P(x,p(x)) \quad \text{for} \quad x \ge a_2, x \notin E). \end{aligned}$$

(iv) *The system* (1.1) *has only finitely many singular points and all of them are contained in the strip* $a_1 < x < a_2, |y| < \infty$.

Then if any singular point of (1.1) *is a saddle or a source,* (1.1) *has a basic cycle. If any singular point is a saddle with positive divergence (i.e.,* $\mathrm{div}(P,Q) > 0$ *at the saddle) or a source, then* (1.1) *has a periodic orbit. The periodic orbit is a limit cycle provided* (1.1) *is analytic.*

As we know, we can use the Dulac theorem to discuss the uniqueness of a limit cycle in an annular region. The following theorem concerns the maximal number of limit cycles of (1.1) on a simply connected region.

Theorem 2.4. *Suppose*

(i) *There exist a* C^1 *function* $H(x,y)$ *and constants* $h_1 < h_2$ *such that the region* $D_2 \equiv \{(x,y) : H(x,y) < h_2\}$ *contains an open subset* D_{20} *which has a simple boundary curve, and that the region* $D_1 \equiv \{(x,y) : H(x,y) \le h_1\}$ *is a closed set contained in* D_{20} *and having* $n-1$ *simple closed boundary curves,* $n \ge 1$.

(ii) *There exist constants* $b \in R$ *and* $h_0 \notin (h_1,h_2)$ *such that the function*

$$F(x,y) = (H(x,y) - h_0)(P_x + Q_y) + b(PH_x + QH_y)$$

has constant sign on the region $D = \{(x,y) : (x,y) \in D_{20},\ h_1 < H(x,y) < h_2\}$ $(= D_{20} - D_1)$ *and does not equal zero identically on any open subset of* D.

Then (1.1) *has at most* $n-1$ *basic cycles entirely in* D. *If, further, the function* $PH_x + QH_y$ *keeps constant sign on* D_1 *and is not zero identically along any basic cycle of* (1.1) *in* D_1, *then* (1.1) *has at most* $n-1$ *basic cycles entirely in* D_{20}. *In particular,* (1.1) *has at most* $n-1$ *limit cycles in* D_{20} *and every two limit cycles surround different singular points.*

Remark 2.1. If $P = P_0 + P_1$, $Q = Q_0 + Q_1$ and P_0, Q_0 and H satisfy

$$P_0H_x + Q_0H_y = 0, \quad \mathrm{div}(P_0,Q_0) = 0,$$

then

$$F(x,y) = (H(x,y) - h_0)\mathrm{div}(P_1,Q_1) + b(P_1H_x + Q_1H_y).$$

Remark 2.2. From the proof of Theorem 2.4 given in Section 3 we know that if the conditions (i)-(ii) are satisfied and F is not equal to zero identically on a periodic orbit of

(1.1) in D, then all limit cycles of (1.1) are located in D entirely, hyperbolic and of the same stability.

Consider the following cubic system

$$\dot{x} = -y + ax(x^2 + y^2 - 1), \quad \dot{y} = x + by(x^2 + y^2 - 1), \tag{2.2}$$

where a and b are constants. It was proved in [5] that the unit circle $\gamma : \ x^2 + y^2 = 1$ is a limit cycle of (2.2) if $ab > -1, \ (a-b)^2 > 4$. Recently, the following conclusions were obtained in [1]:

(i) γ is the unique limit cycle of (2.2) if $a + b = 0$ and $ab \neq 0$;

(ii) γ is a hyperbolic limit cycle, stable for $a+b < 0$ and unstable for $a+b > 0$ if $a+b \neq 0$;

(iii) γ is the unique limit cycle, and the origin is the unique singular point if $a + b \neq 0$ and $ab \geq 0$.

Applying Theorem 2.4 we can solve the problem of the uniqueness of limit cycles in the case of $a + b \neq 0$ and $ab < 0$, and obtain the following proposition which also presents a correction to the conclusion (i) above.

Proposition 2.1. (i) *Let $a + b = 0, \ ab \neq 0$. Then γ is contained in a period annulus of* (2.2). (ii) *Let $a + b \neq 0$. Then γ is the unique limit cycle of* (2.2)*, stable if $a + b < 0$ and unstable if $a + b > 0$.*

§3. Proof of the Main Results

Proof of Theorem 2.1. For the sake of definiteness, we suppose

$$P(x, y) > 0 \quad \text{for} \quad x_0 < x < x_1, \ \ y > y_0. \tag{3.1}$$

Consider the Hamiltonian system $\dot{x} = \frac{1}{R(y)}, \ \dot{y} = S(x)$. It has a positive semiorbit L_0 passing through the point $A(x_0, y_0)$. It can be represented as

$$\int_{x_0}^{x} S(x)dx = \int_{y_0}^{y} \frac{dy}{R(y)} < M, \quad y \geq y_0.$$

Let x_0^* satisfy $M = \int_{x_0}^{x^*} S(x)dx$. Then along L_0 we have $x \to x^* \leq x_1$ as $y \to +\infty$. By the condition (i) and the comparison theorem, the positive semiorbit of system (1.1) passing through the same point A, denoted by L^+, is always located above L_0 in the region $x_0 \leq x < x^*, \ y \geq y_0$. It implies that (1.1) has no basic cycle surrounding the closed curve C_0 given in the theorem. By the condition (ii), (1.1) has no singular point with positive index outside C_0, which implies that (1.1) has no basic cycles outside C_0 and not surrounding it.

Fig. 3.1 The orbit L of (1.1) under (3.1)

Further, by the condition (iii) and (3.1) we have

$$\frac{dF}{dt}\Big|_{(1.1)} = PF_x + QF_y \geq 0 \tag{3.2}$$

for (x, y) in the interior of C_0, and is not zero identically on any basic cycle of (1.1) inside C_0. Hence, by Theorem 1.1 the system (1.1) has no basic cycles inside C_0, and therefore, the negative semiorbit L^- of (1.1) passing through A approaches a singular point inside C_0. And also, by (3.2), the region surrounded by C_0 is negatively invariant with respect to the flow of (1.1), which yields that (1.1) has no basic cycles intersecting C_0 (see Fig. 3.1). Hence, (1.1) has no basic cycles in the whole plane. This ends the proof of Theorem 2.1.

Proof of Theorem 2.2. For definiteness, we suppose

$$\begin{aligned} \mu < 0,\ P(x_0, y) \geq 0 \quad &\text{for} \quad y \geq y_0, \\ Q(x, p(x)) \leq p'(x)P(x, p(x)) \quad &\text{for} \quad x \leq x_0,\ x \notin E. \end{aligned} \tag{3.3}$$

Denote by L_0 the curve $y = p(x)$, $x \leq x_0$, and by L^- the negative semiorbit of (1.1) passing through the endpoint $A(x_0, y_0)$ of L_0. Note that there is no singular point of (1.1) on L_0. It follows from (3.3) and the comparison theorem that L^- is always above L_0 (see Fig. 3.2). Hence, (1.1) has no basic cycles surrounding C_0. Then just the same as before we can verify that (1.1) has no basic cycles which are outside or inside C_0, or intersect it. The proof is completed.

Fig. 3.2 The orbit L of (1.1) under (3.3)

Proof of Corollary 2.1. Since $(x_0, y_0) = (0, 0)$, the endpoint A of the curve L_0 in Fig. 3.1 or Fig. 3.2 is at the origin. Hence, (1.1) has no basic cycles surrounding the origin. By our assumption, all other singular points are of nonpositive index. It follows that (1.1) has no basic cycles not surrounding the origin. If (1.1) has a basic cycle passing through the origin, then the Poincare map is only possiblly well-defined on the inner side of it because of the existence of L_0. Thus, there is a simple closed curve on the inner side whose index with respect to the vector field (P, Q) is equal to 1. It implies that (1.1) has a singular point which is of positive index and not at the origin, a contradiction. This ends the proof.

Proof of Theorem 2.3. For definiteness we suppose (2.1) holds. Let L_1 denote the curve $y = p(x),\ x \leq a_1$. By the condition (iii) we have

$$(p'(x), -1) \cdot (P(x, p(x)), Q(x, p(x))) \geq 0 \quad \text{for} \quad x \leq a_1, \quad x \notin E.$$

Since there are no singular points in the region $x \leq a_1$, any positive semiorbit of (1.1) starting at a point on L_1 goes into lower side of L_1. Let $A_1(a_1, y_1)$ be the endpoint of L_1 and A_2 the point (a_1, y_2) with $y_2 > N$. Since $P(a_1, y) > 0$ for $y > y_1$, any positive semiorbit of (1.1) starting at a point on the segment $L_2 = A_1A_2$ goes into right side of it. Let

$$L_3: \quad \int_{y_2}^{y} \frac{du}{R(u)} = x - a_1, \quad a_1 \le x \le a_2.$$

By the condition (ii) and the comparison theorem, L_3 intersects the line $x = a_2$ at a point $A_3(a_2, y_3)$ with $\lim\limits_{y_2 \to +\infty} y_3 = +\infty$, and any positive semiorbit of (1.1) startiing at a point on L_3 goes into lower side of it. Let

$$L_4: \quad F(x, y) = F(A_3), \quad x \ge a_2.$$

By the condition (i), apart from A_3, L_4 has another endpoint $A_4(a_2, y_4)$ on the line $x = a_2$ with $\lim\limits_{y_3 \to +\infty} y_4 = -\infty$, and any positive semiorbit of (1.1) starting at a point on L_4 goes into left side of it.

Let

$$L_5: \quad \int_{y_4}^{y} \frac{du}{R(u)} = x - a_2, \quad a_1 \le x \le a_2,$$

$$L_6: \quad F(x, y) = F(A_5), \quad x \le a_1,$$

where $A_5(a_1, y_5)$ denotes the left side endpoint of L_5. Then, as above, any positive semiorbit of (1.1) starting at a point of the curve $L_5 \bigcup L_6$ goes into upper side of it. Note that $y_5 \to -\infty$ as $y_2 \to +\infty$ and L_1 is always fixed. The curve L_6 intersects L_1 at a point A_6 for sufficiently large Y_2. Hence, we have constructed a simple closed curve $L = A_1A_2A_3A_4A_5A_6A_1$ such that any positive semiorbit passing through a point on L is always inside it (see Fig. 3.3).

Fig. 3.3

Now we suppose any singular point of (1.1) is a saddle or a source. Since (1.1) has only finitely many singular points, there exists a subset S of L consisting of at most finitely many points such that for any point $A \in L$ the positive semiorbit of A approaches a saddle point if and only if $A \in S$. Then, by the Poincare-Bendixson theorem for any point $B \in L - S$, the positive semiorbit of B approaches a basic cycle of (1.1). The resulting basic cycle is outer stable. If the basic cycle is not a periodic orbit, it is a homoclinic or heteroclinic loop.

Further, suppose (1.1) has positive divergence at all saddle points. Then every homoclinic or heteroclinic loop, if exists, is unstable and inside L. Hence, the obtained basic cycle must be a outer stable periodic orbit. Obviously, the periodic orbit is a limit cycle if (1.1) is analytic. This completes the proof of Theorem 2.3.

Proof of Theorem 2.4. Let $B(x,y) = (H(x,y) - h_0)^b, \quad (x,y) \in D$. Then we have

$$\operatorname{div}(BP, BQ) = F(x,y)[H(x,y) - h_0]^{b-1}, \quad (x,y) \in D. \tag{3.4}$$

By the condition (ii) and Dulac theorem, (1.1) has at most $n-1$ basic cycles entirely in D. The first part of the theorem follows. For the last part we can assume that

$$\frac{dH}{dt}|_{(1.1)} = PH_x + QH_y \geq 0 \qquad \text{for} \qquad (x,y) \in D_1,$$

and $PH_x + QH_y \not\equiv 0$ along any basic cycle in D_1. Then the set D_1 is negatively invariant with respect to the flow of (1.1). Hence, (1.1) has no basic cycles which have points in common with D_1. The proof is completed.

§4. Examples

In this section we present some examples as applications of Theorems 2.1–2.4.

Example 4.1. Consider a system of the form

$$\dot{x} = y, \quad \dot{y} = -(x^2 - a)(y^2 + 1)y - x(1 - bx), \tag{4.1}$$

where $0 \leq b \leq 3/4$. We prove that there exists a constant $a^* \in (0,2)$ such that (4.1) has a limit cycle if and only if $0 < a < a^*$.

First, it is easy to see that the origin is stable (unstable) for $a \leq 0$ ($a > 0$). Then a limit cycle $L(a)$ appears near the origin for $0 < a \ll 1$. Note that (4.1) forms a generalized rotated vector field in a (see [6]). The limit cycle $L(a)$ expands as a increases. We claim that when $a = 2$, $L(a)$ has disappeared. In fact, choose $(x_0, y_0) = (-1, 0)$, $S(x) = a - x^2$, $R(y) = 1 + y^2$, and $F(x,y) = y^2/2 + x^2/2 - bx^3/3$. Since $0 \leq b \leq 3/4$, we have

$$F(x_0,y_0) \leq F(\sqrt{2},0), \quad M = \pi/2 \leq x_0^3/3 - 2x_0,$$
$$PF_x + QF_y = (2 - x^2)(1 + y^2)y^2 \geq 0 \quad \text{for} \quad F(x,y) \leq F(x_0,y_0).$$

Also, $(0, 1/b)$ is a saddle point for $b > 0$. Hence, by Theorem 2.1, (4.1) has no limit cycles or homoclinic loops for $a = 2$. Thus, by the theory of rotated vector fields (4.1) has no limit cycles for all $a \geq 2$. The conclusion follows.

Example 4.2. Consider

$$\dot{x} = y, \quad \dot{y} = -y(a - x^2 - x^3 - y^2) - x. \tag{4.2}$$

We show that there exists a constant $a^* \in (0,2)$ such that (4.2) has a limit cycle if and only if $0 < a < a^*$.

We first prove that (4.2) has no limit cycles for $a = 2$. In fact, the claim follows from Theorem 2.2 by choosing $(x_0, y_0) = (0, 1)$, $p(x) = 1$ and $F(x,y) = x^2 + y^2$ since

$$Q(x,1) = (x+1)^2(x-1) \leq 0 \qquad \text{for} \qquad x \leq 0,$$
$$PF_x + QF_y = -2y^2(2 - x^2 - y^2 - x^3) \leq 0 \qquad \text{for} \quad x^2 + y^2 \leq 1.$$

The rest is just similar to the discussion for (4.1).

Example 4.3. consider

$$\dot{x} = 3(e^y - 1) - x^3 + 3x, \quad \dot{y} = -\frac{2xe^y}{1+x^2}. \tag{4.3}$$

We can prove that (4.3) has a stable limit cycle. In fact, if we choose $a_2 = -a_1 = \sqrt{3}$, $p(x) =$

$x+4$ for $x \le -\sqrt{3}$, and

$$F(x,y) = \ln(1+x^2) + 3(y + e^{-y} - 1),$$
$$R(y) = \frac{e^y}{3|e^y - 1| - 2} \qquad \text{for} \qquad |y| \ge 3,$$

we then have

$$PF_x + QF_y = 2x^2(3-x^2)/(1+x^2) \ge 0 \qquad \text{for} \qquad |x| \le \sqrt{3},$$
$$\frac{-2x}{1+x^2} e^{p(x)} \le e^{p(x)} \qquad \text{for} \quad x \le -\sqrt{3},$$
$$2e^{p(x)} - 3 > -3 > x^3 - 3x \qquad \text{for} \quad x \le -3,$$
$$2e^{p(x)} - 3 > 2e - 3 > 0 \ge x^3 - 3x \qquad \text{for} \quad -3 \le x \le -\sqrt{3}.$$

It follows that

$$Q(x,p(x)) \le p'(x)P(x,p(x)) \qquad \text{for} \quad x \le -\sqrt{3}.$$

Now the conclusion is evident by Theorem 2.3.

Example 4.4. Consider the cubic system

$$\dot{x} = -y + ax(x^2+y^2-1) = P(x,y), \quad \dot{y} = x + by(x^2+y^2-1) = Q(x,y) \tag{4.4}$$

where a and b are constants with $(a,b) \ne (0,0)$. By [1], (4.4) has a periodic orbit γ : $x^2+y^2=1$. First, suppose $a+b=0$, $ab \ne 0$. Then on the region $x^2+y^2 \ne 1$, (4.4) is equivalent to

$$\dot{x} = -\frac{y}{x^2+y^2-1} - bx, \quad \dot{y} = \frac{x}{x^2+y^2-1} + by. \tag{4.5}$$

For $x^2+y^2 \ne 1$, (4.5) is Hamiltonian with the Hamiltonian function

$$H(x,y) = \frac{1}{2}\ln|x^2+y^2-1| + bxy. \tag{4.6}$$

It is easy to see that the origin is a saddle point for $b^2 > 1$ and a center for $b^2 < 1$. By (4.6) for $b^2 = 1$ we have

$$H(x,y) = -\frac{1}{2}\Big[(x-by)^2 + \frac{1}{2}(x^2+y^2)^2(1+O(x^2+y^2))\Big],$$

and hence the origin is a center in this case.

With no loss of generality, we may assume $b>0$ since (4.5) is invariant under the change $(y,t,b) \to (-y,-t,-b)$. It is direct that for $b>1$ (resp., $0<b\le 1$) (4.5) has four (resp., two) singular points $\pm(x_0,x_0)$ and $\pm(x_1,-x_1)$ (resp., $\pm(x_1,-x_1)$) apart from the origin, where $x_0 = \sqrt{\frac{1}{2}(1-\frac{1}{b})}$, $x_1 = \sqrt{\frac{1}{2}(1+\frac{1}{b})}$.

Let

$$J(x,y) = \begin{pmatrix} 3ax^2+ay^2-a & -1+2axy \\ 1+2bxy & 3by^2+bx^2-b \end{pmatrix}, \tag{4.7}$$
$$\det J(x,y) = ab(3x^2+y^2-1)(3y^2+x^2-1) + (1+2bxy)(1-2axy).$$

When $a=-b$, we have

$$\det J(\pm x_0, \pm x_0) = 4(b-1) > 0 \quad \text{for} \quad b>1;$$
$$\det J(\pm x_1, \mp x_1) = -4(b+1) < 0 \quad \text{for} \quad b>0.$$

Hence, $\pm(x_0,x_0)$ are center points and $\pm(x_1,-x_1)$ saddle points. Therefore, in the case of $a=-b\ne 0$ there exists an open set D containing γ and full of periodic orbits of (4.4) such that the outer boundary curve of D is a heteroclinic loop with two saddles and the inner boundary curve is either a double homoclinic loop of the form of eight figure when $b^2 > 1$ or the origin when $b^2 \le 1$. The phase portrait of (4.4) is shown in Fig. 4.1.

(i) $b > 1$ (ii) $0 < b \le 1$

Fig. 4.1 The phase portrait of (4.4)

Now let $a + b \ne 0$. Choose $H = x^2 + y^2$, $h_0 = 1$ and $b = -1$ in Theorem 2.4. Then by Remark 2.1 we have for (4.4)

$$F(x,y) = (a+b)(x^2+y^2-1)^2, \quad \mathrm{div}\Big(\frac{P}{H-1}, \frac{Q}{H-1}\Big) = \frac{F(x.y)}{(H-1)^2}, \tag{4.8}$$
$$PH_x + QH_y = 2(x^2+y^2-1)(ax^2+by^2).$$

By Theorem 2.4 and taking $h_1 = 0$, $h_2 = 1$ and $D_{20} = D_2$ we know that (4.4) has no limit cycles inside γ. By taking $h_1 = 1$, $h_2 = +\infty$ and $D_{20} = D_2$ and using (4.8) we know that if (4.4) has a limit cycle $L \ne \gamma$, then it surrounds γ and is stable if $a + b < 0$, unstable if $a + b > 0$ (see [1]). In order to prove L does not exist, it suffices to show that all singular points outside γ have negative index.

First, if $ab \ge 0$ then the origin is the only singular point of (4.4), which implies that L does not exist. Let $ab < 0$. Then (4.4) has three (resp., five) singular points for $-1 \le ab < 0$ (resp., $ab < -1$) including the origin. Outside γ are only two singular points with coordinates $\pm(x_1, y_1)$ where

$$y_1^2 = \frac{1+K}{1+K^2b^2}, \quad x_1 = -Kby_1, \quad x_1 > 0, \quad K = \sqrt{\frac{-1}{ab}}. \tag{4.9}$$

By (4.7) we have $\det J(\pm x_1, \pm y_1) = J_1 + J_2$, where

$$J_1 = ab[(3K^2b^2+1)y_1^2 - 1][(3+K^2b^2)y_1^2 - 1], \quad J_2 = (1 - 2b^2Ky_1^2)(1 + 2abKy_1^2).$$

By (4.9) we have

$$\begin{aligned} J_1 &= \frac{ab}{(a-b)^2}[(a-3b)K - 2b][(3a-b)K + 2a] \\ &= \frac{1}{(a-b)^2}[-3a^2 - 3b^2 + 10ab - 4a^2b^2 + 2ab(a^2+b^2-6ab)K], \\ J_2 &= \frac{1}{(a-b)^2}(a+b-2ab^2K)(2a^2bK - a - b) \\ &= \frac{1}{(a-b)^2}[4a^2b^2 - (a+b)^2 + 2ab(a+b)^2K]. \end{aligned}$$

Hence

$$J_1 + J_2 = \frac{1}{(a-b)^2}[-4(a-b)^2 + 4ab(a-b)^2K] = 4(abK - 1) = -4(1+K)/K < 0.$$

Therefore, we have proved that γ is the uniqueness limit cycle of (4.4) if $a + b \ne 0$. Now Proposition 2.1 is obvious.

Example 4.5. By choosing $H = x^2 + y^2$, $h_0 = h_1 = 0$, $h_2 = +\infty$, $b = -2$ and $D_{20} = D_2$ in Theorem 2.4, we can prove that the cubic system

$$\dot{x} = y(x^2 + y^2 - 1) + x(ax^2 + by^2 - 1), \quad \dot{y} = -x(x^2 + y^2 - 1) + y(ax^2 + by^2 - 1)$$

has at most one limit cycle. The unique limit cycle exists and is stable if $(a-1)(b-1) > 0$, $a > 0$ and $b > 0$.

Example 4.6. Consider a Lienard system of the form

$$\dot{x} = y, \quad \dot{y} = -(x^2 - a)(2 + x)y - x(1 - x^2). \tag{4.10}$$

We prove that there exists a number $a^* \in (0,1)$ such that (4.10) has a unique limit cycle if and only if $0 < a < a^*$, and the cycle is stable and hyperbolic. First, the origin is stable for $a \leq 0$ and unstable for $a > 0$. Also, apart from the origin (4.10) has saddle points $(\pm 1, 0)$ for $a < 1$, which yields that any limit cycles of (4.10) are contained in the strip $|x| < 1$. Since

$$\operatorname{div}(4.10) = -(x^2 - a)(2 + x) = (1 - x^2)(2 + x) > 0 \quad \text{for} \quad |x| < 1,$$

when $a = 1$, (4.10) has no limit cycles in this case. Note that (4.10) forms a rotated vector field in a. It follows that there exists a number $a^* \in (0,1)$ such that (4.10) has a limit cycle if and only if $0 < a < a^*$. Let $H(x,y) = y^2/2 + x^2/2 - x^4/4$ and choose $h_0 = h_1 = H(\sqrt{a}, 0)$, $h_2 = +\infty$, $b = -1/2$ and $D_{20} = \{(x,y) : \ -1 < x < 1\}$. We have for (4.10)

$$\frac{dH}{dt}\Big|_{(4.10)} = -(x^2 - a)(2 + x)y^2, \quad F(x,y) = -\frac{1}{4}(x^2 - a)^2(2 - x^2 - a^2).$$

Noth that $H(x,y) - H(\sqrt{a}, y) = \frac{1}{4}(x^2 - a)(2 - x^2 - a^2)$ and that $2 - x^2 - a^2 \geq 1 - a^2 > 0$ for $(x,y) \in D_{20}$, $0 < a < a^*$. We have $x^2 - a < 0$ for $H(x,y) < h_1$ since $h_1 \leq H(\sqrt{a}, y)$. Hence, Theorem 2.4 implies that (4.10) has at most one limit cycle for $0 < a < a^*$, and the conclusion follows.

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