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A REMARK ON FORMAL MODELS FOR NONLINEARLY ELASTIC MEMBRANE SHELLS

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Abstract

This paper gives all the two-dimensional membrane models obtained from formal asymptotic analysis of the three-dimensional geometrically exact nonlinear model of a thin elastic shell made with a Saint Venant-Kirchhoff material. Therefore, the other models can be quoted as flexural nonlinear ones. The author also gives the formal equations solved by the associated stress tensor and points out that only one of those models leads, by linearization, to the "classical" linear limiting membrane model, whose justification has already been established by a convergence theorem.

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§1. Introduction

We use the following conventions: Latin indices take their values in the set {1,2,3}, Greek indices in the set {1,2}, and the summation convention is used.

Let ω be an open bounded connected set of $\mathbb{R}^2$ with a Lipschitz boundary γ. We denote by (x_α) a point of $\bar{\omega}$, and $\partial_\alpha = \partial/\partial x_\alpha$. Let $\boldsymbol{\varphi} \in C^3\left(\bar{\omega};\mathbb{R}^3\right)$ be an injective mapping such that the two vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\varphi}$ are linearly independent for all point of $\bar{\omega}$. The vectors $\mathbf{a}_\alpha$ form a covariant basis of the tangent plane to the surface $S = \boldsymbol{\varphi}(\bar{\omega})$. We define the unit vector normal to each point of the middle surface S by $\mathbf{a}_3 = \frac{\mathbf{a}_1\times\mathbf{a}_2}{|\mathbf{a}_1\times\mathbf{a}_2|}$, where "$\times$" represents the vector product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^3$. We define the associated contravariant basis $\left(\mathbf{a}^\beta\right)$ by $\mathbf{a}^\beta\cdot\mathbf{a}_\alpha = \delta_{\alpha\beta}$ ($\delta_{\alpha\beta}$ is the Kronecker symbol), and we complete this contravariant basis with the vector $\mathbf{a}^3$ defined by $\mathbf{a}_3 = \mathbf{a}^3$. Moreover, for all point of S we define the two-dimensional Christoffel symbols $\Gamma^{\rho*}_{\alpha\beta} = \mathbf{a}^\rho\cdot\partial_\alpha\mathbf{a}_\beta$ (which satisfy the relations $\Gamma^{\rho*}_{\alpha\beta} = \Gamma^{\rho*}_{\beta\alpha}$), the metric tensor through its covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha\cdot\mathbf{a}_\beta$ or through its contravariant ones $a^{\alpha\beta} = \mathbf{a}^\alpha\cdot\mathbf{a}^\beta$, the area element $\sqrt{a}dx_1dx_2$ where $a = \det\left(a_{\alpha\beta}\right)$, and the curvature tensor through its covariant components $b_{\alpha\beta} = \left(\partial_\beta\mathbf{a}_\alpha\right)\cdot\mathbf{a}^3 = b_{\beta\alpha}$ or through its mixed ones $b^\beta_\alpha = -\left(\partial_\alpha\mathbf{a}_3\right)\cdot\mathbf{a}^\beta$.

For $\varepsilon > 0$, we consider a shell with thickness 2ε and middle surface S, made of a Saint Venant-Kirchhoff elastic material whose Lamé constants $\lambda > 0$ and $\mu > 0$ are independent

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of ε. Let $\Omega^\varepsilon = \omega \times]-\varepsilon;+\varepsilon[$ and let $\boldsymbol{\phi}$ be the injective mapping (this is the case when ε is small enough[2]), defined by $\boldsymbol{\phi}(x^\varepsilon) = \boldsymbol{\varphi}(x_1^\varepsilon; x_2^\varepsilon) + x_3^\varepsilon \mathbf{a}_3(x_1^\varepsilon; x_2^\varepsilon)$.

The shell whose reference configuration is $\boldsymbol{\phi}(\bar{\Omega}^\varepsilon)$ is clamped on the portion $\boldsymbol{\phi}(\Gamma_0^\varepsilon)$, where $\Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon;+\varepsilon]$, $\gamma_0 \subset \gamma$, of its lateral surface $\boldsymbol{\phi}(\Gamma^\varepsilon)$, where $\Gamma^\varepsilon = \gamma \times [-\varepsilon;+\varepsilon]$. On its upper face $\boldsymbol{\phi}(\Gamma_+^\varepsilon)$, where $\Gamma_+^\varepsilon = \omega \times \{+\varepsilon\}$, and its lower face $\boldsymbol{\phi}(\Gamma_-^\varepsilon)$, where $\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}$, it is subjected to applied surface forces and to applied body forces on $\boldsymbol{\phi}(\bar{\Omega}^\varepsilon)$. Under the action of these forces, the shell undergoes a displacement field.

In order to establish the limiting models of membrane shells, we use the general principles of formal asymptotic analysis. We define an open set $\Omega = \omega \times]-1;+1[$ and the sets $\Gamma_+ = \omega \times \{+1\}$ and $\Gamma_- = \omega \times \{-1\}$ independent of ε. With each function $d^\varepsilon(x^\varepsilon)$ defined on Ω^ε, we associate the function $d(\varepsilon)(x)$ defined on the fixed open set Ω by $d^\varepsilon(x^\varepsilon) = d(\varepsilon)(x)$. Then, the scaled covariant components of the displacement field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ solve the variational problem:

Find $\mathbf{u}(\varepsilon) \in \mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{W}^{1,4}(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 = \gamma_0 \times (-1,+1)\}$ such that

$$\begin{aligned}&\varepsilon \int_\Omega g^{pqij}(\varepsilon)\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon))\,\mathcal{F}_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v})\sqrt{g(\varepsilon)}dx\\ &= \varepsilon \int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+\cup\Gamma_-} h^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\omega,\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where $(f^i(\varepsilon))$ are the contravariant componants of the scaled applied body forces and $(h^i(\varepsilon))$ are the contravariant componants of the scaled applied surface forces in the covariant basis $(\mathbf{a}_i)$, where $g^{ijpq}(\varepsilon)$ are the contravariant components of the scaled three-dimensional elasticity tensor, where $\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon))$ are the covariant components of the scaled symmetric Green-Saint Venant strain tensor whose Gâteaux derivatives at $\mathbf{u}(\varepsilon)$ in the direction $\mathbf{v}$ are $\mathcal{F}_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v})$.

The associated boundary value problem reads

$$\begin{cases} -T^{ij}_{\|j}(\varepsilon)(x) = f^i(\varepsilon)(x), & x \in \Omega,\\ T^{i3}(\varepsilon)(x) = h^i(\varepsilon)(x), & x \in \Gamma_+,\\ -T^{i3}(\varepsilon)(x) = h^i(\varepsilon)(x), & x \in \Gamma_-,\end{cases}$$

where

$$T^{ij}(\varepsilon) = \left(g^{pqij}(\varepsilon) + g^{pqjt}(\varepsilon)g^{si}(\varepsilon)u_{s\|t}(\varepsilon)\right)\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon))$$

are the contravariant components of the first Piola-Kirchhoff scaled stress tensor whose scaled covariant derivatives $T^{ij}_{\|k}(\varepsilon)$ are defined by

$$\begin{aligned}T^{ij}_{\|\alpha}(\varepsilon) &= \partial_\alpha T^{ij}(\varepsilon) + T^{rj}(\varepsilon)\Gamma^i_{r\alpha}(\varepsilon) + T^{ir}(\varepsilon)\Gamma^j_{r\alpha}(\varepsilon),\\ T^{ij}_{\|3}(\varepsilon) &= \frac{1}{\varepsilon}\partial_3 T^{ij}(\varepsilon) + T^{rj}(\varepsilon)\Gamma^i_{r3}(\varepsilon) + T^{ir}(\varepsilon)\Gamma^j_{r3}(\varepsilon),\end{aligned}$$

where $\Gamma^k_{ij}(\varepsilon)$ are the three-dimensional Christoffel symbols expressed on the fixed open set Ω.

We next assume that each geometrical datum that depends on ε can be expanded up to whatever order is necessary in terms of power of x_3 and that there exists a formal asymptotic expansion of the displacement field expressed in the fixed open set Ω by

$$\mathbf{u}(\varepsilon) = \frac{1}{k}\mathbf{u}^{-k} + \cdots + \frac{1}{\varepsilon}\mathbf{u}^{-1} + \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \cdots,$$

where the terms $\mathbf{u}^r$ are independent of ε. Then both tensors $\left(\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon))\right)$ and

$(\mathcal{F}_{i\|j}(\varepsilon)\ (\mathbf{u}(\varepsilon),\mathbf{v}))$ possess asymptotic expansions begining with different powers of ε:

$$\begin{array}{ll}
\mathcal{E}_{\alpha\|\beta}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon^{2k}}\mathcal{E}^{-2k}_{\alpha\|\beta} + \cdots, & \mathcal{F}_{\alpha\|\beta}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) = \frac{1}{\varepsilon^{k}}\mathcal{F}^{-k}_{\alpha\|\beta}(\mathbf{v}) + \cdots, \\
\mathcal{E}_{\alpha\|3}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon^{2k+1}}\mathcal{E}^{-2k-1}_{\alpha\|3} + \cdots, & \mathcal{F}_{\alpha\|3}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) = \frac{1}{\varepsilon^{k+1}}\mathcal{F}^{-k-1}_{\alpha\|3}(\mathbf{v}) + \cdots, \\
\mathcal{E}_{3\|3}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon^{2k+2}}\mathcal{E}^{-2k-2}_{3\|3} + \cdots, & \mathcal{F}_{3\|3}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) = \frac{1}{\varepsilon^{k+2}}\mathcal{F}^{-k-2}_{3\|3}(\mathbf{v}) + \cdots,
\end{array}$$

where the terms $\mathcal{E}^r_{i\|j}$ and $\mathcal{F}^r_{i\|j}$ are independent of ε.

Furthermore, we require that no compatibility conditions on the data are needed for the three-dimensional problem: this means that if at the l-th step, $l \in \mathbb{Z}$, there exist $f^{i,l}$ and $h^{i,l+1}$ independent of ε such as $f^{i,\varepsilon} = f^i(\varepsilon) = \varepsilon^l f^{i,l}$ and $h^{i,\varepsilon} = h^i(\varepsilon) = \varepsilon^{l+1} h^{i,l+1}$ and they must satisfy the compatibility condition

$$\int_\omega p^{i,l}(x_1,x_2)\eta_i(x_1,x_2)dx_1dx_2 = 0$$

for $\boldsymbol{\eta}$ in some functional space on ω, where

$$p^{i,l}(x_1,x_2) = \frac{1}{2}\Big(\int_{-1}^{1} f^{i,l}(x_1,x_2,t)dt + h^{i,l+1}(x_1,x_2,+1) + h^{i,l+1}(x_1,x_2,-1)\Big),$$

then we assume that $f^{i,l} = 0$ and $h^{i,l+1} = 0$.

The identification of the successive powers of ε in the variational problem (cf. [8]) leads us to assume that the applied body forces are of order 0 ($\mathbf{f}^\varepsilon = \mathbf{f}^0$) and that the applied surface forces are of order 1 ($\mathbf{h}^\varepsilon = \varepsilon\mathbf{h}^1$). Then the asymptotic expansion of the displacement field $\mathbf{u}(\varepsilon)$ begins with a term $\mathbf{u}^0$ independent of x_3, and we are next led to solve the following system:

$$\begin{cases}
(\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) + (\lambda+2\mu)\mathcal{E}^0_{3\|3}(\mathbf{u}^0,\mathbf{u}^1))a^{\gamma\delta}u^0_{\delta\|3} \\
+2\mu\mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0,\mathbf{u}^1)(a^{\alpha\gamma} + a^{\alpha\rho}a^{\gamma\delta}u^0_{\delta\|\rho}) = 0, \\
(\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) + (\lambda+2\mu)\mathcal{E}^0_{3\|3}(\mathbf{u}^0,\mathbf{u}^1))(1 + u^0_{3\|3}) \\
+2\mu a^{\alpha\gamma}\mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0,\mathbf{u}^1)u^0_{3\|\gamma} = 0,
\end{cases} \tag{1.1}$$

where the leading term of the asymptotic expansion of the Green-Saint Venant strain tensor can be written as

$$\begin{cases}
2\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) = u^0_{\alpha\|\beta} + u^0_{\beta\|\alpha} + a^{\gamma\delta}u^0_{\gamma\|\alpha}u^0_{\delta\|\beta} + u^0_{3\|\alpha}u^0_{3\|\beta}, \\
2\mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0,\mathbf{u}^1) = u^0_{\alpha\|3} + u^0_{3\|\alpha} + a^{\gamma\delta}u^0_{\gamma\|\alpha}u^0_{\delta\|3} + u^0_{3\|\alpha}u^0_{3\|3}, \\
2\mathcal{E}^0_{3\|3}(\mathbf{u}^0,\mathbf{u}^1) = 2u^0_{3\|3} + a^{\gamma\delta}u^0_{\gamma\|3}u^0_{\delta\|3} + u^0_{3\|3}u^0_{3\|3},
\end{cases} \tag{1.2}$$

with

$$\begin{array}{ll}
u^0_{\alpha\|\beta} = \partial_\beta u^0_\alpha - \Gamma^{\zeta *}_{\alpha\beta}u^0_\zeta - b_{\alpha\beta}u^0_3, & u^0_{\alpha\|3} = \partial_3 u^1_\alpha + b^\zeta_\alpha u^0_\zeta, \\
u^0_{3\|\beta} = \partial_\beta u^0_3 + b^\zeta_\beta u^0_\zeta, & u^0_{3\|3} = \partial_3 u^1_3.
\end{array}$$

The purpose of this work is to find all the solutions to the system (1.1) which is an algebraic system with unknowns $u^0_{i\|3}$, and to give the associated variational equations and the boundary value problems.

The solution given by B. Miara (Section 2)

$$\begin{cases}
\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + (\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right) = 0, \\
\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right) = 0,
\end{cases} \tag{1.3}$$

allows us to find (Section 3) the classical variational problem:

Find $\mathbf{u}^0 \in \mathbf{V}(\omega) = \left\{\boldsymbol{\eta} = (\eta_i) \in \mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\right\}$ such that for all $\boldsymbol{\eta}(x_1,x_2) \in \mathbf{V}(\omega)$

$$\int_\omega b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0)\mathcal{F}^0_{\gamma\|\delta}(\mathbf{u}^0,\boldsymbol{\eta})\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega,$$

where $b^{\alpha\beta\gamma\delta}$ is the two-dimensional elasticity tensor and where $\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)$ is the Gâteaux derivative of the two-dimensional change of metric tensor $\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)$ at the point $\mathbf{u}^0$ in the direction $\boldsymbol{\eta}$.

The associated boundary value problem reads

$$\begin{cases} -T^{i\delta,0}_{\|\delta} = p^{i,0} & \text{in } \omega, \\ T^{i\delta,0} n_\delta = 0 & \text{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \gamma_0, \end{cases}$$

where $T^{i\delta,0}$ are the first non vanishing terms of the asymptotic expansion of the first Piola-Kirchhoff stress tensor[5] given by

$$T^{\gamma\delta,0} = \left\{ b^{\alpha\beta\gamma\delta} + b^{\alpha\beta\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho} \right\} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right),$$
$$T^{3\delta,0} = b^{\alpha\beta\delta\rho} u^0_{3\|\rho} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right),$$

and where the "divergence" of a bidimensional tensor is given by

$$T^{i\delta,0}_{\|\delta} = \partial_\delta T^{i\delta,0} + \Gamma^{\delta *}_{r\delta} T^{ir,0} + \Gamma^{i*}_{r\delta} T^{r\delta,0}$$

with $\Gamma^{3*}_{k3} = 0$, $\Gamma^{k*}_{33} = 0$, $\Gamma^{3*}_{\gamma\delta} = b_{\gamma\delta}$ and $\Gamma^{\gamma *}_{\delta 3} = -b^\gamma_\delta$.

We note that the linearization of the variational problem written for a shell completely clamped on its lateral boundary (i.e. $\gamma_0 = \gamma$) leads to the "classical" linear model whose existence and uniqueness of the solution has been proved in [3]:

Find $\mathbf{u}^0 \in \mathbf{V}_m(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$ such that for all $\boldsymbol{\eta}(x_1,x_2) \in \mathbf{V}_m(\omega)$,

$$\int_\omega b^{\alpha\beta\gamma\delta} e^0_{\alpha\|\beta}(\mathbf{u}^0) e^0_{\gamma\|\delta}(\boldsymbol{\eta}) \sqrt{a} d\omega = \int_\omega p^{i,0} \eta_i \sqrt{a} d\omega,$$

where $e^0_{\alpha\|\beta}(\mathbf{u}^0)$ is the linearized change of metric tensor.

A second solution to the system (1.1) (Section 4)

$$\begin{cases} u^0_{\alpha\|3} = 0, \\ u^0_{3\|3} = -1, \end{cases} \tag{1.4}$$

gives the new limiting variational problem (Theorem 4.1):

Find $\mathbf{u}^0 \in \mathbf{V}(\omega)$ such that for all $\boldsymbol{\eta}(x_1,x_2) \in \mathbf{V}(\omega)$,

$$\int_\omega a^{\alpha\beta\gamma\delta} \tilde{\mathcal{E}}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) \tilde{\mathcal{F}}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right) \sqrt{a} d\omega = \int_\omega p^{i,0} \eta_i \sqrt{a} d\omega,$$

where $\tilde{\mathcal{E}}^0_{\alpha\|\beta}(\mathbf{u}^0) = \mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) - \frac{1}{2}\frac{\lambda}{\lambda+2\mu} a_{\alpha\beta}$ and where

$$\tilde{\mathcal{F}}^0_{\alpha\|\beta}(\mathbf{u}^0,\boldsymbol{\eta}) = \left(\tilde{\mathcal{E}}^0_{\alpha\|\beta}(\mathbf{u}^0)\right)' \boldsymbol{\eta} = \mathcal{F}^0_{\alpha\|\beta}(\mathbf{u}^0,\boldsymbol{\eta}).$$

The associated boundary value problem reads

$$\begin{cases} -\tilde{T}^{\gamma\delta,0}_{\|\delta} = p^{\gamma,0} & \text{in } \omega, \\ -\tilde{T}^{3\delta,0}_{\|\delta} = p^{3,0} - \frac{1}{2}\lambda b_{\gamma\delta} a^{\gamma\delta} & \text{in } \omega, \\ \tilde{T}^{\gamma\delta,0} n_\delta = \frac{\lambda}{2} a^{\gamma\delta} n_\delta & \text{on } \gamma/\gamma_0, \\ \tilde{T}^{3\delta,0} n_\delta = 0 & \text{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \gamma_0, \end{cases}$$

where $\tilde{T}^{i\delta,0}$ (which is not the leading term of the asymptotic expansion of the first Piola-Kirchoff stress tensor) is given by

$$\tilde{T}^{\gamma\delta,0} = (a^{\alpha\beta\gamma\delta} + a^{\alpha\beta\tau\delta} a^{\rho\gamma} u^0_{\rho\|\tau})\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) - \frac{\lambda}{2} a^{\delta\tau} a^{\gamma\rho} u^0_{\rho\|\tau},$$
$$\tilde{T}^{3\delta,0} = a^{\alpha\beta\gamma\delta} u^0_{3\|\gamma}\, \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - \frac{\lambda}{2} a^{\gamma\delta} u^0_{3\|\gamma}.$$

The linearization of the variational equation reads:

Find $\mathbf{u}^0 \in \mathbf{V}_l(\omega) = \left\{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega);\ \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\right\}$ such that for all $\boldsymbol{\eta}(x_1, x_2) \in \mathbf{V}_l(\omega)$

$$\int_\omega a^{\alpha\beta\gamma\delta} e^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) e^0_{\gamma\|\delta}\left(\boldsymbol{\eta}\right) \sqrt{a} d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau} u^0_{\rho\|\gamma}\, \eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\, \eta^0_{3\|\delta}\right)\sqrt{a} d\omega$$
$$= \frac{1}{2}\int_\omega \lambda a^{\gamma\delta} e^0_{\gamma\|\delta}\left(\boldsymbol{\eta}\right) \sqrt{a} d\omega + \int_\omega p^{i,0}\eta_i \sqrt{a} d\omega.$$

So, it does not lead to the linear model. In each case, the linear one and the nonlinear one, a linear term in $\mathbf{u}^0$ and perturbation terms of volume and surface forces appear.

Finally, the system (1.1) has also the set of solutions (Section 5)

$$\begin{cases} \left(u^0_{\alpha\|3}\right)_\tau = \left(a_{\alpha\beta} + u^0_{\alpha\|\beta}\right) w^\beta_\tau, \\ \left(u^0_{3\|3}\right)_\tau = u^0_{3\|\alpha}\, w^\alpha_\tau - 1, \end{cases} \qquad \tau = 1, 2$$

where $\begin{pmatrix} w^1_1 \\ w^2_1 \end{pmatrix}$ and $\begin{pmatrix} w^1_2 \\ w^2_2 \end{pmatrix}$ are the components of the eigenvectors (defined up to a multiplicative constant) of the 2×2 matrix

$$AE := \begin{pmatrix} a^{\alpha 1}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 1}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0) \\ a^{\alpha 2}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 2}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0) \end{pmatrix}.$$

The new variational formulations associated with these solutions are then given by (Theorem 5.2):

Find $\mathbf{u}^0 \in \tilde{\mathbf{V}}(\omega) = \left\{\boldsymbol{\eta} = (\eta_i) \in \mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\right\}$ such that for all $\boldsymbol{\eta} \in \tilde{\mathbf{V}}(\omega)$,

$$\int_\omega \left\{b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) + d_1\Big(\frac{\lambda}{\lambda + 2\mu} a^{\gamma\delta} - w^\gamma_1 w^\delta_1\Big)\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0, \boldsymbol{\eta}\right)\sqrt{a} d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a} d\omega,$$

where d_1 is the eigenvalue of the matrix $-\mu(I + 2AE)$ associated with the eigenvector $\mathbf{w}_1$. We get the same kind of models with the eigenvalue d_2 of the matrix $-\mu(I + 2AE)$ associated with the eigenvector $\mathbf{w}_2$.

The associated boundary value problems read

$$\begin{cases} -\bar{T}^{i\delta,0}_{\|\delta} = p^{i,0} & \text{in } \omega, \\ \bar{T}^{i\delta,0} n_\delta = 0 & \text{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \gamma_0, \end{cases}$$

where the contravariant components of the limiting stress tensor associated with these limiting membrane shell models are given by

$$\bar{T}^{\gamma\delta,0} = (b^{\alpha\beta\gamma\delta} + b^{\alpha\beta\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho})\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - d_\tau(w^\gamma_\tau w^\delta_\tau + w^\delta_\tau w^\rho_\tau a^{\sigma\gamma} u^0_{\sigma\|\rho})$$
$$+ \frac{d_\tau\lambda}{\lambda + 2\mu}(a^{\gamma\delta} + a^{\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}),$$
$$\bar{T}^{3\delta,0} = \left(b^{\alpha\beta\delta\rho}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - d_\tau w^\delta_\tau w^\rho_\tau + \frac{d_\tau\lambda}{\lambda + 2\mu} a^{\delta\rho}\right) u^0_{3\|\rho}$$

with no summation on τ indices.

Once again, the linearization of these new variational problems does not lead to the linear model.

§2. About the Asymptotic Analysis of Shells

Let $\Omega^\varepsilon = \omega \times]-\varepsilon; +\varepsilon[$, we denote $x^\varepsilon = (x_i^\varepsilon)$ a point of $\bar{\Omega}^\varepsilon$ and let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$. Let $\boldsymbol{\phi}$ be the injective mapping (this is the case when ε is small enough[2]), defined by

$$\boldsymbol{\phi} : x^\varepsilon = (x_i^\varepsilon) \in \bar{\Omega}^\varepsilon \longmapsto \boldsymbol{\phi}(x^\varepsilon) = \boldsymbol{\varphi}(x_1^\varepsilon; x_2^\varepsilon) + x_3^\varepsilon \mathbf{a}_3 (x_1^\varepsilon; x_2^\varepsilon).$$

When ε is small enough, the three vectors defined by $\mathbf{g}_i^\varepsilon = \partial_i^\varepsilon \boldsymbol{\phi}$ are linearly independent, therefore they define a covariant basis at each point of $\bar{\Omega}^\varepsilon$. The associated contravariant basis $(\mathbf{g}^{j,\varepsilon})$ is defined by $\mathbf{g}^{j,\varepsilon}.\mathbf{g}_i^\varepsilon = \delta_{ij}$. We can then define the covariant components $g_{ij}^\varepsilon = \mathbf{g}_i^\varepsilon.\mathbf{g}_j^\varepsilon$ and the contravariant ones $g^{ij,\varepsilon} = \mathbf{g}^{i,\varepsilon}.\mathbf{g}^{j,\varepsilon}$ of the metric tensor. We also define the volume element $\sqrt{g^\varepsilon}dx^\varepsilon$ and the surface element $\det(\nabla\boldsymbol{\phi})\,|\nabla\boldsymbol{\phi}^{-t}\mathbf{n}|\,ds^\varepsilon$ where $g^\varepsilon = \det(g_{ij}^\varepsilon)$, as well as the three-dimensional Christoffel symbols $\Gamma_{ij}^{p,\varepsilon} = \mathbf{g}^{p,\varepsilon}.\partial_j^\varepsilon \mathbf{g}_i^\varepsilon = \Gamma_{ji}^{p,\varepsilon}$. Note that, for the mapping $\boldsymbol{\phi}$ defined above, $\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{k,\varepsilon} = 0$.

The shell whose reference configuration $\boldsymbol{\phi}(\bar{\Omega}^\varepsilon)$ is a natural state, is clamped on the portion $\boldsymbol{\phi}(\Gamma_0^\varepsilon)$ of its lateral surface $\boldsymbol{\phi}(\Gamma^\varepsilon)$. On its upper face $\boldsymbol{\phi}(\Gamma_+^\varepsilon)$ and its lower face $\boldsymbol{\phi}(\Gamma_-^\varepsilon)$, it is subjected to applied surface forces whose contravariant components are $h^{i,\varepsilon} : x^\varepsilon \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \to \mathbb{R}^3$, and to body forces whose contravariant components are $f^{i,\varepsilon} : x^\varepsilon \in \Omega^\varepsilon \to \mathbb{R}^3$. Under the action of these forces, the shell undergoes a displacement field $u_i^\varepsilon \mathbf{g}^{i,\varepsilon}$.

We next write the problem on a fixed open set $\Omega = \omega \times]-1; +1[$ (i.e. independent of ε) and we use the general principles of formal asymptotic analysis[6,7,1]:

(i) We denote by $x = (x_i)$ a point of $\bar{\Omega}$, and let $\partial_i = \partial/\partial x_i$. With the point $x \in \bar{\Omega}$, we associate the point $x^\varepsilon = (x_i^\varepsilon) \in \bar{\Omega}^\varepsilon$ defined by $x_\alpha^\varepsilon = x_\alpha$ and $x_3^\varepsilon = \varepsilon x_3$, hence $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$.

(ii) With each function d^ε defined on Ω^ε, we associate the function $d(\varepsilon)$ defined on Ω by

$$d^\varepsilon(x^\varepsilon) = d(\varepsilon)(x),\ x^\varepsilon \in \Omega^\varepsilon,\ x \in \Omega.$$

Then the "scaled" displacement field satisfies the variational problem:

Find $\mathbf{u}(\varepsilon) = (u_i(\varepsilon)) \in \mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{W}^{1,4}(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$ such that

$$\begin{aligned} &\varepsilon \int_\Omega g^{pqij}(\varepsilon)\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon))\,\mathcal{F}_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v})\sqrt{g(\varepsilon)}dx \\ &= \varepsilon \int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+\cup\Gamma_-} h^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\omega \end{aligned} \tag{2.1}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where

$$g^{pqij}(\varepsilon) = \lambda g^{pq}(\varepsilon)g^{ij}(\varepsilon) + \mu\left(g^{pi}(\varepsilon)g^{qj}(\varepsilon) + g^{pj}(\varepsilon)g^{qi}(\varepsilon)\right)$$

are the contravariant components of the "scaled" three-dimensional elasticity tensor, where

$$\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{2}\left\{u_{p\|q}(\varepsilon) + u_{q\|p}(\varepsilon) + g^{sr}(\varepsilon)u_{s\|p}(\varepsilon)u_{r\|q}(\varepsilon)\right\}$$

are the covariant components of the "scaled" symmetric Green-Saint Venant strain tensor, and where $\mathcal{F}_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v})$ are their Gâteaux derivatives at the point $\mathbf{u}(\varepsilon)$ in the direction $\mathbf{v}$, i.e.

$$\begin{aligned} &\mathcal{F}_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon),\mathbf{v}) = \mathcal{E}'_{i\|j}(\varepsilon)(\mathbf{u}(\varepsilon))\,\mathbf{v} \\ &= \frac{1}{2}\left\{v_{i\|j}(\varepsilon) + v_{j\|i}(\varepsilon) + g^{sr}(\varepsilon)\left(u_{s\|i}(\varepsilon)v_{r\|j}(\varepsilon) + v_{s\|i}(\varepsilon)u_{r\|j}(\varepsilon)\right)\right\} \end{aligned}$$

with $v_{i\|\beta}(\varepsilon) = \partial_\beta v_i - \Gamma_{i\beta}^k(\varepsilon)v_k,\ v_{i\|3}(\varepsilon) = \frac{1}{\varepsilon}\partial_3 v_i - \Gamma_{i3}^\sigma(\varepsilon)v_\sigma$.

Let us recall the principle of formal asymptotic analysis used by B. Miara & E. Sanchez-Palencia[9] in the case of linearly elastic thin shells:

(iii) We assume that each geometrical datum that depends on ε can be expanded up to whatever order is necessary in terms of power of x_3:

$$\mathbf{g}^i(\varepsilon) = \mathbf{a}^i + \varepsilon x_3 \mathbf{g}^{i,1} + \cdots ,$$
$$g^{ij}(\varepsilon) = a^{ij} + \varepsilon x_3 g^{ij,1} + \cdots ,$$
$$g^{pqij}(\varepsilon) = a^{pqij} + \varepsilon x_3 g^{pqij,1} + \cdots ,$$

where $a^{pqij} = \lambda a^{pq} a^{ij} + \mu \left\{ a^{pi} a^{qj} + a^{pj} a^{qi} \right\}$, $a^{ij} = \mathbf{a}^i . \mathbf{a}^j$ and $a^{\alpha 3} = \mathbf{a}^\alpha . \mathbf{a}_3 = 0$,

$$\Gamma^k_{ij}(\varepsilon) = \Gamma^{k,0}_{ij} + \varepsilon x_3 \Gamma^{k,1}_{ij} + \cdots ,$$
$$\Gamma^{\gamma,0}_{\alpha\beta} = \Gamma^{\gamma *}_{\alpha\beta}, \quad \Gamma^{3,0}_{\alpha\beta} = b_{\alpha\beta}, \quad \Gamma^{\beta,0}_{\alpha 3} = -b^\beta_\alpha, \quad \Gamma^{3,0}_{i3} = \Gamma^{i,0}_{33} = 0.$$

(iv) We assume that there exists a formal asymptotic expansion of the unknown

$$\mathbf{u}(\varepsilon) = \frac{1}{\varepsilon^k} \mathbf{u}^{-k} + \cdots + \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \cdots .$$

In her article, B. Miara[8] demonstrates that, for "membrane" thin elastic shells, the applied body forces are of order 0, $\mathbf{f}(\varepsilon) = \mathbf{f}^0$, the applied surface forces are of order 1, $\mathbf{h}(\varepsilon) = \varepsilon \mathbf{h}^1$, and that the displacement field is of order 0, $\mathbf{u}(\varepsilon) = \mathbf{u}^0$.

The identification of the successive powers of ε proves that

$$\partial_3 \mathbf{u}^0 = 0,$$

and then, by identifying the coefficients of ε^{-1} in (2.1), we get

$$\int_\Omega \left(\lambda a^{\alpha\beta} \mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) + (\lambda + 2\mu) \mathcal{E}^0_{3\|3}(\mathbf{u}^0, \mathbf{u}^1) \right) \mathcal{F}^{-1}_{3\|3}(\mathbf{u}^0, \mathbf{u}^1, \mathbf{v}) \sqrt{a} dx$$
$$+ \int_\Omega 4\mu a^{\alpha\gamma} \mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0, \mathbf{u}^1) \mathcal{F}^{-1}_{\gamma\|3}(\mathbf{u}^0, \mathbf{v}) \sqrt{a} dx = 0$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where

$$\begin{cases} \mathcal{F}^{-1}_{\gamma\|3}(\mathbf{u}^0, \mathbf{v}) = \frac{1}{2} \left(\partial_3 v_\gamma + a^{\beta\sigma} u^0_{\beta\|\gamma} \partial_3 v_\sigma \right) + \frac{1}{2} u^0_{3\|\gamma} \partial_3 v_3, \\ \mathcal{F}^{-1}_{3\|3}(\mathbf{u}^0, \mathbf{u}^1, \mathbf{v}) = a^{\gamma\delta} u^0_{\delta\|3} \partial_3 v_\gamma + \left(1 + u^0_{3\|3} \right) \partial_3 v_3. \end{cases}$$

If we choose test functions $\mathbf{v} = (v_1, v_2, 0)$ and then $\mathbf{v} = (0, 0, v_3)$, the variational problem above leads us to solve the nonlinear algebraic system in $u^0_{i\|3}$ (1.1) which can be written as a matrix in the following way.

With the notations:

$$A = \left(a^{\alpha\beta} \right), \quad E := E(\mathbf{u}^0) = \left(\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) \right), \quad U = \begin{pmatrix} u^0_{1\|1} & u^0_{1\|2} \\ u^0_{2\|1} & u^0_{2\|2} \end{pmatrix},$$
$$\mathcal{E}_0 := \mathcal{E}\left(\mathbf{u}^0, \mathbf{u}^1 \right) = \left(\mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0, \mathbf{u}^1) \right), \quad \mathbf{z} = \left(u^0_{3\|\alpha} \right), \quad \mathbf{y} = \left(u^0_{\alpha\|3} \right),$$

we get $U^t A U = \left(a^{\gamma\delta} u^0_{\gamma\|\alpha} u^0_{\delta\|\beta} \right)$ and $\mathbf{z}\mathbf{z}^t = \left(u^0_{3\|\alpha} u^0_{3\|\beta} \right)$, so that we can express the leading term of the asymptotic expansion of the Green-Saint Venant strain tensor (1.2) by

$$2E = U + U^t + U^t A U + \mathbf{z}\mathbf{z}^t,$$
$$2\mathcal{E}_0 = \left(I + U^t A \right) \mathbf{y} + \left(1 + u^0_{3\|3} \right) \mathbf{z},$$
$$2\mathcal{E}^0_{3\|3} = \mathbf{y}^t A \mathbf{y} + \left(1 + u^0_{3\|3} \right)^2 - 1.$$

Therefore, the system of equations (1.1) reads

$$\left\{\begin{array}{l}\left[\lambda tr\,(AE)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\right]A\mathbf{y}+2\mu A\,(I+UA)\,\mathcal{E}_0=\mathbf{0},\\ \left[\lambda tr\,(AE)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\right]\left(1+u^0_{3\|3}\right)+2\mu\mathbf{z}^tA\mathcal{E}_0=0.\end{array}\right. \tag{2.2}$$

We are now going to find all the solutions $u^0_{i\|3}$ of this system.

§3. Revisiting the First Limiting Model of Membrane Shells

Let us first recall the results obtained by B. Miara[8] leading to the classical limiting membrane shell model.

3.1. The Variational Problem

A first solution (1.3) of system (2.2) leads to the limiting membrane variational problem:

Find $\mathbf{u}^0\in\mathbf{V}(\omega)=\left\{\boldsymbol{\eta}=(\eta_i)\in\mathbf{W}^{1,4}(\omega);\ \boldsymbol{\eta}=\mathbf{0}\text{ on }\gamma_0\right\}$ such that for all $\boldsymbol{\eta}(x_1,x_2)\in\mathbf{V}(\omega)$ independent of x_3,

$$\int_\omega b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0)\mathcal{F}^0_{\gamma\|\delta}(\mathbf{u}^0,\boldsymbol{\eta})\sqrt{a}d\omega=\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega, \tag{3.1}$$

where the two-dimensional elasticity tensor is given by

$$b^{\alpha\beta\gamma\delta}=\frac{2\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\gamma\delta}+\mu\left(a^{\alpha\gamma}a^{\beta\delta}+a^{\alpha\delta}a^{\beta\gamma}\right),$$

and where the Gâteaux derivative of the two-dimensional change of metric tensor $\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)$ at the point $\mathbf{u}^0$ in the direction $\boldsymbol{\eta}$ is given by

$$\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)=\frac{1}{2}\left(\eta^0_{\gamma\|\delta}+\eta^0_{\delta\|\gamma}+a^{st}u^0_{s\|\delta}\eta^0_{t\|\gamma}+a^{st}u^0_{s\|\gamma}\eta^0_{t\|\delta}\right),$$

with the covariant derivatives of the vector $\boldsymbol{\eta}$

$$\eta^0_{\gamma\|\delta}=\partial_\delta\eta_\gamma-\Gamma^{\rho*}_{\gamma\delta}\eta_\rho-b_{\gamma\delta}\eta_3,\quad \eta^0_{3\|\delta}=\partial_\delta\eta_3+b^\rho_\delta\eta_\rho.$$

3.2. The Boundary Value Problem

The limiting contravariant components of the "scaled" stress tensor

$$T^{ij}(\varepsilon)=\left\{g^{pqij}(\varepsilon)+g^{pqjt}(\varepsilon)g^{si}(\varepsilon)u_{s\|t}(\varepsilon)\right\}\mathcal{E}_{p\|q}(\varepsilon)(\mathbf{u}(\varepsilon)) \tag{3.2}$$

associated with the solution (1.3) have been computed in [4, 5]; they are of order 0 for the terms $T^{i\delta}$ and of order 1 for the terms T^{i3}:

$$\begin{aligned}
T^{\gamma\delta,0}&=\left\{b^{\alpha\beta\gamma\delta}+b^{\alpha\beta\delta\rho}a^{\sigma\gamma}u^0_{\sigma\|\rho}\right\}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right),\\
T^{3\delta,0}&=b^{\alpha\beta\delta\rho}u^0_{3\|\rho}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right),\\
T^{\gamma3,1}&=-h^{\gamma,1}(x_1,x_2,-1)-\int_{-1}^{x_3}f^{\gamma,0}(x_1,x_2,t)dt\\
&\quad-(x_3+1)\left(\partial_\delta T^{\gamma\delta,0}+T^{\gamma\delta,0}\Gamma^{\rho*}_{\delta\rho}+T^{\rho\delta,0}\Gamma^{\gamma*}_{\rho\delta}-T^{3\delta,0}b^\gamma_\delta\right),\\
T^{33,1}&=-h^{3,1}(x_1,x_2,-1)-\int_{-1}^{x_3}f^{3,0}(x_1,x_2,t)dt\\
&\quad-(x_3+1)\left(\partial_\delta T^{3\delta,0}+T^{3\delta,0}\Gamma^{\rho*}_{\delta\rho}+T^{\gamma\delta,0}b_{\gamma\delta}\right).
\end{aligned} \tag{3.3}$$

Theorem 3.1. (i) *The limiting "membrane" problem associated with* (3.1) *is of the* *divergence form*

$$\begin{cases} -T^{i\delta,0}_{\|\delta} = p^{i,0} & \textit{in } \omega, \\ T^{i\delta,0} n_\delta = 0 & \textit{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \textit{on } \gamma_0, \end{cases}$$

where

$$T^{i\delta,0}_{\|\delta} = \partial_\delta T^{i\delta,0} + \Gamma^{\delta *}_{r\delta} T^{ir,0} + \Gamma^{i*}_{r\delta} T^{r\delta,0}.$$

(ii) *Hence we can rewrite the terms $T^{i3,1}$ as follow*:

$$T^{i3,1} = -h^{i,1}(x_1, x_2, -1) - \int_{-1}^{x_3} f^{i,0}(x_1, x_2, t)dt$$
$$+ \frac{x_3+1}{2}\Big\{ h^{i,1}(x_1, x_2, +1) + h^{i,1}(x_1, x_2, -1) + \int_{-1}^{1} f^{i,0}(x_1, x_2, t)dt \Big\}.$$

Proof. We rewrite the variational formulation (3.1) as follows:

$$\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega = \int_\omega b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0)\left(\eta^0_{\gamma\|\delta} + a^{\sigma\tau}u^0_{\sigma\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega,$$
$$= \int_\omega \left\{ T^{\gamma\delta,0}\eta^0_{\gamma\|\delta} + T^{3\delta,0}\eta^0_{3\|\delta} \right\}\sqrt{a}d\omega.$$

If we assume $T^{i\delta,0} \in \mathbf{H}^1(\omega)$, using a Green formula and the relation $\partial_\delta\sqrt{a} = \Gamma^{\rho *}_{\delta\rho}\sqrt{a}$ we get

$$\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega$$
$$= -\int_\omega \left(\partial_\delta T^{\gamma\delta,0} + \Gamma^{\rho *}_{\delta\rho}T^{\gamma\delta,0} + T^{\sigma\delta,0}\Gamma^{\gamma *}_{\sigma\delta} - T^{3\delta,0}b^\gamma_\delta\right)\eta_\gamma\sqrt{a}d\omega + \int_{\gamma/\gamma_0} T^{\gamma\delta,0}n_\delta\eta_\gamma\sqrt{a}d\gamma$$
$$- \int_\omega \left(\partial_\delta T^{3\delta,0} + \Gamma^{\rho *}_{\delta\rho}T^{3\delta,0} + T^{\gamma\delta,0}b_{\gamma\delta}\right)\eta_3\sqrt{a}d\omega + \int_{\gamma/\gamma_0} T^{3\delta,0}n_\delta\eta_3\sqrt{a}d\gamma$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$. Hence, we deduce that

$$-T^{i\delta,0}_{\|\delta} = p^{i,0} \text{ in } L^2(\omega),$$

and that

$$T^{i\delta,0}n_\delta = 0 \text{ on } \gamma/\gamma_0.$$

3.3. Linearization of the First Limiting Model

By linearizing the problem (3.1) written for a completely clamped shell, we find the "classical" variational formulation of linearly elastic membrane shells:

Find $\mathbf{u}^0 \in \mathbf{V}_m(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$ such that for all $\boldsymbol{\eta}(x_1, x_2) \in \mathbf{V}_m(\omega)$ independent of x_3,

$$\int_\omega b^{\alpha\beta\gamma\delta}e^0_{\alpha\|\beta}(\mathbf{u}^0)e^0_{\gamma\|\delta}(\boldsymbol{\eta})\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega, \tag{3.4}$$

where $e^0_{\alpha\|\beta}(\mathbf{u}^0)$ is the linearized change of metric tensor defined by

$$e^0_{\alpha\|\beta}(\mathbf{u}^0) = \frac{1}{2}\left(u^0_{\alpha\|\beta} + u^0_{\beta\|\alpha}\right),$$

with

$$u^0_{\alpha\|\beta} = \frac{1}{2}\left(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha\right) - \Gamma^{\zeta *}_{\alpha\beta}u^0_\zeta - b_{\alpha\beta}u^0_3.$$

Theorem 3.2. *The associated equations of equilibrium read*

$$\begin{cases} -\sigma^{\gamma\delta,0}_{\|\delta} = -\partial_\delta \sigma^{\gamma\delta,0} - \Gamma^{\rho *}_{\delta\rho}\sigma^{\gamma\delta,0} - \Gamma^{\gamma *}_{\rho\delta}\sigma^{\rho\delta,0} = p^{\gamma,0} \quad in \ \ \omega, \\ -\sigma^{3\delta,0}_{\|\delta} = -b_{\gamma\delta}\sigma^{\gamma\delta,0} = p^{3,0} \quad in \ \ \omega, \\ u^0_\alpha = 0 \quad on \ \gamma_0 = \gamma, \end{cases} \tag{3.5}$$

with $\sigma^{\gamma\delta,0} = b^{\alpha\beta\gamma\delta} e^0_{\alpha\|\beta}(\mathbf{u}^0)$.

Proof. If we assume that $\sigma^{\gamma\delta,0} \in \mathbf{H}^1(\omega)$, then by applying a Green formula to the variational formulation (3.4) which reads

$$\int_\omega \sigma^{\gamma\delta,0}\eta^0_{\gamma\|\delta}\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega$$

for all $\boldsymbol{\eta} \in \mathbf{V}_m(\omega)$, we deduce the boundary value problem.

§4. A Second Limiting Model of Membrane Shells

4.1. The Variational Problem

Theorem 4.1. *A second solution to the system* (1.1) *is*

$$\begin{cases} u^0_{\alpha\|3} = 0, \\ u^0_{3\|3} = -1, \end{cases}$$

and the associated limiting variational problem reads: *Find* $\mathbf{u}^0 \in \mathbf{V}(\omega)$ *such that for all* $\boldsymbol{\eta}(x_1,x_2) \in \mathbf{V}(\omega)$ *independent of* x_3,

$$\int_\omega a^{\alpha\beta\gamma\delta}\tilde{\mathcal{E}}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)\tilde{\mathcal{F}}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega \tag{4.1}$$

with

$$\begin{aligned} \tilde{\mathcal{E}}^0_{\alpha\|\beta}(\mathbf{u}^0) &= \mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) - \frac{1}{2}\frac{\lambda}{\lambda+2\mu}a_{\alpha\beta} \\ &= \frac{1}{2}(u^0_{\alpha\|\beta} + u^0_{\beta\|\alpha} + a^{\gamma\delta}u^0_{\gamma\|\alpha}u^0_{\delta\|\beta} + u^0_{3\|\alpha}u^0_{3\|\beta}) - \frac{1}{2}\frac{\lambda}{\lambda+2\mu}a_{\alpha\beta}, \end{aligned}$$

$$\tilde{\mathcal{F}}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right) = \mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right) = \frac{1}{2}(\eta^0_{\gamma\|\delta} + \eta^0_{\delta\|\gamma} + a^{st}u^0_{s\|\gamma}\eta^0_{t\|\delta} + a^{st}u^0_{s\|\delta}\eta^0_{t\|\gamma}).$$

The formal solution $\mathbf{u}^0$ *of the variational formulation is a stationary point of the functional*

$$J(\boldsymbol{\eta}) = \frac{1}{2}\int_\omega a^{\alpha\beta\gamma\delta}\tilde{\mathcal{E}}^0_{\alpha\|\beta}(\boldsymbol{\eta})\tilde{\mathcal{E}}^0_{\gamma\|\delta}(\boldsymbol{\eta})\sqrt{a}d\omega - \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega.$$

Remark. This problem can also be written as follows:

$$\begin{aligned} &\int_\omega a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega \\ &= \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\eta^0_{\gamma\|\delta}\sqrt{a}d\omega + \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega, \\ &= -\int_\omega \lambda\left(\Gamma^{\gamma *}_{\delta\rho}a^{\rho\delta} + \Gamma^{\delta *}_{\rho\delta}a^{\gamma\rho}\right)\eta_\gamma\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}b_{\gamma\delta}\eta_3\sqrt{a}d\omega \\ &\quad + \frac{1}{2}\int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}\eta_\gamma\nu_\delta\sqrt{a}d\gamma + \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega, \end{aligned} \tag{4.2}$$

using the relations

$$\eta^0_{\gamma\|\delta} = \partial_\delta\eta_\gamma - \Gamma^{\rho *}_{\gamma\delta}\eta_\rho - b_{\gamma\delta}\eta_3, \quad \partial_\delta\left(a^{\gamma\delta}\right) = \Gamma^{\gamma *}_{\delta\rho}a^{\rho\delta} + \Gamma^{\delta *}_{\rho\delta}a^{\gamma\rho} \ \text{ and } \ \partial_\delta\sqrt{a} = \Gamma^{\rho *}_{\delta\rho}\sqrt{a}.$$

So we obtain the nonlinear classical part of membrane shells:

$$\int_\omega a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}d\omega$$

with a linear term in $\mathbf{u}^0$: $\frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta}+u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega$ and perturbation terms on the body forces

$$-\int_\omega \lambda\left(\Gamma^{\gamma*}_{\delta\rho}a^{\rho\delta}+\Gamma^{\delta*}_{\rho\delta}a^{\gamma\rho}\right)\eta_\gamma\sqrt{a}d\omega-\frac{1}{2}\int_\omega \lambda a^{\gamma\delta}b_{\gamma\delta}\eta_3\sqrt{a}d\omega$$

and surface forces $\frac{1}{2}\int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}\eta_\gamma\nu_\delta\sqrt{a}d\gamma$ which only depend on the geometry of the shell.

Proof. Let us assume that, as in the first model,

$$\lambda\mathrm{tr}\,(AE)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}=0,$$

the system (2.2) reads

$$\begin{cases}2\mu(I+U^tA)^tA\mathcal{E}_0=\mathbf{0},\\ 2\mu\mathbf{z}^tA\mathcal{E}_0=0.\end{cases}$$

When the displacement field $\mathbf{u}^0$ vanishes, the matrix $(I+U^tA)=I$ is invertible. So we assume that at least in a suitable neighbourhood of $\mathbf{u}^0=\mathbf{0}$ (i.e. for small displacements), the matrix $(I+U^tA)$ is invertible. We then deduce that the vector $\mathcal{E}_0$ vanishes and thus we are led back to the system (1.3).

In order to find other solutions of the system (2.2) and therefore other limiting models, we must choose $\lambda\mathrm{tr}\,(AE)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\neq 0$. Moreover, if we assume that $\mathcal{E}_0=\mathbf{0}$, and since the matrix A is invertible, the system (2.2) then reads

$$\begin{cases}\mathbf{y}=\left(u^0_{\alpha\|3}\right)=\mathbf{0},\\ 1+u^0_{3\|3}=0.\end{cases}$$

We still have to check that this solution actually cancels the vector $\mathcal{E}_0$:

$$2\mathcal{E}_0=\left(I+U^tA\right)\mathbf{y}+(1+u^0_{3\|3})\mathbf{z}=\mathbf{0}.$$

So, the solution we have found is a solution to the system (2.2).

Thus, we are now able to give the limiting variational problem associated with this solution. To do so, we identify the terms ε^0 in the variational formulation (2.1) (see [8]):

$$\begin{aligned}&\int_\Omega x_3\left(g^{pqij,1}-b^\lambda_\lambda a^{pqij}\right)\mathcal{E}^0_{p\|q}(\mathbf{u}^0,\mathbf{u}^1)\mathcal{F}^{-1}_{i\|j}(\mathbf{u}^0,\mathbf{u}^1,\mathbf{v})\sqrt{a}dx\\&+\int_\Omega a^{pqij}\mathcal{E}^0_{p\|q}(\mathbf{u}^0,\mathbf{u}^1)\mathcal{F}^0_{i\|j}(\mathbf{u}^0,\mathbf{u}^1,\mathbf{u}^2,\mathbf{v})\sqrt{a}dx\\&+\int_\Omega a^{pqij}\mathcal{E}^1_{p\|q}(\mathbf{u}^0,\mathbf{u}^1,\mathbf{u}^2)\mathcal{F}^{-1}_{i\|j}(\mathbf{u}^0,\mathbf{u}^1,\mathbf{v})\sqrt{a}dx\\=&\int_\Omega f^{i,0}v_i\sqrt{a}dx+\int_{\Gamma_+\cup\Gamma_-}h^{i,1}v_i\sqrt{a}d\omega\end{aligned}\tag{4.3}$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$, where

$$\begin{aligned}\mathcal{F}^0_{i\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\mathbf{u}^2,\mathbf{v}\right)=\frac{1}{2}\Big\{&v^0_{i\|3}+v^0_{3\|i}+a^{st}\left(u^0_{s\|i}v^0_{t\|3}+u^0_{s\|3}v^0_{t\|i}+u^1_{s\|i}v^{-1}_{t\|3}+u^1_{s\|3}v^{-1}_{t\|i}\right)\\&+x_3\left(b^\alpha_\rho a^{\rho\beta}+b^\beta_\rho a^{\alpha\rho}\right)\left(u^0_{\alpha\|i}v^{-1}_{\beta\|3}+u^0_{\alpha\|3}v^{-1}_{\beta\|i}\right)\Big\},\end{aligned}$$

with $v^{-1}_{t\|\alpha} = 0$ and $v^{-1}_{t\|3} = \partial_3 v_t$. Using test functions $\mathbf{v}(x_1, x_2, x_3) = \boldsymbol{\eta}(x_1, x_2) \in \mathbf{V}(\omega)$ independent of x_3, we get $\mathcal{F}^{-1}_{i\|j}(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}) = 0$, and since we study the solution for which $\mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0) = 0$, the variational formulation reduces to

$$\int_\Omega \left\{a^{\alpha\beta ij}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + a^{33ij}\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right)\right\}\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right)\sqrt{a}dx = 2\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega.$$

Moreover, $2\mathcal{E}^0_{3\|3} = \mathbf{y}^t A\mathbf{y} + \left(1 + u^0_{3\|3}\right)^2 - 1 = -1$, hence we get

$$\begin{aligned}&\int_\Omega \left(a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - \frac{1}{2}\lambda a^{\gamma\delta}\right)\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0, \boldsymbol{\eta}\right)\sqrt{a}dx \\ &+ \int_\Omega \left(\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - \frac{\lambda + 2\mu}{2}\right)\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right)\sqrt{a}dx = 2\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega.\end{aligned}$$

Furthermore, $\eta^0_{3\|3} = 0$. Since $\boldsymbol{\eta}$ is independent of x_3, we get $\eta^{-1}_{t\|3} = \partial_3\eta_t = 0$. We then have $\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right) = a^{\gamma\delta}u^0_{\gamma\|3}\eta^0_{\delta\|3}$. Moreover, $u^0_{\gamma\|3} = 0$ is a solution to the system (2.2). Thus we get $\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right) = 0$, which leads to the result given in the theorem.

Remark. We note that when the transverse resultant of forces $p^{i,0}$ is equal to zero the problem reads: Find $\mathbf{u}^0 \in \mathbf{V}(\omega)$ such that for all $\boldsymbol{\eta}(x_1, x_2) \in \mathbf{V}(\omega)$ independent of x_3,

$$\begin{aligned}&\int_\omega a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0, \boldsymbol{\eta}\right)\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega \\ &= -\int_\omega \lambda\left(\Gamma^{\gamma *}_{\delta\rho}a^{\rho\delta} + \Gamma^{\delta *}_{\rho\delta}a^{\gamma\rho}\right)\eta_\gamma\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}b_{\gamma\delta}\eta_3\sqrt{a}d\omega + \frac{1}{2}\int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}\eta_\gamma\nu_\delta\sqrt{a}d\gamma.\end{aligned}$$

This problem have a nonvanishing solution.

4.2. The Boundary Value Problem

Theorem 4.2. *The boundary value problem associated with the variational formulation* (4.1) *reads*

$$\begin{cases} -\tilde{T}^{\gamma\delta,0}_{\|\delta} = p^{\gamma,0} & \text{in } \omega, \\ -\tilde{T}^{3\delta,0}_{\|\delta} = p^{3,0} - \frac{1}{2}\lambda b_{\gamma\delta}a^{\gamma\delta} & \text{in } \omega, \\ \tilde{T}^{\gamma\delta,0}n_\delta = \frac{\lambda}{2}a^{\gamma\delta}n_\delta & \text{on } \gamma/\gamma_0, \\ \tilde{T}^{3\delta,0}n_\delta = 0 & \text{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \gamma_0, \end{cases}$$

where $\tilde{T}^{i\delta,0}$ (*which is not the leading term of the asymptotic expansion of the first Piola-Kirchoff stress tensor*) *is given by*

$$\begin{aligned}\tilde{T}^{\gamma\delta,0} &= \left(a^{\alpha\beta\gamma\delta} + a^{\alpha\beta\tau\delta}a^{\rho\gamma}u^0_{\rho\|\tau}\right)\mathcal{E}^0_{\alpha\|\beta}(\mathbf{u}^0) - \frac{\lambda}{2}a^{\delta\tau}a^{\gamma\rho}u^0_{\rho\|\tau}, \\ \tilde{T}^{3\delta,0} &= a^{\alpha\beta\gamma\delta}u^0_{3\|\gamma}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - \frac{\lambda}{2}a^{\gamma\delta}u^0_{3\|\gamma}.\end{aligned}$$

Proof. We write the variational problem (4.1) as (4.2).

By applying a Green formula, we get

$$\begin{aligned}&\int_\omega \lambda a^{\gamma\delta}\eta^0_{\gamma\|\delta}\sqrt{a}d\omega \\ &= \int_\omega \lambda\left[-\partial_\delta\left(a^{\gamma\delta}\sqrt{a}\right)\eta_\gamma - \Gamma^{\gamma *}_{\rho\delta}a^{\rho\delta}\sqrt{a}\eta_\gamma - b_{\gamma\delta}a^{\gamma\delta}\sqrt{a}\eta_3\right]d\omega + \int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}n_\delta\eta_\gamma\sqrt{a}d\gamma\end{aligned}$$

for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$. Since $\partial_\beta \mathbf{a}^\alpha = -\Gamma^{\alpha *}_{\rho\beta} \mathbf{a}^\rho + b^\alpha_\beta \mathbf{a}^3$ (Gauss formula), we get

$$\partial_\delta \left(a^{\gamma\delta}\sqrt{a}\right) = \left(-\Gamma^{\gamma *}_{\delta\rho} a^{\rho\delta} - \Gamma^{\delta *}_{\delta\rho} a^{\gamma\rho} + \Gamma^{\rho *}_{\delta\rho} a^{\gamma\delta}\right)\sqrt{a} = -\Gamma^{\gamma *}_{\delta\rho} a^{\rho\delta}\sqrt{a},$$

hence

$$\int_\omega \lambda a^{\gamma\delta} \eta^0_{\gamma\|\delta}\sqrt{a}d\omega = -\int_\omega \lambda b_{\gamma\delta} a^{\gamma\delta}\eta_3\sqrt{a}d\omega + \int_{\gamma/\gamma_0} \lambda a^{\gamma\delta} n_\delta \eta_\gamma \sqrt{a}d\gamma.$$

Moreover,

$$2\int_\omega a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)\left(\eta^0_{\gamma\|\delta} + a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega$$
$$-\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega = 2\int_\omega \left\{\tilde{T}^{\gamma\delta,0}\eta^0_{\gamma\|\delta} + \tilde{T}^{3\delta,0}\eta^0_{3\|\delta}\right\}\sqrt{a}d\omega.$$

The variational formulation (4.2) then reads

$$\int_\omega \left\{\tilde{T}^{\gamma\delta,0}\eta^0_{\gamma\|\delta} + \tilde{T}^{3\delta,0}\eta^0_{3\|\delta}\right\}\sqrt{a}d\omega$$
$$= \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda b_{\gamma\delta}a^{\gamma\delta}\eta_3\sqrt{a}d\omega + \frac{1}{2}\int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}n_\delta\eta_\gamma\sqrt{a}d\gamma.$$

If we assume that $\tilde{T}^{i\delta,0} \in \mathbf{H}^1(\omega)$, then by applying a Green formula we get for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$

$$-\int_\omega \left(\partial_\delta \tilde{T}^{\gamma\delta,0} + \Gamma^{\rho *}_{\delta\rho}\tilde{T}^{\gamma\delta,0} + \tilde{T}^{\sigma\delta,0}\Gamma^{\gamma *}_{\sigma\delta} - \tilde{T}^{3\delta,0}b^\gamma_\delta\right)\eta_\gamma\sqrt{a}d\omega + \int_{\gamma/\gamma_0}\tilde{T}^{\gamma\delta,0}n_\delta\eta_\gamma\sqrt{a}d\gamma$$
$$-\int_\omega \left(\partial_\delta \tilde{T}^{3\delta,0} + \Gamma^{\rho *}_{\delta\rho}\tilde{T}^{3\delta,0} + \tilde{T}^{\gamma\delta,0}b_{\gamma\delta}\right)\eta_3\sqrt{a}d\omega + \int_{\gamma/\gamma_0}\tilde{T}^{3\delta,0}n_\delta\eta_3\sqrt{a}d\gamma$$
$$= \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda b_{\gamma\delta}a^{\gamma\delta}\eta_3\sqrt{a}d\omega + \frac{1}{2}\int_{\gamma/\gamma_0}\lambda a^{\gamma\delta}n_\delta\eta_\gamma\sqrt{a}d\gamma.$$

4.3. Linearization of the Second Limiting Model

By linearizing, the variational problem (4.1) reads:

Find $\mathbf{u}^0 \in \mathbf{V}_l(\omega) = \{\boldsymbol{\eta} \in \mathbf{H}^1(\omega);\ \boldsymbol{\eta} = 0 \text{ on } \gamma_0\}$ such that

$$\int_\omega a^{\alpha\beta\gamma\delta}e^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)e^0_{\gamma\|\delta}\left(\boldsymbol{\eta}\right)\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}\left(a^{\rho\tau}u^0_{\rho\|\gamma}\eta^0_{\tau\|\delta} + u^0_{3\|\gamma}\eta^0_{3\|\delta}\right)\sqrt{a}d\omega$$
$$= \int_\omega \tilde{a}^{\alpha\beta\gamma\delta}e^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)e^0_{\gamma\|\delta}\left(\boldsymbol{\eta}\right)\sqrt{a}d\omega - \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}u^0_{3\|\gamma}\eta^0_{3\|\delta}\sqrt{a}d\omega$$
$$= \frac{1}{2}\int_\omega \lambda a^{\gamma\delta}e^0_{\gamma\|\delta}\left(\boldsymbol{\eta}\right)\sqrt{a}d\omega + \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega \tag{4.4}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_l(\omega)$, where $\tilde{a}^{\alpha\beta\gamma\delta} = \lambda a^{\alpha\beta}a^{\gamma\delta} + \left(\mu - \frac{\lambda}{2}\right)\left(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma}\right)$, and so we are not led to the linear model (3.4) of membrane shells.

Remark. The symmetric matrix $\tilde{a}$ is positive definite for materials like steel ($\lambda = 10.10^5\text{kg/cm}^2$ and $\mu = 8,2.10^5\text{kg/cm}^2$), iron ($\lambda = 9,9.10^5\text{kg/cm}^2$ and $\mu = 7,8.10^5\text{kg/cm}^2$), bronze ($\lambda = 6,2.10^5\text{kg/cm}^2$ and $\mu = 3,8.10^5\text{kg/cm}^2$), glass ($\lambda = 2,2.10^5\text{kg/cm}^2$ and $\mu = 2,2.10^5\text{kg/cm}^2$), nickel ($\lambda = 1,3.10^5\text{kg/cm}^2$ and $\mu = 0,85.10^5\text{kg/cm}^2$).

Theorem 4.3. *The limiting problem associated with the linearized variational formula-*

tion (4.4) *reads*

$$\begin{cases} -\tilde{t}^{\gamma\delta,0}_{\|\delta} = -\partial_\delta \tilde{t}^{\gamma\delta,0} - \Gamma^{\rho *}_{\delta\rho}\tilde{t}^{\gamma\delta,0} - \Gamma^{\gamma *}_{\sigma\delta}\tilde{t}^{\sigma\delta,0} + \tilde{t}^{3\delta,0} b^\gamma_\delta = p^{\gamma,0} \quad in \ \ \omega, \\ -\tilde{t}^{3\delta,0}_{\|\delta} = -\partial_\delta \tilde{t}^{3\delta,0} - \Gamma^{\rho *}_{\delta\rho}\tilde{t}^{3\delta,0} - b_{\gamma\delta}\tilde{t}^{\gamma\delta,0} = p^{3,0} - \frac{1}{2}\lambda b_{\gamma\delta} a^{\gamma\delta} \quad in \ \ \omega, \\ \tilde{t}^{\gamma\delta,0} n_\delta = \frac{\lambda}{2} a^{\gamma\delta} n_\delta \ \ on \ \gamma/\gamma_0, \\ \tilde{t}^{3\delta,0} n_\delta = 0 \ \ on \ \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} \ \ on \ \gamma_0, \end{cases}$$

where the linearized stress tensor $\tilde{t}^{\gamma\delta,0}$ *is given by*

$$\tilde{t}^{\gamma\delta,0} = \tilde{a}^{\alpha\beta\gamma\delta} e^0_{\alpha\|\beta}(\mathbf{u}^0), \quad \tilde{t}^{3\delta,0} = -\frac{\lambda}{2} a^{\delta\rho} u^0_{3\|\rho}.$$

§5. The Other Limiting Models of Membrane Shells

In order to find the other membrane shell models, we must now assume that

$$\begin{cases} \lambda a^{\alpha\beta} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + (\lambda + 2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) \neq 0, \\ \mathcal{E}^0_{\alpha\|3}(\mathbf{u}^0, \mathbf{u}^1) \neq 0. \end{cases}$$

Theorem 5.1. *The other solutions of the system* (1.1) *are given by the relations*

$$\begin{cases} \left(u^0_{\alpha\|3}\right) = \left(a_{\alpha\gamma} + u^0_{\alpha\|\gamma}\right) w^\gamma, \\ \left(u^0_{3\|3}\right) = u^0_{3\|\gamma} w^\gamma - 1, \end{cases} \tag{5.1}$$

where $\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ *when they exist are the components of each eigenvector* (*defined up to a multiplicative constant*) *of the* 2×2 *matrix*

$$\begin{pmatrix} a^{\alpha 1}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 1}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0) \\ a^{\alpha 2}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 2}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0) \end{pmatrix}.$$

Proof. If we denote $d = \lambda \mathrm{tr}(AE) + (\lambda + 2\mu)\mathcal{E}^0_{3\|3}$, the system (2.2) then reads

$$\begin{cases} 2\mu\,(I + UA)\,\mathcal{E}_0 + d\mathbf{y} = \mathbf{0}, \\ 2\mu \mathbf{z}^t A \mathcal{E}_0 + d\left(1 + u^0_{3\|3}\right) = 0. \end{cases} \tag{5.2}$$

The first equation of the system (5.2) also reads

$$2\mu A\mathcal{E}_0 + d\left(A^{-1} + U\right)^{-1}\mathbf{y} = \mathbf{0},$$

hence, when we multiply this equation by $\mathbf{z}^t$, we get

$$2\mu \mathbf{z}^t A\mathcal{E}_0 + d\mathbf{z}^t\left(A^{-1} + U\right)^{-1}\mathbf{y} = 0,$$

and when we substract the second equation of the system (5.2), we get

$$d\mathbf{z}^t\left(A^{-1} + U\right)^{-1}\mathbf{y} = d(1 + u^0_{3\|3}).$$

Moreover, since we consider the case $d \neq 0$, we get

$$\left(1 + u^0_{3\|3}\right) = \mathbf{z}^t\left(A^{-1} + U\right)^{-1}\mathbf{y}. \tag{5.3}$$

In order to express the values of $\mathbf{y}$ we rewrite $\mathcal{E}_0$ as

$$\begin{aligned}2\mathcal{E}_0 &= \left(I + U^t A\right)\mathbf{y} + \left(1 + u^0_{3\|3}\right)\mathbf{z}\\ &= \left\{\left(I + U^t A\right)\left(A^{-1} + U\right) + \mathbf{z}\mathbf{z}^t\right\}\left(A^{-1} + U\right)^{-1}\mathbf{y}\\ &= \left\{A^{-1} + U + U^t + U^t AU + \mathbf{z}\mathbf{z}^t\right\}\left(A^{-1} + U\right)^{-1}\mathbf{y}\\ &= \left(A^{-1} + 2E\right)\left(A^{-1} + U\right)^{-1}\mathbf{y}.\end{aligned}$$

By replacing $\mathcal{E}_0$ in the first equation of the system (5.2) by the expression found above, we get

$$\left\{\mu\left(I + UA\right)\left(A^{-1} + 2E\right)\left(A^{-1} + U\right)^{-1} + dI\right\}\mathbf{y} = \mathbf{0},$$

hence

$$\mu\left(A^{-1} + U\right)\left(I + 2AE\right)\left(A^{-1} + U\right)^{-1}\mathbf{y} = -d\mathbf{y}.$$

Let $M = A^{-1} + U$. Then, the equation above reads

$$\left(I + 2AE\right)M^{-1}\mathbf{y} = -\frac{d}{\mu}M^{-1}\mathbf{y}.$$

Let $M^{-1}\mathbf{y}$ be an eigenvector of the matrix $(I + 2AE)$ associated with the eigenvalue $-\frac{d}{\mu}$.

Then, if $\mathbf{w}$ is one of the two eigenvectors (defined up to a multiplicative constant) of the 2×2 matrix AE and χ is its associated eigenvalue, and if we note $M^{-1}\mathbf{y} = \mathbf{w}$, then we get $-\frac{d}{\mu} = 1 + 2\chi$. Thus, we just have to compute the eigenvalues and the eigenvectors of AE.

Next, we can give the expression of the vector $\mathbf{y} = M\mathbf{w}$:

$$u^0_{\alpha\|3} = y_\alpha = (a_{\alpha\gamma} + u^0_{\alpha\|\gamma})w^\gamma.$$

And then, using the equation (5.3) we get

$$1 + u^0_{3\|3} = \mathbf{z}^t\mathbf{w} = u^0_{3\|\gamma}w^\gamma.$$

5.1. The Variational Problems

Associated with each eigenvector $\mathbf{w}$ of the matrix AE given in Theorem 5.1, we get a displacement field whose covariant components solve the following variational problem.

Theorem 5.2. *The limiting variational problems associated with the solutions* (5.1) *read*: *Find* $\mathbf{u}^0 \in \tilde{\mathbf{V}}(\omega)$ *such that for all* $\boldsymbol{\eta} \in \tilde{\mathbf{V}}(\omega)$,

$$\int_\omega \left\{b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + d\Big(\frac{\lambda}{\lambda + 2\mu}a^{\gamma\delta} - w^\gamma w^\delta\Big)\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0, \boldsymbol{\eta}\right)\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega, \quad (5.4)$$

where (w^1, w^2) *are the components of the eigenvector* (*defined up to a multiplicative constant*) *of the matrix* $\begin{pmatrix} a^{\alpha 1}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 1}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0) \\ a^{\alpha 2}\mathcal{E}^0_{\alpha\|1}(\mathbf{u}^0) & a^{\alpha 2}\mathcal{E}^0_{\alpha\|2}(\mathbf{u}^0)\end{pmatrix}$ *associated with the eigenvalue* χ, *and where* $d = -\mu(1 + 2\chi)$.

Proof. As we did in the proof of Theorem 4.1, we use the variational formulation obtained by identifying the terms in ε^0 in (2.1), and we get (4.3).

For test functions $\mathbf{v}(x_1, x_2, x_3) = \boldsymbol{\eta}(x_1, x_2)$ independent of x_3,

$$\mathcal{F}^{-1}_{i\|j}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right) = 0$$

and the variational formulation reduces to

$$\int_\Omega a^{pqij}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0, \mathbf{u}^1\right)\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0, \mathbf{u}^1, \boldsymbol{\eta}\right)\sqrt{a}dx = 2\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega.$$

Let us expand the left part of this relation:

$$
\begin{aligned}
&\int_\Omega a^{pqij}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
=&\int_\Omega \left\{a^{\alpha\beta ij}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+2a^{\alpha 3ij}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)+a^{33ij}\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
=&\int_\Omega \left\{a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+\lambda a^{\gamma\delta}\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&+\int_\Omega 4\mu a^{\alpha\gamma}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{\gamma\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&+\int_\Omega \left\{\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx.
\end{aligned}
$$

Let us now write the $\mathcal{F}^0_{i\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)$ in the following way:

$$
\begin{aligned}
2\mathcal{F}^0_{\gamma\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)&=\left(\eta^0_{\gamma\|3}+\eta^0_{3\|\gamma}\right)+x_3g^{st,1}\left(u^0_{s\|\gamma}\eta^{-1}_{t\|3}+u^0_{s\|3}\eta^{-1}_{t\|\gamma}\right)\\
&\quad+a^{st}\left(u^0_{s\|\gamma}\eta^0_{t\|3}+u^0_{s\|3}\eta^0_{t\|\gamma}+u^1_{s\|\gamma}\eta^{-1}_{t\|3}+u^1_{s\|3}\eta^{-1}_{t\|\gamma}\right)\\
&=\left(\eta^0_{\gamma\|3}+\eta^0_{3\|\gamma}\right)+a^{\sigma\rho}\left(u^0_{\sigma\|\gamma}\eta^0_{\rho\|3}+u^0_{\sigma\|3}\eta^0_{\rho\|\gamma}\right)+u^0_{3\|3}\eta^0_{3\|\gamma},
\end{aligned}
$$

hence if we denote $\boldsymbol{\xi}=\begin{pmatrix}\eta^0_{3\|1}\\ \eta^0_{3\|2}\end{pmatrix}$, $\boldsymbol{\tau}=\begin{pmatrix}\eta^0_{1\|3}\\ \eta^0_{2\|3}\end{pmatrix}$, $\mathcal{F}_0=\begin{pmatrix}\mathcal{F}^0_{1\|3}\\ \mathcal{F}^0_{2\|3}\end{pmatrix}$ and $H=\begin{pmatrix}\eta^0_{1\|1} & \eta^0_{1\|2}\\ \eta^0_{2\|1} & \eta^0_{2\|2}\end{pmatrix}$, we get

$$
2\mathcal{F}_0=\left(I+U^tA\right)\boldsymbol{\tau}+\left(1+u^0_{3\|3}\right)\boldsymbol{\xi}+H^tA\mathbf{y},
$$

and in the same way,

$$
\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)=\eta^0_{3\|3}+x_3g^{st,1}u^0_{s\|3}\eta^{-1}_{t\|3}+a^{st}u^0_{s\|3}\eta^0_{t\|3}+a^{st}u^1_{s\|3}\eta^{-1}_{t\|3}=a^{\gamma\delta}u^0_{\gamma\|3}\eta^0_{\delta\|3},
$$

hence

$$
\mathcal{F}^0_{3\|3}=\boldsymbol{\tau}^tA\mathbf{y}.
$$

So we deduce

$$
\begin{aligned}
&\int_\Omega \left\{\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
=&\int_\Omega d\boldsymbol{\tau}^tA\mathbf{y}\sqrt{a}dx,\\
&\int_\Omega 4\mu a^{\alpha\gamma}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{\gamma\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
=&\int_\Omega 2\mu\left\{\boldsymbol{\tau}^t\left(I+AU\right)+\left(1+u^0_{3\|3}\right)\boldsymbol{\xi}^t+\mathbf{y}^tAH\right\}A\mathcal{E}_0\sqrt{a}dx.
\end{aligned}
$$

The first equation of the system (5.2) multiplied by $\boldsymbol{\tau}^tA$ reads

$$
2\mu\boldsymbol{\tau}^t\left(I+AU\right)A\mathcal{E}_0+d\boldsymbol{\tau}^tA\mathbf{y}=0,
$$

so by adding the integrals above, we deduce

$$\begin{aligned}
&\int_\Omega \left\{\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+(\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&+\int_\Omega 4\mu a^{\alpha\gamma}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{\gamma\|3}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&=\int_\Omega 2\mu\left\{\left(1+u^0_{3\|3}\right)\boldsymbol{\xi}^t+\mathbf{y}^tAH\right\}A\mathcal{E}_0\sqrt{a}dx\\
&=\int_\Omega \left\{\mathbf{z}^t\mathbf{w}\boldsymbol{\xi}^t+\mathbf{w}^t\left(A^{-1}+U^t\right)AH\right\}2\mu A\mathcal{E}_0\sqrt{a}dx.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_\Omega a^{pqij}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&=\int_\Omega \left\{a^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+\lambda a^{\gamma\delta}\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&+\int_\Omega \left\{\mathbf{z}^t\mathbf{w}\boldsymbol{\xi}^t+\mathbf{w}^t\left(A^{-1}+U^t\right)AH\right\}2\mu A\mathcal{E}_0\sqrt{a}dx.
\end{aligned}$$

To obtain the variational formulations, we still have to give the expressions of $2\mu A\mathcal{E}_0$ and $\mathcal{E}^0_{3\|3}$. By using the first equation of the system (5.2),

$$2\mu A\mathcal{E}_0=-d\left(A^{-1}+U\right)^{-1}\mathbf{y}=-d\mathbf{w}.$$

By using the definition of d,

$$\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)=\frac{d}{\lambda+2\mu}-\frac{\lambda}{\lambda+2\mu}\mathrm{tr}\left(AE\right).$$

The integral above then reads

$$\begin{aligned}
&\int_\Omega a^{pqij}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0,\mathbf{u}^1\right)\mathcal{F}^0_{i\|j}\left(\mathbf{u}^0,\mathbf{u}^1,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&=\int_\Omega \left\{b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+\frac{\lambda}{\lambda+2\mu}a^{\gamma\delta}d\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}dx\\
&-\int_\Omega d\mathbf{w}^t\left\{\mathbf{z}\boldsymbol{\xi}^t+\left(I+U^tA\right)H\right\}\mathbf{w}\sqrt{a}dx.
\end{aligned}$$

Moreover,

$$\left(\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\right)=\frac{1}{2}\left(H+H^t+U^tAH+H^tAU+\mathbf{z}\boldsymbol{\xi}^t+\boldsymbol{\xi}\mathbf{z}^t\right),$$

and consequently, if we notice that

$$\mathbf{w}^t\left(\mathbf{z}\boldsymbol{\xi}^t+\left(I+U^tA\right)H\right)\mathbf{w}=\mathbf{w}^t\left(\boldsymbol{\xi}\mathbf{z}^t+H^t\left(I+AU\right)\right)\mathbf{w},$$

we get

$$\mathbf{w}^t\left\{\mathbf{z}\boldsymbol{\xi}^t+\left(I+U^tA\right)H\right\}\mathbf{w}=\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)w^\gamma w^\delta.$$

The variational formulations then read

$$\int_\Omega \left\{b^{\alpha\beta\gamma\delta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)+d\left(\frac{\lambda}{\lambda+2\mu}a^{\gamma\delta}-w^\gamma w^\delta\right)\right\}\mathcal{F}^0_{\gamma\|\delta}\left(\mathbf{u}^0,\boldsymbol{\eta}\right)\sqrt{a}dx=2\int_\omega p^{i,0}\eta_i\sqrt{a}d\omega.$$

5.2. The Boundary Value Problems

Theorem 5.3. *The boundary value problems associated with the variational formulation* (5.4) *read*

$$\begin{cases} -\bar{T}^{i\delta,0}_{\|\delta} = p^{i,0} & \text{in } \omega, \\ \bar{T}^{i\delta,0} n_\delta = 0 & \text{on } \gamma/\gamma_0, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \gamma_0, \end{cases}$$

where the contravariant components of the limiting stress tensor associated with these limiting membrane shell models are given by

$$\begin{aligned} \bar{T}^{\gamma\delta,0} &= (b^{\alpha\beta\gamma\delta} + b^{\alpha\beta\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}) \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - d(w^\gamma w^\delta + w^\delta w^\rho a^{\sigma\gamma} u^0_{\sigma\|\rho}) \\ &\quad + \frac{d\lambda}{\lambda + 2\mu}(a^{\gamma\delta} + a^{\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}), \\ \bar{T}^{3\delta,0} &= \Big(b^{\alpha\beta\delta\rho} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) - d w^\delta w^\rho + \frac{d\lambda}{\lambda + 2\mu} a^{\delta\rho}\Big) u^0_{3\|\rho}, \\ \bar{T}^{i3,1} &= -h^{i,1}(x_1, x_2, -1) - \int_{-1}^{x_3} f^{i,0}(x_1, x_2, t)dt + (x_3 + 1) p^{i,0}. \end{aligned}$$

Proof. By identifying the terms of order 0 of the "scaled" first Piola-Kirchhoff stress tensor, we get

$$\bar{T}^{\gamma\delta,0} = (a^{pq\gamma\delta} + a^{pq\delta t} a^{s\gamma} u^0_{s\|t}) \mathcal{E}^0_{p\|q}\left(\mathbf{u}^0, \mathbf{u}^1\right),$$

and therefore

$$\begin{aligned} \bar{T}^{\gamma\delta,0} &= (a^{\alpha\beta\gamma\delta} + a^{\alpha\beta\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}) \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu a^{\alpha\delta} a^{\sigma\gamma} u^0_{\sigma\|3} \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) \\ &\quad + \lambda(a^{\gamma\delta} + a^{\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}) \mathcal{E}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right). \end{aligned}$$

Moreover $\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) = \frac{d}{\lambda+2\mu} - \frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)$, so

$$\begin{aligned} \bar{T}^{\gamma\delta,0} &= (b^{\alpha\beta\gamma\delta} + b^{\alpha\beta\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}) \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu a^{\alpha\delta} a^{\sigma\gamma} u^0_{\sigma\|3} \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) \\ &\quad + \frac{d\lambda}{\lambda + 2\mu}(a^{\gamma\delta} + a^{\delta\rho} a^{\sigma\gamma} u^0_{\sigma\|\rho}). \end{aligned}$$

Furthermore $2\mu A\mathcal{E}_0 = -d\mathbf{w}$ and $\mathbf{y} = \left(A^{-1} + U\right)\mathbf{w}$, hence

$$\begin{aligned} 2\mu a^{\alpha\delta} a^{\sigma\gamma} u^0_{\sigma\|3} \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) &= A\mathbf{y}\left(2\mu A\mathcal{E}_0\right)^t = -d\left(I + AU\right)\mathbf{w}\mathbf{w}^t \\ &= -d\begin{pmatrix} w^1 w^1 + w^1 w^\rho a^{\sigma 1} u^0_{\sigma\|\rho} & w^1 w^2 + w^2 w^\rho a^{\sigma 1} u^0_{\sigma\|\rho} \\ w^1 w^2 + w^1 w^\rho a^{\sigma 2} u^0_{\sigma\|\rho} & w^2 w^2 + w^2 w^\rho a^{\sigma 2} u^0_{\sigma\|\rho} \end{pmatrix} \\ &= -d(w^\gamma w^\delta + w^\delta w^\rho a^{\sigma\gamma} u^0_{\sigma\|\rho}). \end{aligned}$$

We thus get the expression of the $\bar{T}^{\gamma\delta,0}$.

In the same way,

$$\begin{aligned} \bar{T}^{3\delta,0} &= \{a^{pq3\delta} + a^{pq\delta t} a^{s3} u^0_{s\|t}\} \mathcal{E}^0_{p\|q}\left(\mathbf{u}^0, \mathbf{u}^1\right) \\ &= a^{\alpha\beta\delta\rho} u^0_{3\|\rho} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu a^{\alpha\delta}(1 + u^0_{3\|3}) \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) + \lambda a^{\delta\rho} u^0_{3\|\rho} \mathcal{E}^0_{3\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right). \end{aligned}$$

Moreover $\mathcal{E}^0_{3\|3}(\mathbf{u}^0, \mathbf{u}^1) = \frac{d}{\lambda+2\mu} - \frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right)$, so

$$\bar{T}^{3\delta,0} = b^{\alpha\beta\delta\rho} u^0_{3\|\rho} \mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu a^{\alpha\delta}(1 + u^0_{3\|3}) \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) + \frac{d\lambda}{\lambda + 2\mu} a^{\delta\rho} u^0_{3\|\rho}.$$

Furthermore, by using the relations $2\mu A\mathcal{E}_0 = -d\mathbf{w}$ and $(1 + u^0_{3\|3}) = \mathbf{z}^t\mathbf{w}$, we get

$$2\mu a^{\alpha\delta}\left(1 + u^0_{3\|3}\right) \mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0, \mathbf{u}^1\right) = (1 + u^0_{3\|3}) 2\mu A\mathcal{E}_0 = -d\mathbf{w}\mathbf{z}^t\mathbf{w} = -d w^\delta w^\rho u^0_{3\|\rho}.$$

Finally, we get the expression of the $\bar{T}^{3\delta,0}$.

$$\begin{aligned}\bar{T}^{\gamma 3,0} &= \{a^{pq\gamma 3} + a^{pq3t}a^{s\gamma}u^0_{s\|t}\}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &= \lambda a^{\alpha\beta}a^{\sigma\gamma}u^0_{\sigma\|3}\,\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu\{a^{\alpha\gamma} + a^{\alpha\rho}a^{\sigma\gamma}u^0_{\sigma\|\rho}\}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &\quad + (\lambda+2\mu)a^{\sigma\gamma}u^0_{\sigma\|3}\,\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &= (\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + (\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right))a^{\sigma\gamma}u^0_{\sigma\|3}\\ &\quad + 2\mu\left\{a^{\alpha\gamma} + a^{\alpha\rho}a^{\sigma\gamma}u^0_{\sigma\|\rho}\right\}\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right),\end{aligned}$$

which is exactly the first equation of the system (1.1), hence $\bar{T}^{\gamma 3,0} = 0$.

$$\begin{aligned}\bar{T}^{33,0} &= \{a^{pq33} + a^{pq3t}a^{s3}u^0_{s\|t}\}\mathcal{E}^0_{p\|q}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &= \lambda a^{\alpha\beta}(1+u^0_{3\|3})\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + 2\mu a^{\alpha\rho}u^0_{3\|\rho}\,\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &\quad + (\lambda+2\mu)(1+u^0_{3\|3})\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right)\\ &= (\lambda a^{\alpha\beta}\mathcal{E}^0_{\alpha\|\beta}\left(\mathbf{u}^0\right) + (\lambda+2\mu)\mathcal{E}^0_{3\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right))(1+u^0_{3\|3}) + 2\mu a^{\alpha\rho}u^0_{3\|\rho}\,\mathcal{E}^0_{\alpha\|3}\left(\mathbf{u}^0,\mathbf{u}^1\right),\end{aligned}$$

which is exactly the second equation of the system (1.1), hence $\bar{T}^{33,0} = 0$.

The expressions of the $\bar{T}^{i3,1}$ are obtained in the same way as in [5].

The variational formulations (5.4) also read

$$\int_\omega \{\bar{T}^{\gamma\delta,0}\eta^0_{\gamma\|\delta} + \bar{T}^{3\delta,0}\eta^0_{3\|\delta}\}\sqrt{a}d\omega = \int_\omega p^{i,0}\eta_i\sqrt{a}d\omega,$$

and if we assume that $\bar{T}^{i\delta,0} \in \mathbf{H}^1(\omega)$ then the end of the proof resembles that of Theorem 3.1.

Remark. The linearization of the variational formulation has still to be established.

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