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SEMI-GLOBAL C^1 SOLUTION TO THE MIXED INITIAL-BOUNDARY VALUE PROBLEM FOR QUASILINEAR HYPERBOLIC SYSTEMS**

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Abstract

By means of an equivalent invariant form of boundary conditions, the authors get the existence and uniqueness of semi-global C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems with general nonlinear boundary conditions.

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§1. Introduction

A systematic theory on the local C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems can be found in Li Ta-tsien & Yu Wenci[1] and Li Ta-tsien, Yu Wenci & Shen Weixi[2]. In order to study the exact boundary controllability for quasilinear hyperbolic systems (cf. [3–5]), it is necessary to consider the semi-global C^1 solution, i.e., the C^1 solution on the time interval $0 \le t \le T_0$, where $T_0 > 0$ is a preassigned and possibly quite large number. M. Cirina[6,7] considered this kind of problem for special boundary conditions, but he imposed very strong hypotheses on the coefficients of the system (globally bounded and globally Lipschitz continuous), which is a grave restriction to applications. In this paper we first improve the original theory of local C^1 solution, and then, by establishing a uniform a priori estimate on the C^1 norm of the solution, the successive extension of local C^1 solution will lead to the existence and uniqueness of semi-global C^1 solution for the mixed initial-boundary value problem with general nonlinear boundary conditions.

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

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where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of $(t, x), A(u)$ is a given $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n), F(u)$ is a given vector function with suitably smooth components $f_i(u)(i = 1, \cdots, n)$ and

$$F(0) = 0. \tag{1.2}$$

By the definition of hyperbolicity, on the domain under consideration the matrix $A(u)$ has n real eigenvalues $\lambda_i(u)(i = 1, \cdots, n)$ and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))(i = 1, \cdots, n)$ and, correspondingly, a complete set of right eigenvectors $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (i = 1, \cdots, n), \tag{1.3}$$

$$A(u)r_i(u) = \lambda_i(u)r_i(u) \quad (i = 1, \cdots, n). \tag{1.4}$$

We have

$$\det|l_{ij}(u)| \neq 0 \qquad (\text{resp.} \quad \det|r_{ij}(u)| \neq 0). \tag{1.5}$$

Without loss of generality, we may assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.6}$$

$$r_i^T(u)r_i(u) \equiv 1 \qquad (i = 1, \cdots, n), \tag{1.7}$$

where δ_{ij} stands for the Kronecker symbol.

In this paper we assume that on the domain under consideration, the eigenvalues satisfy the following conditions:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m; s = m + 1, \cdots, n). \tag{1.8}$$

We consider the following mixed initial-boundary value problem (Problem I) for the quasilinear hyperbolic system (1.1) on the domain

$$R(T) = \Big\{(t, x)\Big|0 \leq t \leq T, \quad 0 \leq x \leq 1\Big\} \quad (T > 0)$$

with the initial condition

$$t = 0 : u = \varphi(x) \qquad (0 \leq x \leq 1) \tag{1.9}$$

and the boundary conditions

$$x = 0 : \tilde{v}_s = g_s(t, \tilde{v}_1, \cdots, \tilde{v}_m) + h_s(t) \quad (s = m + 1, \cdots, n), \tag{1.10}$$

$$x = 1 : \tilde{v}_r = g_r(t, \tilde{v}_{m+1}, \cdots, \tilde{v}_n) + h_r(t) \quad (r = 1, \cdots, m), \tag{1.11}$$

where

$$\tilde{v}_i = l_i(\varphi(x))u \quad (i = 1, \cdots, n) \tag{1.12}$$

and without loss of generality, we assume that

$$g_i(t, 0, \cdots, 0) \equiv 0 \quad (i = 1, \cdots, n). \tag{1.13}$$

Moreover, the conditions of C^1 compatibility are supposed to be satisfied at the points (0,0) and (0,1) respectively.

The mixed initial-boundary value problem (1.1) and (1.9)–(1.11) (Problem I) admits a unique local C^1 solution $u = u(t, x)$ on $R(T)$ for $T > 0$ suitably small (see [1, 2]), however, since (1.10)–(1.11) are not of an invariant form in the course of the successive extension of local C^1 solution, this kind of boundary conditions is not convenient for the study of semi-global (or global) C^1 solution. In order to get the semi-global C^1 solution, instead of (1.10)–(1.11) we consider the following boundary conditions:

$$x = 0 : v_s = G_s(t, v_1, \cdots, v_m) + H_s(t) \qquad (s = m + 1, \cdots, n), \tag{1.14}$$

$$x = 1 : v_r = G_r(t, v_{m+1}, \cdots, v_n) + H_r(t) \qquad (r = 1, \cdots, m), \tag{1.15}$$

where

$$v_i = l_i(u)u \qquad (i = 1, \cdots, n) \tag{1.16}$$

and without loss of generality, we assume that

$$G_i(t, 0, \cdots, 0) \equiv 0 \qquad (i = 1, \cdots, n). \tag{1.17}$$

Obviously, the boundary conditions (1.14)–(1.15) are invariant under the successive extension of local C^1 solution, then the corresponding mixed initial-boundary value problem (1.1), (1.9) and (1.14)–(1.15) (Problem II) has an advantage for the study of semi-global (or global) C^1 solution. Of course, we suppose that the conditions of C^1 compatibility are still satisfied at the points (0,0) and (0,1) respectively for Problem II.

We first prove in §2 that when u is suitably small, Problem I is equivalent to problem II; then in §3 the existence and uniqueness of local C^1 solution to Problem II follows from the well-known result on the existence and uniqueness of local C^1 solution to Problem I; finally, by means of a uniform a priori estimate on the C^1 norm of the solution to Problem II, we get the existence and uniqueness of semi-global C^1 solution to both Problem I and Problem II, provided that the C^1 norm of φ and H (resp. h) is small enough.

§1. Equivalence of Problem I and Problem II

In order to prove the equivalence of Problem I and Problem II, it suffices to show that the boundary conditions (1.10)-(1.11) can be equivalently replaced by the boundary conditions (1.14)-(1.15), provided that u is suitably small.

Theorem 2.1. *Suppose that l_{ij} (resp. r_{ij}), g_i, h_i, G_i and H_i $(i, j = 1, \cdots, n)$ are all C^1 functions with respect to their arguments. When*

$$|u| \le \varepsilon_0, \tag{2.1}$$

where $\varepsilon_0 > 0$ is a suitably small number, the boundary conditions (1.10)–(1.11) *can be equivalently replaced by the boundary conditions* (1.14)–(1.15), *then Problem* I *is equivalent to Problem* II.

Proof. Let

$$L(u) = (l_{ij}(u)) \tag{2.2}$$

be the matrix composed by the left eigenvectors. By (1.6), for the matrix composed by the right eigenvectors

$$R(u) = (r_{ij}(u)), \tag{2.3}$$

we have

$$R(u) = L^{-1}(u). \tag{2.4}$$

By (1.16) we have

$$v = L(u)u, \tag{2.5}$$

$$u = R(u)v. \tag{2.6}$$

Similarly, by (1.12) we have

$$\tilde{v} = L(\varphi(x))u, \tag{2.7}$$

$$u = R(\varphi(x))\tilde{v}. \tag{2.8}$$

We now prove that the boundary condition (1.11) on $x = 1$ can be equivalently replaced by the boundary condition (1.15). Similarly, we can prove that the boundary condition (1.10) on $x = 1$ can be equivalently replaced by the boundary condition (1.14).

Let $x=1$. By (2.7)–(2.8) we have

$$\tilde{v}=L(\varphi(1))u, \tag{2.9}$$

$$u=R(\varphi(1))\tilde{v}, \tag{2.10}$$

namely,

$$\tilde{v}_i=\sum_{j=1}^{n} l_{ij}(\varphi(1))u_j \quad (i=1,\cdots,n), \tag{2.9$'$}$$

$$u_i=\sum_{j=1}^{n} r_{ij}(\varphi(1))\tilde{v}_j \quad (i=1,\cdots,n). \tag{2.10$'$}$$

Then, it follows from (2.5) that

$$v=L(R(\varphi(1))\tilde{v})R(\varphi(1))\tilde{v} \tag{2.11}$$

or

$$v_i=\sum_{k=1}^{n} l_{ik}(u)u_k=\sum_{k,h=1}^{n} l_{ik}(R(\varphi(1)\tilde{v})r_{kh}(\varphi(1))\tilde{v}_h. \tag{2.11$'$}$$

Hence, noting (2.10), we get

$$\frac{\partial v_i}{\partial \tilde{v}_j}=\sum_{k=1}^{n} l_{ik}(u)r_{kj}(\varphi(1))+\sum_{k,l=1}^{n}\frac{\partial l_{ik}(u)}{\partial u_l}r_{lj}(\varphi(1))u_k. \tag{2.12}$$

Thus, noting (1.6), when $u=0$ (then $\varphi(1)=0$), $\dfrac{\partial v_i}{\partial \tilde{v}_j}=\delta_{ij}$, then, under the hypothesis (2.1), the inverse of (2.11) can be obtained as

$$\tilde{v}=B(v) \tag{2.13}$$

or

$$\tilde{v}_i=b_i(v_1,\cdots,v_n) \quad (i=1.\cdots,n). \tag{2.13$'$}$$

Suppose that (1.15) holds on $x=1$. Noting (2.11)$'$, on $x=1$ we have

$$\begin{aligned}\tilde{v}_r&=b_r(v_1,\cdots,v_n)\\&=b_r(G_1(t,v_{m+1},\cdots,v_n)+H_1(t),\cdots,G_m(t,v_{m+1},\cdots,v_n)+H_m(t),v_{m+1},\cdots,v_n)\\&=\tilde{b}_r(t,v_{m+1},\cdots,v_n)\\&=\tilde{b}_r\Big(t,\sum_{k,h=1}^{n} l_{m+1,k}(R(\varphi(1)\tilde{v})r_{kh}(\varphi(1))\tilde{v}_k,\cdots,\sum_{k,h=1}^{n} l_{nk}(R(\varphi(1))\tilde{v})r_{kh}(\varphi(1))\tilde{v}_k\Big)\\&\qquad (r=1,\cdots,m).\end{aligned} \tag{2.14}$$

Hence, noting (2.12), for $r,\bar{r}=1,\cdots,m$ we get

$$\frac{\partial \tilde{b}_r}{\partial \tilde{v}_{\bar{r}}}=\sum_{s=m+1}^{n}\frac{\partial \tilde{b}_r}{\partial v_s}(t,v_{m+1},\cdots,v_n)\Big[\sum_{k=1}^{n} l_{sk}(u)r_{k\bar{r}}(\varphi(1))+\sum_{k,l=1}^{n}\frac{\partial l_{sk}(u)}{\partial u_l}r_{l\bar{r}}(\varphi(1))u_k\Big]. \tag{2.15}$$

Noting (1.6), we have

$$\sum_{k=1}^{n} l_{sk}(u)r_{k\bar{r}}(\varphi(1))=\sum_{k=1}^{n}\Big(l_{sk}(u)-l_{sk}(\varphi(1))\Big)r_{k\bar{r}}(\varphi(1)) \quad (\bar{r}=1,\cdots,m;s=m+1,\cdots,n), \tag{2.16}$$

then it follows from (2.15) that when $u = 0$ (then $\varphi(1) = 0$),

$$\frac{\partial \tilde{b}_r}{\partial \tilde{v}_{\bar{r}}} = 0 \quad (r, \bar{r} = 1, \cdots, m). \tag{2.17}$$

Hence, it is easy to see that under the hypothesis (2.1), on $x = 1$ (2.14) then the boundary condition (1.15) on $x = 1$ can be rewritten in a form of (1.11).

Similarly, the boundary condition (1.11) on $x = 1$ can be rewritten in a from of (1.15). This finishes the proof.

Theorem 2.2. *Under the hypotheses of Theorem* 2.1*, the functions* $h(t) = (h_1(t), \cdots, h_n(t))$ *and* $H(t) = (H_1(t), \cdots, H_n(t))$ *in two equivalent boundary conditions* (1.10)–(1.11) *and* (1.14)–(1.15) *satisfy the following relationships: for any given* $l_{ij}(u)$ *and* $g_i(t, \cdot)$ (*resp.* $G(t, \cdot)$) $(i, j = 1, \cdots, n)$*, there exist two positive constants* C_1 *and* C_2 *depending only on* ε_0*, such that on the domain under consideration we have*

$$C_1 ||h||_0 \leq ||H||_0 \leq C_2 ||h||_0, \tag{2.18}$$

$$||h||_1 \to 0 \Leftrightarrow ||H||_1 \to 0, \tag{2.19}$$

where $|| \ ||_0$ *and* $|| \ ||_1$ *stand for the* C^0 *norm and the* C^1 *norm respectively:*

$$||h||_0 = \sup_{\substack{i=1,\cdots,n \\ t}} |h_i(t)|, \quad ||h||_1 = \sup_{\substack{i=1,\cdots,n \\ t}} (|h_i(t)| + |h_i'(t)|), \quad etc. \tag{2.20}$$

Proof. We still consider the situation on $x = 1$. By Theorem 2.1, under the hypothesis (2.1), (1.11) is equivalent to (1.15).

We take

$$\tilde{v}_s = 0 \quad (s = m+1, \cdots, n) \tag{2.21}$$

on $x = 1$, then, noting (1.13), it follows from (1.11) that

$$\tilde{v}_r = h_r(t) \quad (r = 1, \cdots, m). \tag{2.22}$$

By (2.10), we have

$$u = \sum_{p=1}^{m} r_p(\varphi(1)) h_p(t), \tag{2.23}$$

namely,

$$u_k = \sum_{p=1}^{m} r_{kp}(\varphi(1)) h_p(t) \quad (k = 1, \cdots, n), \tag{2.23$'$}$$

then it follows from (2.11)$'$ that

$$v_i = \sum_{k=1}^{n} l_{ik}(u) u_k = \sum_{k=1}^{n} l_{ik}\Big(\sum_{p=1}^{m} r_p(\varphi(1)) h_p(t)\Big) \sum_{q=1}^{m} r_{kq}(\varphi(1)) h_q(t). \tag{2.24}$$

Hence, by (1.15) we get

$$\begin{aligned} H_r(t) = &\sum_{k=1}^{n} l_{rk}(u) \sum_{q=1}^{m} r_{kq}(\varphi(1)) h_q(t) - G_r\Big(t, \sum_{k=1}^{n} l_{m+1,k}(u) \sum_{q=1}^{m} r_{kq}(\varphi(1)) h_q(t), \cdots, \\ &\sum_{k=1}^{n} l_{nk}(u) \sum_{q=1}^{m} r_{kq}(\varphi(1)) h_q(t)\Big) \quad (r = 1, \cdots, m), \end{aligned} \tag{2.25}$$

then

$$
\begin{aligned}
H_r'(t)=&\sum_{k,h=1}^{n}\frac{\partial l_{rk}(u)}{\partial u_h}\sum_{p=1}^{m}r_{hp}(\varphi(1))h_p'(t)u_k+\sum_{k=1}^{n}l_{rk}(u)\sum_{q=1}^{m}r_{kq}(\varphi(1))h_q'(t)\\
&-\frac{\partial G_r}{\partial t}-\sum_{s=m+1}^{n}\frac{\partial G_r}{\partial v_s}\Big\{\sum_{k,h=1}^{n}\frac{\partial l_{sk}(u)}{\partial u_h}\sum_{p=1}^{m}r_{hp}(\varphi(1))h_p'(t)u_k\\
&+\sum_{k=1}^{n}l_{sk}(u)\sum_{q=1}^{m}r_{kq}(\varphi(1))h_q'(t)\Big\}\qquad(r=1,\cdots,m). \qquad (2.26)
\end{aligned}
$$

Noting (1.17), we have

$$\frac{\partial G_i}{\partial t}(t,0,\cdots,0)\equiv 0\qquad(i=1,\cdots,n). \qquad (2.26)$$

Hence, under the hypothesis (2.1), it follows immediately from (2.25) and (2.26) that

$$||H||_0\le C_2||h||_0, \qquad (2.28)$$

$$||H'||_0\le B(||h||_1), \qquad (2.29)$$

where $B=B(y)$ is an increasing continuous function with $B(0)=0$, then we get

$$||h||_1\to 0\Rightarrow ||H||_1\to 0. \qquad (2.30)$$

Similarly, taking

$$v_s=0\qquad(s=m+1,\cdots,n) \qquad (2.31)$$

on $x=1$ and noting (1.17), it following from (1.15) that

$$v_r=H_r(t)\qquad(r=1,\cdots,m). \qquad (2.32)$$

By (2.5) and (2.7) and noting (2.6), on $x=1$ we have

$$v_i-\tilde v_i=\sum_{k=1}^{n}\Big(l_{ik}(u)-l_{ik}(\varphi(1))\Big)u_k, \qquad (2.33)$$

in which

$$u=\sum_{p=1}^{m}r_p(u)H_p(t), \qquad (2.34)$$

namely,

$$u_k=\sum_{p=1}^{m}r_{kp}(u)H_p(t)\quad(k=1,\cdots,n). \qquad (2.34)'$$

Thus, by (1.11) we get

$$
\begin{aligned}
h_r(t)=&H_r(t)-\sum_{k=1}^{n}\Big(l_{rk}(u)-l_{rk}(\varphi(1))\Big)u_k-g_r\Big(t,-\sum_{k=1}^{n}\Big(l_{m+1,k}(u)-l_{m+1,k}(\varphi(1))\Big)u_k,\\
&\cdots,-\sum_{k=1}^{n}\Big(l_{nk}(u)-l_{nk}(\varphi(1))\Big)u_k\Big)\quad(r=1,\cdots,m), \qquad (2.35)
\end{aligned}
$$

then

$$h'_r(t) = H'_r(t) - \sum_{k,h=1}^{n} \frac{\partial l_{rk}(u)}{\partial u_h}\frac{\partial u_h}{\partial t}u_k - \sum_{k=1}^{n}\Big(l_{rk}(u) - l_{rk}(\varphi(1))\Big)\frac{\partial u_k}{\partial t} - \frac{\partial g_r}{\partial t}$$
$$+ \sum_{s=m+1}^{n}\frac{\partial g_r}{\partial \tilde{v}_s}\Big(\sum_{k,h=1}^{n}\frac{\partial l_{sk}(u)}{\partial u_h}\frac{\partial u_h}{\partial t}u_k + \sum_{k=1}^{n}\Big(l_{sk}(u) - l_{sk}(\varphi(1))\Big)\frac{\partial u_k}{\partial t}\Big)$$
$$(r = 1, \cdots, m). \tag{2.36}$$

By (2.5)–(2.6) and noting (2.31)–(2.32), under the hypothesis (2.1), it is easy to see that there exist two positive constants C_3 and C_4 depending only on ε_0, such that on $x = 1$ we have

$$C_3||H||_0 \le ||u||_0 \le C_4||H||_0. \tag{2.37}$$

Moreover, differentiating $(2.34)'$ with respect to t, on $x = 1$ we get

$$\frac{\partial u_k}{\partial t} = \sum_{p=1}^{m} r_{kp}(u)H'_p(t) + \sum_{p=1}^{m}\sum_{h=1}^{n}\frac{\partial r_{kp}(u)}{\partial u_h}\frac{\partial u_h}{\partial t}H_p(t), \tag{2.38}$$

then, noting (2.37), under the hypothesis (2.1) it is easy to see that on $x = 1$ we have

$$\Big\|\frac{\partial u}{\partial t}\Big\|_0 \le C_5||H'||_0, \tag{2.39}$$

where C_5 is a positive constant depending only on ε_0.

Noting (1.13), we have

$$\frac{\partial g_i}{\partial t}(t, 0, \cdots, 0) \equiv 0 \qquad (i = 1, \cdots, n), \tag{2.40}$$

then, under the hypothesis (2.1), it follows from (2.35) and (2.36) that

$$C_1||h||_0 \le ||H||_0, \tag{2.41}$$
$$||h'||_0 \le b(||H||_1), \tag{2.42}$$

where $b = b(y)$ is an increasing continuous function with $b(0) = 0$, hence we get

$$||H||_1 \to 0 \Rightarrow ||h||_1 \to 0. \tag{2.43}$$

The proof of Theorem 2.2 is complete.

§3. Local C^1 Solution to the Mixed Initial-Boundary Value Problem

By means of the theory on the local C^1 solution to the mixed initial-boundary value problem in [1, 2], we can obtain the following

Theorem 3.1. *Suppose that* $l_{ij}(w)$, $\lambda_i(u)$, $f_i(u)$, $g_i(t, .)$, $h_i(t)$ $(i, j = i, \cdots, n)$ *and* $\varphi(x)$ *are all* C^1 *functions with respect to their arguments. Suppose furthermore that* (1.2), (1.5), (1.8) *and* (1.13) *hold. Suppose finally that the corresponding conditions of* C^1 *compatibility are satisfied at points* $(0, 0)$ *and* $(0, 1)$ *respectively. Then, for any given* $A(u)$, $F(u)$ *and* $g_i(t, \cdot)$ $(i = 1, \cdots, n)$, *there exists a positive constant* $\delta = \delta(||\varphi||_1, ||h||_1)$ *depending only on the* C^1 *norms* $||\varphi||_1$ *and* $||h||_1$, *such that Problem* I *admits a unique* C^1 *solution* $u = u(t, x)$ *on the domain*

$$R(\delta) = \Big\{(t, x)\Big|0 \le t \le \delta, \quad 0 \le x \le 1\Big\}. \tag{3.1}$$

Moreover, when $\|\varphi\|_1$ *and* $\|h\|_1$ *are suitably small, we have*

$$|u(t, x)| \le \varepsilon_0, \quad \forall (t, x) \in R(\delta), \tag{3.2}$$

where ε_0 *is the small positive contant given in Theorem* 2.1.

Thus, by means of Theorem 2.1 and Theorem 2.2 we have

Theorem 3.2. *Suppose that* $l_{ij}(u)$, $\lambda_i(u)$, $f_i(u)$, $G_i(t,.)$, $H_i(t)$ $(i,j = 1,\cdots,n)$ *and* $\varphi(x)$ *are all* C^1 *functions with respect to their arguments. Suppose furthermore that* (1.2), (1.5), (1.8) *and* (1.17) *hold. Suppose finally the corresponding conditions of* C^1 *compatibility are satisfied at points* $(0,0)$ *and* $(0,1)$ *respectively. Then for any given* $A(u)$, $F(u)$ *and* $G_i(t,.)$ $(i = 1,\cdots,n)$, *if the* C^1 *norms* $\|\varphi\|_1$ *and* $\|H\|_1$ *are suitably small, Problem* II *admits a unique* C^1 *solution* $u = u(t,x)$ *on the domain* (3.1), *where* δ *is a positive constant depending only on* $\|\varphi\|_1$ *and* $\|H\|_1$: $\delta = \delta(\|\varphi\|_1,\ \|H\|_1)$.

Proof. We only consider the solution $u = u(t,x)$ satisfying $|u| \le \varepsilon_0$. By Theorem 2.2, when $\|\varphi\|_1$ and $\|H\|_1$ are suitably small, $\|h\|_1$ is also small, then by Theorem 3.1, the corresponding Problem I admits a unique C^1 solution $u = u(t,x)$ and (3.2) holds. Hence, by Theorem 2.1, Problem II is equivalent to Problem I, then $u = u(t,x)$ is also the unique C^1 solution to Problem II on the domain (3.1). Moreover, noting (2.28)–(2.29) and (2.41)–(2.42), from $\delta = \delta(\|\varphi\|_1,\ \|h\|_1)$ we get $\delta = \delta(\|\varphi\|_1,\ \|H\|_1)$.

§4. Semi-Global C^1 Solution to the Mixed Initial-Boundary Value Problem

In this section we will prove the following two main theorems.

Theorem 4.1. *Under the hypotheses of Theorem* 3.1, *for any given* $T_0 > 0$, *the mixed initial-boundary value problem* (1.1) *and* (1.9)–(1.11) (*Problem* I) *admits a unique* C^1 *solution* $u = u(t,x)$ *on the domain*

$$R(T_0) = \{(t,x)\big|0 \le t \le T_0, 0 \le x \le 1\}, \tag{4.1}$$

provided that $||\varphi||_{C^1[0,1]}$ *and* $||h||_{C^1[0,T_0]}$ *are suitably small* (*depending on* T_0).

Theorem 4.2. *Under the hypotheses of Theorem* 3.2, *for any given* $T_0 > 0$, *the mixed initial-boundary value problem* (1.1), (1.9) *and* (1.14)–(1.15) (*Problem* II) *admits a unique* C^1 *solution* $u = u(t,x)$ *on the domain* (4.1), *provided that* $\|\varphi\|_{C^1[0,1]}$ *and* $\|H\|_{C^1[0,T_0]}$ *are suitably small* (*depending on* T_0).

We refer to these solutions as semi-global C^1 solutions.

We first prove Theorem 4.2. By Theorem 3.2, for this purpose it is only necessary to prove the following

Lemma 4.1. *Under the hypotheses of Theorem* 3.2, *for any given* $T_0 > 0$, *if* $\|\varphi\|_{C^1[0,1]}$ *and* $\|H\|_{C^1[0,T_0]}$ *are suitably small* (*depending on* T_0), *then, for any* C^1 *solution* $u = u(t,x)$ *to Problem* II *on the domain*

$$R(T) = \Big\{(t,x)\Big|0 \le t \le T, 0 \le x \le 1\Big\} \tag{4.2}$$

with $0 < T \le T_0$, *we have the following uniform a priori estimate*:

$$||u(t,.)||_1 \triangleq ||u(t,.)||_0 + ||u_x(t,.)||_0 \le C(T_0), \quad \forall 0 \le t \le T, \tag{4.3}$$

where $C(T_0)$ *is a sufficiently small positive constant independent of* T *but possibly depending on* T_0.

Proof. Let $v = (v_1,\cdots,v_n)$ be defined by (1.16) and

$$w_i = l_i(u)u_x \quad (i = 1,\cdots,n). \tag{4.4}$$

By (1.6), we have

$$u = \sum_{i=1}^{n} v_i r_i(u), \tag{4.5}$$

$$u_x = \sum_{i=1}^{n} w_i r_i(u). \tag{4.6}$$

Noting (1.7), it suffices to estimate $\|v(t,\cdot)\|_0$ and $\|w(t,\cdot)\|_0$.

It is easy to see that (cf. [8–10])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k + \sum_{j=1}^{n} \tilde{\beta}_{ij}(u) f_j(u), \tag{4.7}$$

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n} \tilde{\gamma}_{ij}(u) w_j, \tag{4.8}$$

where

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x} \tag{4.9}$$

denotes the directional derivative along the i-th characteristic,

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u), \tag{4.10}$$

$$\gamma_{ijk}(u) = \frac{1}{2}\Big\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik} + (j|k)\Big\}, \tag{4.11}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms , and

$$\tilde{\beta}_{ij}(u) = l_{ij}(u) - \sum_{h,k=1}^{n} (l_i(u) \nabla r_k(u) r_k(u))(l_h(u) u) l_{kj}(u), \tag{4.12}$$

$$\tilde{\gamma}_{ij}(u) = l_i(u) \nabla F(u) r_j(u) - \sum_{k=1}^{n} \Big(l_i(u) \nabla r_j(u) r_k(u)\Big)(l_k(u) F(u)). \tag{4.13}$$

For the time being we assume that on the domain $R(T)$

$$|v(t,x)| \le \frac{\eta_0}{n}, \quad |w(t,x)| \le \eta_1, \tag{4.14}$$

where η_0 and η_1 are suitably small positive constants. Then, by (4.5) and noting (1.7), we have

$$|u(t,x)| \le \eta_0, \qquad \forall (t,x) \in R(T). \tag{4.15}$$

At the end of the proof, we will show the validity of hypothesis (4.14).

Let

$$T_1 = \max_{\substack{i=1,\cdots,n \\ |u| \le \eta_0}} \frac{1}{|\lambda_i(u)|} > 0, \tag{4.16}$$

$$v(\tau) = \sup_{0 \le t \le \tau} ||v(t,.)||_0, \quad w(\tau) = \sup_{0 \le t \le \tau} ||w(t,.)||_0. \tag{4.17}$$

For any given point $(t,x) \in R(T_1)$, we draw down the r-th characteristic $(r = 1, \cdots, m)$ passing through (t,x). Noting (1.8) and (4.16), there are only two possibilities:

(a) This r-th characteristic intersects the x-axis at a point $(0,\alpha)$. Integrating the r-th equation in (4.7) along this characteristic from 0 to t, and noting (1.2), (1.7) and (4.14)–

(4.15), we get

$$|v_r(t,x)| \le ||v(0,.)||_0 + C_1 \int_0^t v(\tau)d\tau, \tag{4.18}$$

here and hereafter, C_i $(i = 1, 2, \cdots)$ denote positive constants.

(b) This r-characteristic intersects $x = 1$ at a point $(t_r, 1)$, and all s-th characteristics passing through $(t_r, 1)$ intersect the x-axis at point $(0, \beta_s)$ $(s = m+1, \cdots, n)$ respectively. Similarly to (4.18), we have

$$|v_r(t,x)| \le |v_r(t_r,1)| + C_1 \int_{t_r}^t v(\tau)d\tau. \tag{4.19}$$

Moreover, by means of the boundary conditions (1.15) it is easy to get that

$$|v_r(t_r,1)| \le K_1 \max_{m+1\le s\le n} |v_s(t_r,1)| + ||H||_0, \tag{4.20}$$

henceforth K_i $(i = 1, 2, \cdots)$ denote positive constants depending only on T_0, and, without loss of generality, we may suppose $K_1 \ge 1$. Similarly to (4.18), integrating the s-th equation in (4.7) along the s-th characteristic gives

$$|v_s(t_r,1)| \le ||v(0,.)||_0 + C_2 \int_0^{t_r} v(\tau)d\tau \quad (s = m+1, \cdots, n). \tag{4.21}$$

The combination (4.19)–(4.21) leads to

$$|v_r(t,x)| \le K_1||v(0,.)||_0 + ||H||_0 + C_3 \int_0^t v(\tau)d\tau. \tag{4.22}$$

Thus, it follows from (4.18) and (4.22) that

$$|v_r(t,x)| \le K_1||v(0,.)||_0 + ||H||_0 + C_4 \int_0^t v(\tau)d\tau \quad (r = 1, \cdots, m). \tag{4.23}$$

Similar estimates can be obtained for $v_s(t,x)$ $(s = m+1, \cdots, n)$. Hence, we have

$$v(t) \le K_1||v(0,.)||_0 + ||H||_0 + C_5 \int_0^t v(\tau)d\tau, \qquad \forall t \in [0, T_1], \tag{4.24}$$

then, using Gronwall's inequality we get

$$v(t) \le K_2 \max\Big\{||v(0,.)||_0, ||H||_0\Big\}, \qquad \forall t \in [0, T_1], \tag{4.25}$$

in which we may assume that $K_2 \ge 1$.

Taking $v(T_1, x)$ as initial data on $t = T_1$ and repeating the previous procedure, we obtain

$$v(t) \le K_2 \max\Big\{||v(T_1,.)||_0, ||H||_0\Big\} \le K_2^2 \max\Big\{||v(0,.)||_0, ||H||_0\Big\}, \quad \forall t \in [T_1, 2T_1]. \tag{4.26}$$

Repeating this procedure at most $N \le [\frac{T_0}{T_1}] + 1$ times, we get

$$v(t) \le K_2^N \max\Big\{||v(0,.)||_0, ||H||_0\Big\}, \quad \forall t \in [0, T]. \tag{4.27}$$

Noting (1.16) and (1.9), we finally get

$$v(t) \le K_3 \max\Big\{||\varphi||_{C^0[0,1]}, ||H||_{C^0[0,T_0]}\Big\}, \quad \forall t \in [0, T]. \tag{4.28}$$

Then, by (4.5) and noting (1.7), we have

$$|u(t,x)| \le K_4 \max\Big\{||\varphi||_{C^0[0,1]}, ||H||_{C^0[0,T_0]}\Big\}, \quad \forall (t,x) \in R(T). \tag{4.29}$$

We now estimate $w(t)$.

As before, for any given point $(t,x) \in R(T_1)$, there are still two possibilities for the r-th characteristic $(r = 1, \cdots, m)$ passing through (t,x).

In case (a), integrating the r-th equation in (4.8) along this r-th characteristic yields

$$|w_r(t,x)| \le ||w(0,.)||_0 + C_6 \int_0^t w(\tau)d\tau. \tag{4.30}$$

In case (b), similarly to (4.19), we have

$$|w_r(t,x)| \le |w_r(t_r,1)| + C_6 \int_{t_r}^t w(\tau)d\tau. \tag{4.31}$$

In order to estimate $|w_r(t_r,1)|$, we seek the boundary conditions satisfied by w on $x = 1$. Differentiating (1.15) with respect to t, we get

$$x = 1 : \qquad \frac{\partial v_r}{\partial t} = \frac{\partial G_r}{\partial t} + \sum_{s=m+1}^{n} \frac{\partial G_r}{\partial v_s}\frac{\partial v_s}{\partial t} + H_r'(t) \quad (r = 1, \cdots, m), \tag{4.32}$$

where $G_r = G_r(t, v_{m+1}, \cdots, v_n)$.

By (1.16) and using (1.1), (1.3) and (4.6), we have

$$\begin{aligned}
\frac{\partial v_i}{\partial t} &= l_i(u)\frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial l_i(u)}{\partial u_j}\frac{\partial u_j}{\partial t} u \\
&= l_i(u)\Big(F(u) - A(u)\frac{\partial u}{\partial x}\Big) + \sum_{j=1}^{n} \frac{\partial l_i(u)}{\partial u_j}\Big(f_j(u) - \sum_{k=1}^{n} a_{jk}(u)\frac{\partial u_k}{\partial x}\Big)u \\
&= -\lambda_i(u)w_i + l_i(u)F(u) + \sum_{j=1}^{n} \frac{\partial l_i(u)}{\partial u_j}\Big(f_j(u) - \sum_{k,h=1}^{n} a_{jk}(u)r_{kh}(u)w_h\Big)u \\
&= -\lambda_i(u)w_i + \sum_{h=1}^{n} b_{ih}(u)w_h + \bar{b}_i(u) \quad (i = 1, \cdots, n),
\end{aligned} \tag{4.33}$$

where $b_{ih}, \bar{b}_i$ $(i, h = 1, \cdots, n)$ are continuous functions of u and, noting (1.2), when $|u| \le \eta_0$,

$$|b_{ih}(u)|, \quad |\bar{b}_i(u)| \le C_7|u| \quad (i, h = 1, \cdots, n). \tag{4.34}$$

Hence, for $\eta_0 > 0$ small enough, by (4.33) and noting (1.8) and (2.27), (4.32) can by rewritten as

$$x = 1 : w_r = \sum_{s=m+1}^{n} C_{rs}(t,u)w_s + \bar{C}_r(t,u) + \sum_{\bar{r}=1}^{m} \bar{\bar{C}}_{r\bar{r}}(t,u)H_{\bar{r}}'(t) \quad (r = 1, \cdots, m), \tag{4.35}$$

where $C_{rs}, \bar{C}_r$ and $\bar{\bar{C}}_{r\bar{r}}$ $(r, \bar{r} = 1, \cdots, m; s = m+1, \cdots, n)$ are continuous functions of t and u, moreover, as $|u| \to 0$,

$$d(u) = \sup_{\substack{0 \le t \le T_0 \\ r=1,\cdots,m}} |\bar{C}_r(t,u)| \to 0. \tag{4.36}$$

By (4.35), we have

$$|w_r(t_r,1)| \le K_5 \max_{s=m+1,\cdots,n} |w_s(t_r,1)| + K_6(d(u) + ||H'||_0) \ \ (r = 1, \cdots, m). \tag{4.37}$$

Integrating the s-th equation in (4.8) along the corresponding s-th characteristic gives

$$|w_s(t_r,1)| \le ||w(0,.)||_0 + C_8 \int_0^{t_r} w(\tau)d\tau \quad (s = m+1, \cdots, n). \tag{4.38}$$

Combining (4.31) and (4.37)–(4.38) yields

$$|w_r(t,x)| \le K_5||w(0,.)||_0 + K_6\Big(d(u) + ||H'||_0\Big) + C_9\int_0^t w(\tau)d\tau \quad (r = 1,\cdots,m). \quad (4.39)$$

Similar estimates can be obtained for $w_s(t,x)$ $(s = m+1,\cdots,n)$. Hence we have

$$w(t) \le K_5||w(0,.)||_0 + K_6\Big(d(u) + ||H'||_0\Big) + C_{10}\int_0^t w(\tau)d\tau, \quad \forall t \in [0,T_1], \quad (4.40)$$

then, using Gronwall's inequality we get

$$w(t) \le K_7 \max\Big\{||w(0,.)||_0, d(u) + ||H'||_0\Big\}, \quad \forall t \in [0,T_1], \quad (4.41)$$

in which we may assume that $K_7 \ge 1$.

Repeating the previous procedure, similarly to (4.27), we have

$$w(t) \le K_7^N \max\Big\{||w(0,.)||_0, d(u) + ||H'||_0\Big\}, \quad \forall t \in [0,T], \quad (4.42)$$

then, noting (4.4) and using (1.9), we get

$$w(t) \le K_8 \max\Big\{||\varphi'||_{C^0[0.1]}, \quad d(u) + ||H'||_{C^1[0,T_0]}\Big\}, \quad \forall t \in [0,T]. \quad (4.43)$$

Noting (4.36) and (4.29), when $||\varphi||_{C^1[0,1]}$ and $||H||_{C^1[0,T_0]}$ are small enough, for any T with $0 < T \le T_0$, $v(t)$ and $w(t)$ are sufficiently small on $0 \le t \le T$. This implies not only (4.3) but also the validity of hypothesis (4.14). The proof is finished.

We now proof Theorem 4.1.

By Theorem 2.1, under the hypothesis (2.1), Problem I is equivalent to Problem II. Consider the C^1 solution $u = u(t,x)$ satisfying $|u| \le \varepsilon_0$ on the domain under consideration. By Theorem 2.2, when $||\varphi||_{C^1[0,1]}$ and $||h||_{C^1[0,T_0]}$ are small, $||\varphi||_{C^1[0,1]}$ and $||H||_{C^1[0,T_0]}$ are also small. Then by Theorem 4.2, the corresponding Problem II admits a unique semi-global C^1 solution $u = u(t,x)$ on the domain $R(T_0)$, moreover, the C^1 norm $\|u(t,.)\|_1$ is small enough on $0 \le t \le T_0$, then (2.1) holds. Thus, $u = u(t,x)$ is the semi-global C^1 solution to Problem I on the domain $R(T_0)$. This proves Theorem 4.1.

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