Chin. Ann. of Math.
22B: 3(2001),337-342.

ON A KIND OF GENERALIZED QUASI-VARIATIONAL INEQUALITIES AND FAN's MINIMAX INEQUALITY WITHOUT CONVEXITY***

SHEN Zifei* WU Xian**

Abstract

This paper gives a Fan's type minimax theorem, a nearest point theorem and two existence theorems of solutions for a kind of generalized quasi-variational inequalities in H-spaces without any linear structure.

Keywords Locally convex H-space, H-convex set, Acyclic set, Almost upper semi-continuous multivalued mapping

2000 MR Subject Classification 47H04, 47H10, 49J35, 52A99, 58E35

Chinese Library Classification O177.95 **Document Code** A

Article ID 0252-9599(2001)03-0337-06

§1. Introduction

In 1972, Ky Fan researched the minimax inequality

$$\min_{y\in C}\sup_{x\in C} f(x,y) \le \sup_{x\in C} f(x,x)$$

and estabished the following minimax theorem:

Theorem 1.1.[6] *Let E be a Hausdorff topological vector space, X a nonempty compact convex subset of E and $\phi : X\times X \longrightarrow R$ a function. If the following conditions are fulfilled:*

(i) *for each $y\in X$, $\phi(\cdot, y)$ is lower semicontinuous,*

(ii) *for each $x\in X$, $\phi(x,\cdot)$ is quasi-concave,*

then there exists a point $x_0 \in X$ such that

$$\sup_{y\in X}\phi(x_0,y) = \min_{x\in X}\sup_{y\in X}\phi(x,y) \le \sup_{x\in X}\phi(x,x).$$

Since then, this interesting result has attracted much attention (see e.g. [2–5,12]). In a recent paper[3], Chu gave the following result on Fan's minimax inequality:

Manuscript received February 18, 2000. Revised August 17, 2000.

*Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.

**Department of Mathematics, Yunnan Normal University, Kunming 650092, China.

***Project supported by the Foundation of the Technology Commission of Zhejiang Province, China. (No. 19990500), the National Natural Science Foundation of China, and the Foundation of the Science and Technology Commission of Yunnan Province, China.

Theorem 1.2.[3] *Let C be a nonempty compact convex subset of a locally convex topological vector space. If $f : C \times C \longrightarrow R$ is a continuous function such that for each fixed $y \in C$ the set $\{x \in C : f(x, y) > t\}$ is acyclic or empty for all $t \in R$, then there exists a point $\bar{y} \in C$ such that*

$$\min_{y\in C} \max_{x\in C} f(x, y) \leq \max_{x\in C} f(x, \bar{y}) \leq \max_{x\in C} f(x, x).$$

The objects of the note are to research the Fan's minimax inequality under different conditions, nearest point problem and existence problem of solutions for a kind of generalized quasi-variational inequalities in H-spaces without any linear structure.

In order to establish our main results, we give some concepts and notations.

To begin with we explain the notion of an H-space introduced by Horvath and Bardaro-Ceppitelli[1,9−11] and related concepts on H-spaces.

Let X be a topological space and $\mathcal{F}(X)$ the family of all nonempty finite subset of X. Let $\{\Gamma_A\}$ be a family of some nonempty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space. Given an H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called

(1) H-convex if $\Gamma_A \subset D$ for all $A \in \mathcal{F}(D)$;

(2) weakly H-convex if $\Gamma_A \cap D$ is nonempty contractible for each $A \in \mathcal{F}(D)$;

(3) H-compact if for each $A \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset D_A of X such that $D \cup A \subset D_A$.

An H-space $(X, \{\Gamma_A\})$ is called

(4) a locally convex H-space if X is a uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure $\mathcal{U}$ such that for each $i \in I$, $V_i(x) = \{y \in X : (y, x) \in V_i\}$ is H-convex for each $x \in X$ (see [15]);

(5) an l.c.-space (see [8]) if X is a uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X : E \cap V_i[x] \neq \emptyset\}$ is H-convex whenever E is H-convex, where $V_i[x] = \{y \in X : (x, y) \in V_i\}$.

Remark 1.1. The concept of an l.c.-space is different from a locally convex H-space. But an l.c.-space $(X, \{\Gamma_A\})$ with $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$ must be a locally convex H-space. Otherwise, a nonempty convex subset X of a locally convex topological vector space must be an l.c.- space with $\Gamma_A = \mathrm{co}\, A$ for all $A \in \mathcal{F}(X)$, and hence $(X, \{\mathrm{co}\, A\})$ must be a locally convex H-space.

Let X be a topological space. We denote by 2^X the family of all subsets of X. If $A \subset X$, we shall denote by $\mathrm{cl}(A)$ the closure of A. A topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic.

Let X, Y be two topological spaces, $f : X \longrightarrow R$ and $S, T : X \longrightarrow 2^Y$ two multivalued mappings.

(6) f is called upper semicontinuous (resp. lower semicontinuous) if for each $r \in R$, the set $\{x \in X : f(x) \geq r\}$ (resp. $\{x \in X : f(x) \leq r\}$) is closed;

(7) if X is an H-space, f is called H-quasiconcave (resp. H-quasiconvex), if for each $r \in R$, the set $\{x \in X : f(x) > r\}$ (resp. $\{x \in X : f(x) < r\}$) is H-convex;

(8) T is called upper semicontinuous if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$, there exists an open neighborhood U of x such that $T(z) \subset V$ for each $z \in U$;

(9) T is called almost upper semicontinuous if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$, there exists an open neighborhood U of x such that $T(z) \subset \mathrm{cl}\, V$ for all $z \in U$;

(10) for each $y \in Y$, we denote $T^{-1}(y) = \{x \in X : y \in T(x)\}$, which is called a lower section of T;

(11) the multivalued mappings $S \cap T$, $\mathrm{cl}\, T : X \longrightarrow 2^Y$ are defined by

$$S \cap T(x) = S(x) \cap T(x), \quad \mathrm{cl}\, T(x) = \mathrm{cl}\,(T(x)), \quad \forall x \in X.$$

§2. Main Results

Now, we establish our main results.

Theorem 2.1. *Let $(X, \{\Gamma_A\})$ be a compact Hausdorff locally convex H-space, $A : X \longrightarrow 2^X$ an almost upper semicontinuous multivalued mapping with nonempty H-convex values and open lower sections and $\phi : X \times X \longrightarrow R \cup \{\pm\infty\}$ a function. If the following conditions are fulfilled:*

(i) *for each $y \in X$, $\phi(x, y)$ is lower semicontinuous in x,*

(ii) *for each $x \in X$, $\phi(x, y)$ is H-quasiconcave in y and* $\mathrm{cl}\, A(x)$ *is acyclic,*

(iii) *for each $x \in X$, $\phi(x, x) \le \gamma (\gamma \in R)$,*

(iv) *the set $W = \{x \in X : \sup_{y \in A(x)} \phi(x, y) > \gamma\}$ is paracompact,*

then there exists a point $\bar{x} \in X$ such that $\bar{x} \in \mathrm{cl}(A(\bar{x}))$ and

$$\phi(\bar{x}, y) \le \gamma$$

for all $y \in A(\bar{x})$.

Proof. Define a multivalued mapping $P : X \longrightarrow 2^X$ by

$$P(x) = \{y \in A(x) : \phi(x, y) > \gamma\}, \qquad \forall x \in X.$$

Then $P(x)$ is H-convex for all $x \in X$ by (ii) and the H-convexity of $A(x)$. Moreover, for each $y \in X$,

$$P^{-1}(y) = \{x \in X : y \in P(x)\} = A^{-1}(y) \bigcap \{x \in X : \phi(x, y) > \gamma\}$$

is open in X since A has open lower sections and $\phi(x, y)$ is lower semicontinuous in x. Consequently, the set

$$W = \Big\{x \in X : \sup_{y \in A(x)} \phi(x, y) > \gamma\Big\} = \{x \in X : P(x) \ne \emptyset\} = \bigcup_{y \in X} P^{-1}(y)$$

is open. By virtue of Theorem 2[8], there exists a continuous mapping $f : W \longrightarrow X$ such that $f(x) \in P(x)$ for all $x \in W$.

Define a multivalued mapping $T : X \longrightarrow 2^X$ by

$$T(x) = \begin{cases} \{f(x)\}, & \text{if } x \in W, \\ \mathrm{cl}\, A(x), & \text{if } x \in X \setminus W. \end{cases}$$

Since $A : X \longrightarrow 2^X$ is almost upper semicontinuous and X is compact, the mapping $\mathrm{cl}\, A : X \longrightarrow 2^X$ is upper semicontinuous. Hence for each open subset V of X, the set

$$\begin{aligned} \{x \in X : T(x) \subset V\} &= \{x \in W : f(x) \in V\} \bigcup \{x \in X \setminus W : \mathrm{cl}\, A(x) \subset V\} \\ &= \{x \in W : f(x) \in V\} \bigcup \{x \in X : \mathrm{cl}\, A(x) \subset V\} \end{aligned}$$

is open. Consequently, $T : X \longrightarrow 2^X$ is an upper semicontinuous multivalued mapping with closed acyclic values. By virtue of Lemma 2.1[15], there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$. By (iii), we know that $\bar{x} \in \mathrm{cl}\, A(\bar{x})$ and $P(\bar{x}) = \emptyset$, i.e.

$$\phi(\bar{x}, y) \le \gamma$$

for all $y \in A(\bar{x})$. This completes the proof.

Theorem 2.2. *Let $(X, \{\Gamma_A\})$ be an Hausdorff locally convex H-space and D a H-compact subset of X. Let $A : X \longrightarrow 2^D$ be an almost upper semicontinuous multivalued mapping with nonempty H-convex values and open lower sections and $\phi : X \times X \longrightarrow R$ a function. If the following conditions are fulfilled:*

(i) *for each $y \in X$, $\phi(x, y)$ is lower semicontinuous in x,*

(ii) *for each $x \in X$, $\phi(x, y)$ is H-quasiconcave in y and $\mathrm{cl}\, A(x)$ is acyclic,*

(iii) *for each $x \in X$, $\phi(x, x) \le \gamma (\gamma \in R)$,*

(iv) *the set $W = \{x \in X : \sup\limits_{y \in A(x)} \phi(x, y) > \gamma\}$ is paracompact,*

then there exists a point $\bar{x} \in \mathrm{cl}\, D$ such that $\bar{x} \in \mathrm{cl}(A(\bar{x}))$ and

$$\phi(\bar{x}, y) \le \gamma$$

for all $y \in A(\bar{x})$.

Proof. Since D is an H-compact subset of X, there exists a compact weakly H-convex subset E of X such that $D \subset E$. Note that $(E, \{E \bigcap \Gamma_A\})$ is a compact Hausdorff locally convex H-space and the set

$$\begin{aligned} W_1 &= \Big\{ x \in E : \sup_{y \in A(x)} \phi(x, y) > \gamma \Big\} \\ &= E \bigcap \Big\{ x \in X : \sup_{y \in A(x)} \phi(x, y) > \gamma \Big\} \\ &= E \bigcap W \end{aligned}$$

is closed in W. Hence W_1 is paracompact since W is paracompact. By Theorem 2.1 there exists a point $\bar{x} \in E \subset X$ such that $\bar{x} \in \mathrm{cl}(A(\bar{x})) \subset \mathrm{cl}\, D$ and

$$\phi(\bar{x}, y) \le \gamma$$

for all $y \in A(\bar{x})$. This completes the proof.

Remark 2.1. Theorem 2.1 and Theorem 2.2 are two new existence theorems for solutions of generalized quasi-variational inequalities.

Theorem 2.3. *Let $(Y, \{\Gamma_A\})$ be a Hausdorff locally convex H-space and X a compact weakly H-convex subset of Y. Let $\phi : X \times Y \longrightarrow R$ be an upper semicontinuous function such that for each $y \in Y$, the set $\{x \in X : \phi(x, y) > t\}$ is acyclic or empty for all $t \in R$. Then*

$$\inf_{y \in X} \sup_{x \in X} \phi(x, y) \le \sup_{x \in X} \phi(x, x).$$

If, in addition, $\phi(x, .)$ is lower semicontinuous on X for each $x \in X$, then there exists a point $\bar{y} \in X$ such that

$$\sup_{x \in X} \phi(x, \bar{y}) \le \sup_{x \in X} \phi(x, x).$$

Proof. Let $M = \sup_{x\in X} \phi(x,x)$. If $M = +\infty$, the conclusion holds, obviously. Now, we assume $M < +\infty$. If

$$\inf_{y\in X} \sup_{x\in X} \phi(x,y) > M,$$

we may take a real number $r \in R$ such that

$$\inf_{y\in X} \sup_{x\in X} \phi(x,y) > r > M.$$

Define a multivalued mapping $T : Y \longrightarrow 2^X$ by

$$T(y) = \{x \in X : \phi(x,y) \geq r\}, \qquad \forall y \in Y.$$

Then $T(y) \neq \emptyset$ for all $y \in X$. By the upper semicontinuity of ϕ, T has closed graph in $Y \times X$ so that T is upper semitinuous because X is compact. For each $y \in X$, since the set $\{x \in X : \phi(x,y) > t\}$ is acyclic or empty for all $t \in R$, $T(y)$ is acyclic by Lemma 1.3[12]. By Lemma 2.1[15] there exists a point $y_0 \in X$ such that $y_0 \in T(y_0)$, i.e.

$$\phi(y_0, y_0) \geq r > M = \sup_{x\in X} \phi(x,x).$$

It is a contradiction. Hence

$$\inf_{y\in X} \sup_{x\in X} \phi(x,y) \leq M = \sup_{x\in X} \phi(x,x).$$

If, in addition, $\phi(x,.)$ is lower semicontinuous on X for each $x \in X$, then so is the function

$$g(y) = \sup_{x\in X} \phi(x,y) \text{ (in } X),$$

and hence there exists a point $\bar{y} \in X$ such that

$$\sup_{x\in X} \phi(x,\bar{y}) = \inf_{y\in X} \sup_{x\in X} \phi(x,y) \leq \sup_{x\in X} \phi(x,x)$$

since X is compact. This completes the proof.

Remark 2.2. Theorem 2.3 improves and extends Theorem 2.7[3] (i.e. Theorem 1.2) to H-spaces.

Theorem 2.4. *Let $(Y, \{\Gamma_A\})$ be a Hausdorff locally convex H-space and X a compact metrizable weakly H-convex subset of Y with the metric d. Let $f : Y \longrightarrow X$ be a continuous mapping such that for each $y \in Y$, the set $\{x \in X : d(x, f(y)) < t\}$ is acyclic or empty for all $t \in R$. Then there exists a point $\bar{y} \in X$ such that*

$$d(\bar{y}, f(\bar{y})) = \min_{x\in X} d(x, f(\bar{y})).$$

Proof. Let $\phi(x,y) = d(y, f(y)) - d(x, f(y))$ for all $(x,y) \in X \times X$. Then $\phi : X \times X \longrightarrow R$ is a continuous function. For each $r \in R$ and each $y \in X$, the set

$$\{x \in X : \phi(x,y) > r\} = \{x \in X : d(x, f(y)) < d(y, f(y)) - r\}$$

is acyclic or empty. By Theorem 2.3 there exists a point $\bar{y} \in X$ such that

$$\sup_{x\in X} \phi(x,\bar{y}) \leq \sup_{x\in X} \phi(x,x),$$

i.e.

$$\sup_{x\in X} [d(\bar{y}, f(\bar{y})) - d(x, f(\bar{y}))] \leq 0,$$

i.e.

$$d(\bar{y}, f(\bar{y})) \leq \inf_{x \in X} d(x, f(\bar{y})).$$

Hence

$$d(\bar{y}, f(\bar{y})) = \min_{x \in X} d(x, f(\bar{y})).$$

This completes the proof.

Remark 2.3. Theorem 2.4 improves and extends Theorem 1 and Theorem 2 of [7].

References

[1] Bardaro, C. & Ceppitelli R., Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities [J], *J. Math. Anal. Appl.,* **132**(1988), 484–490.

[2] Browder, F. E., Coincidence theorems, minimax theorems and variational inequalities [J], *Contemp. Math.,* **26**(1984), 67–80.

[3] Chu, L. J., On Fan's minimax inequality [J], *J. Math. Anal. Appl.,* **201**(1996), 103–113.

[4] Chang, S. S., Theory of variational inequality and complementarity problem with applications [M], Shanghai Scientific and Technological Literature Publishing House, 1991.

[5] Chang, S. S. & Zhang, Y., Generalized KKM theorem and variational inequalities [J], *J. Math. Anal. Appl.,* **158**(1991), 10–25.

[6] Fan, K., A minimax inequality and applications [A], in Inequalities 3 (O. Shisha, Ed.) [C], Academic Press, New York, 1972, 103–113.

[7] Fan, K., Extension of two fixed point theorems of F. E. Browder [J], *Math. Z.,* **112**(1969), 234–240.

[8] Horvath, C. D., Contractibility and generalized convexity [J], *J. Math. Anal. Appl.,* **156**(1991), 341–357.

[9] Horvath, C. D., Some results on multivalued mappings and inequalitis without convexity [A], Nonlinear and Convex Analysis (Edited by B. L. Lin and Simons) [C], Marcel Dekker, New York, 1987, 99–106.

[10] Horvath, C. D., Points fixes et coincidences pour les applications multivoques sans convexite [J], *C. R. Acad. Sci. Paris,* **296**(1983), 403–406.

[11] Horvath, C. D., Points fixes et coincidences dans les espaces topologoques compacts contractiles [J], *C. R. Acad. Sci. Paris,* **299**(1984), 519–521.

[12] McClendon, J. F., Minimax and variational inequalities for compact spaces [J], *Proc. Amer. Math. Soc.,* **89**(1983), 717–721.

[13] Park, S., Bae, J. S. & Kang, H. K., Geometric properties, minimax inequalities, and fixed point theorems on convex spaces [J], *Proc. Amer. Math. Soc.,* **121**(1994), 429–439.

[14] Tarafdar, E., Fixed point theorems in H-spaces and equilibrium point of abstract economies [J], *J. Austral. Math. Soc.* (*Ser.A*), **53**(1992), 252–260.

[15] Wu, X. & Shen, Z. F., On Ky Fan's minimax inequality [J], *Adv. in Math.* (*China*), **27**(1998), 324–330.