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EVANS FUNCTIONS AND ASYMPTOTIC STABILITY OF TRAVELING WAVE SOLUTIONS

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Abstract

This paper studies the asymptotic stability of traveling wave solutions of nonlinear systems of integral-differential equations. It has been established that linear stability of traveling waves is equivalent to nonlinear stability and some "nice structure" of the spectrum of an associated operator implies the linear stability. By using the method of variation of parameter, the author defines some complex analytic function, called the Evans function. The zeros of the Evans function corresponds to the eigenvalues of the associated linear operator. By calculating the zeros of the Evans function, the asymptotic stability of the travling wave solutions is established.

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§1. Introduction

Stability of traveling wave solutions has been one of the main issues in applied mathematics. In this paper, we are concerned with asymptotic stability of traveling wave solutions of the nonlinear system of integral-differential equations

$$\frac{\partial u_i}{\partial t} = f_i(u) + \alpha_i \int_{-\infty}^{\infty} K_i(x-y) H(u_i(y,t) - \theta_i) dy, \ i = 1, 2, \cdots, n. \tag{1.1}$$

The nonlinear systems are derived from neuronal networks[7−10]. The nonlocal integral terms often represent interactions between neurons. These systems also arise from other interesting backgrounds, such as phase transitions[15]. In the system (1.1), $x \in (-\infty, \infty)$ denotes the spatial variable and $t \in (0, \infty)$ denotes the temporal variable, $u = (u_1, u_2, \cdots, u_n)^T$ is a real vector-valued function of x and t. The functions $f_1, f_2, \cdots, f_n$ are assumed to be sufficiently smooth. Moreover each kernel function K_i is nonnegative, even, piecewisely smooth, such that

$$\int_{-\infty}^{\infty} K_i(x) dx = 1, \ \int_{-\infty}^{\infty} |K_i'(x)| dx < \infty.$$

We also assume that there are positive constants C_i and ρ_i such that $K_i(x) \le C_i \exp(-\rho_i |x|)$, for all $x \in R$ and $i = 1, 2, \cdots, n$. This should include functions with compact support, e.g.

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$K(x) = 1$ if $|x| \leq 1/2$; and $K(x) = 0$ if $|x| > 1/2$. The function H is the Heaviside step function, $H(x) = 1$ if $x > 0$; and $H(x) = 0$ if $x < 0$. In addition the system $f_1(u) = f_2(u) = \cdots = f_n(u) = 0$ has a unique solution $u = \phi_0$. We further assume that $\alpha_i \geq 0$ and $\theta_i > 0$ are suitable parameters such that the nonlinear integral differential system (1.1) supports a traveling wave. Each component ϕ_i of the traveling wave $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T$ crosess its threshold θ_i properly if $\alpha_i > 0$, namely there are finitely many points z_{ij} such that $\phi_i(z_{ij}) = \theta_i$ and $\phi_i'(z_{ij}) \neq 0$, where $j = 1, 2, \cdots, m_i$ and $i = 1, 2, \cdots, n$. We require that $\sum_{i=1}^{n} m_i = n$. Interesting cases include $m_i = 1$ for all $i = 1, 2, \cdots, n$ and $m_{2i-1} = 2, m_{2i} = 0$ for all $i = 1, 2, \cdots, [n/2]$.

The traveling wave solutions $\phi = \phi(z)$, with $z = x + vt$, for some constant v, satisfy

$$\begin{aligned} v\frac{d\phi_i}{dz} &= f_i(\phi) + \alpha_i \int_{-\infty}^{\infty} K_i(z-y)H(\phi_i(y) - \theta_i)dy, \\ \phi_i(z_{ij}) &= \theta_i,\ \phi_i'(z_{ij}) \neq 0, \\ \lim_{|z|\to\infty} \phi(z) &= \phi_0,\ j = 1, 2, \cdots, m_i,\ i = 1, 2, \cdots, n. \end{aligned}$$

The real equations describing the neuronal networks would consist of billions of equations, involving both nonlocal integral terms and differential terms. Certainly these equations contain more parameters and possibly more thresholds. Moreover these equations may support multipulse solutions. For brevity, we assume that the existence of traveling wave solutions have been solved. Thus we will only be concerned with the stability of the traveling wave solutions.

§2. Preliminary Results

In this section we list relevant results from ordinary differential equations.

Theorem 2.1. *If $A = (a_{ij})$ is an $n \times n$ constant matrix, then the fundamental matrix of the linear homogeneous system $\psi_z = A\psi$ is $\exp(Az)$, and the general solutions are given by $\psi(z) = \exp(Az)c$, where $c = (c_1, c_2, c_3, \cdots, c_n)^T$ is an arbitrary constant vector, or is given by $\psi(z) = \exp(\lambda z)\xi$, where λ is an eigenvalue of A and ξ is a corresponding eigenvector. Let f be a vector-valued function defined on R, then the general solution of the nonhomogeneous system $\psi_z = A\psi + f(z)$ is given by $\psi(z) = \exp(Az)\psi(0) + \int_0^z \exp[A(z-s)]f(s)ds$. Suppose that A has r eigenvalues with positive real parts and s eigenvalues with negative real parts, such that $r + s = n$, and suppose that $P_\pm$ are the matrix projections corresponding to the eigenvalues with positive and negative real parts. If f is a vector-valued bounded continuous function on R, then the unique bounded continuous solution of the nonhomogeneous system $\psi_z = A\psi + f$ is*

$$\psi(z) = \int_{-\infty}^{z} P_- \exp[A(z-x)]f(x)dx - \int_{z}^{\infty} P_+ \exp[A(z-x)]f(x)dx.$$

Probably the most interesting case is $n = 2$. Let the constant matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have one eigenvalue with positive real part and one eigenvalue with negative real part. Let $f(z)$ and $g(z)$ be bounded continuous functions. Then the unique bounded solution to the linear differential equations

$$\frac{d}{dz}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

is given by

$$\begin{aligned}&\int_{-\infty}^{z}\exp[\lambda_-(z-x)]P_-\begin{pmatrix}f(x)\\g(x)\end{pmatrix}dx-\int_{z}^{\infty}\exp[\lambda_+(z-x)]P_+\begin{pmatrix}f(x)\\g(x)\end{pmatrix}dx\\&=-\frac{1}{\sqrt{(a-d)^2+4bc}}\begin{pmatrix}\lambda_--d & b\\ c & \lambda_--a\end{pmatrix}\int_{-\infty}^{z}\exp[\lambda_-(z-x)]\begin{pmatrix}f(x)\\g(x)\end{pmatrix}dx\\&\quad-\frac{1}{\sqrt{(a-d)^2+4bc}}\begin{pmatrix}\lambda_+-d & b\\ c & \lambda_+-a\end{pmatrix}\int_{z}^{\infty}\exp[\lambda_+(z-x)]\begin{pmatrix}f(x)\\g(x)\end{pmatrix}dx.\end{aligned}$$

Theorem 2.2. *If A is an $n\times n$ matrix-valued function and $X(z)$ is a fundamental matrix of the system $\psi_z=A(z)\psi$, then the general solutions of the nonhomogeneous system $\psi_z=A(z)\psi+f(z)$ are*

$$\psi(z)=X(z)\left(X(0)^{-1}\psi(0)+\int_0^z X(s)^{-1}f(s)ds\right).$$

Suppose that there exist r linearly independent solutions $\psi_i(z), i=1,\cdots,r$, such that they approach zero exponentially fast as $z\to-\infty$ and there exist $s=n-r$ linearly independent solutions $\psi_i(z), i=r+1,\cdots,n$, such that they approach zero exponentially fast as $z\to+\infty$. We define $X(z)=(\psi_1(z),\psi_2(z),\cdots,\psi_n(z))$. Suppose that $\det X(z)\neq 0$, for all $z\in R$. Then $X(z)$ is a fundamental matrix. Let f be a vector-valued bounded continuous function on R. Then the unique bounded continuous solution of the nonhomogeneous system $\psi_z=A(z)\psi+f$ is

$$\begin{aligned}\psi(z)&=(\psi_1(\lambda,\varepsilon,z),\psi_2(\lambda,\varepsilon,z),\cdots,\psi_n(\lambda,\varepsilon,z))\begin{pmatrix}c_1(\lambda,\varepsilon,z)\\c_2(\lambda,\varepsilon,z)\\\cdots\\c_n(\lambda,\varepsilon,z)\end{pmatrix}\\&=c_1(\lambda,\varepsilon,z)\psi_1(\lambda,\varepsilon,z)+c_2(\lambda,\varepsilon,z)\psi_2(\lambda,\varepsilon,z)+\cdots+c_n(\lambda,\varepsilon,z)\psi_n(\lambda,\varepsilon,z),\end{aligned}$$

where

$$c_i(\lambda,\varepsilon,z)=-\frac{1}{E(\lambda,\varepsilon)}\int_z^{\infty}\exp\Big[-\int_0^s \mathrm{Tr}A(\lambda,\varepsilon,\tau)d\tau\Big]D_i(\lambda,\varepsilon,s)ds \quad \text{for } i\le r,$$

$$c_i(\lambda,\varepsilon,z)=+\frac{1}{E(\lambda,\varepsilon)}\int_{-\infty}^{z}\exp\Big[-\int_0^s \mathrm{Tr}A(\lambda,\varepsilon,\tau)d\tau\Big]D_i(\lambda,\varepsilon,s)ds \quad \text{for } i> r,$$

$$D_i(\lambda,\varepsilon,s)=\det\left(\psi_1(\lambda,\varepsilon,s),\cdots,\psi_{i-1}(\lambda,\varepsilon,s),f(s),\psi_{i+1}(\lambda,\varepsilon,s),\cdots,\psi_n(\lambda,\varepsilon,s)\right),$$

$$E(\lambda,\varepsilon)=\exp\Big[-\int_0^z \mathrm{Tr}A(\lambda,\varepsilon,s)ds\Big]\det X(\lambda,\varepsilon,z).$$

Moreover

$$F(f)\equiv\exp\Big[-\int_0^z \mathrm{Tr}A(\lambda,\varepsilon,s)ds\Big]D_i(\lambda,\varepsilon,z)$$

is a continuous linear functional on C^n. By Riesz representation theorem, there exists a unique function $\varphi_i(\lambda,\varepsilon,z)$, such that for all point $f\in C^n$, the above linear functional $F(f)=(f,\varphi_i(\lambda,\varepsilon,z))$. A simple calculation with (J) shows that

$$(f,\varphi_{iz})=(f,\varphi_i)_z=-(Af,\varphi_i)=(f,-A^T\varphi_i).$$

Because f is an arbitrary vector in C^n, we must have $\varphi_{iz}+A^T\varphi_i=0$, namely φ_i is the solution of the adjoint system $\psi_z+A^T\psi=0$, where $i=1,2,\cdots,n$. Moreover, we have $(\psi_i,\varphi_i)=E(\lambda,\varepsilon)$ and $(\psi_j,\varphi_i)=0$, for all $j\neq i$. In addition, $c_i(z)=-\frac{1}{E(\lambda,\varepsilon)}\int_z^{\infty}(f(s),\varphi_i(s))ds$ for

$i \le r$ and $c_i(z) = \frac{1}{E(\lambda,\varepsilon)} \int_{-\infty}^{z} (f(s), \varphi_i(s))ds$ for $i > r$, and all the products $c_i(z)\psi_i(z), 1 \le i \le n$, are bounded on R.

Theorem 2.3. *Let A be an $n \times n$ constant matrix and let $B(z)$ be an $n \times n$ matrix-valued function satisfying $\|B(z)\| \le C \exp(-\rho|z|)$, for some positive constants C and ρ. Suppose that $\mu_1 < \cdots < \mu_r < 0 < \mu_{r+1} < \cdots < \mu_m$ are the distinct real parts of all the eigenvalues of A and that n_k is the number of eigenvalues counting algebraic multiplicities with real part μ_k, for all $1 \le k \le m$. Then $\sum_{k=1}^{m} \mu_k = n$.*

We will first consider solutions to the systems of ordinary differential equations

$$\frac{\partial \psi}{\partial z} - A\psi = 0, \ \frac{\partial \psi}{\partial z} + A^T \psi = 0.$$

Then we will construct related solutions to the perturbed systems of ordinary differential equations

$$\frac{\partial \psi}{\partial z} - [A + B(z)]\psi = 0, \ \frac{\partial \psi}{\partial z} + [A + B(z)]^T \psi = 0.$$

Finally we will use the method of variation of parameter to construct all bounded solutions to the nonhomogeneous systems of ordinary differential equations

$$\frac{\partial \psi}{\partial z} - [A + B(z)]\psi = f(z), \ \frac{\partial \psi}{\partial z} + [A + B(z)]^T \psi = g(z),$$

where f and g are bounded uniformly continuous functions defined on R.

First of all, there exists an invertible matrix T such that

$$A_0 \equiv T^{-1}AT = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_m \end{pmatrix},$$

where

$$J_k = \begin{pmatrix} \mu_k + i\nu_{k1} & * & 0 & \cdots & 0 & 0 \\ 0 & \mu_k + i\nu_{k2} & * & \cdots & 0 & 0 \\ 0 & 0 & \mu_k + i\nu_{k3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mu_k + i\nu_{kn_k - 1} & * \\ 0 & 0 & 0 & \cdots & 0 & \mu_k + i\nu_{kn_k} \end{pmatrix}$$

is an $n_k \times n_k$ matrix with $\mathrm{Re}(\mu_k + i\nu_{kl}) = \mu_k$ and each of the $* = 0$ or 1. Moreover

$$\exp(+A_0 z) = \begin{pmatrix} \exp(+J_1 z) & 0 & \cdots & 0 \\ 0 & \exp(+J_2 z) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \exp(+J_m z) \end{pmatrix},$$

$$\exp(-A_0^T z) = \begin{pmatrix} \exp(-J_1^T z) & 0 & \cdots & 0 \\ 0 & \exp(-J_2^T z) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \exp(-J_m^T z) \end{pmatrix}.$$

If one of the Jordan blocks in A_0 takes the form $J = \mu I$, then $\exp(Jz) = \exp(\mu z)I$ and

$\exp(-J^Tz) = \exp(-\mu z)I$. Suppose that one of the Jordan blocks in A_0 takes the form

$$J = \begin{pmatrix} \mu & 1 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 1 & \cdots & 0 & 0 \\ 0 & 0 & \mu & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mu & 1 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{pmatrix} = \mu I + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \equiv \mu I + C.$$

Then $C^p \neq 0$ but $C^{p+1} = 0$, for some unique integer $p \geq 1$, and

$$\exp(Jz) = \exp(\mu z)\exp(Cz) = \exp(\mu z)\Big(I + Cz + \frac{1}{2!}C^2z^2 + \cdots + \frac{1}{p!}C^pz^p\Big)$$

$$= \exp(\mu z)\begin{pmatrix} 1 & z & \frac{1}{2!}z^2 & \cdots & \frac{1}{(p-1)!}z^{p-1} & \frac{1}{p!}z^p \\ 0 & 1 & z & \cdots & \frac{1}{(p-2)!}z^{p-2} & \frac{1}{(p-1)!}z^{p-1} \\ 0 & 0 & 1 & \cdots & \frac{1}{(p-3)!}z^{p-3} & \frac{1}{(p-2)!}z^{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & z \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then the original systems of ordinary differential equations are reduced to

$$\frac{\partial\psi}{\partial z} - A_0\psi = 0, \ \frac{\partial\psi}{\partial z} + A_0^T\psi = 0.$$

The fundamental matrices for these systems are $\exp(+A_0z)$ and $\exp(-A_0^Tz)$. The following linearly independent solutions are basis of the general solution spaces of the above systems, respectively

$$\exp(+A_0z)e_1, \exp(+A_0z)e_2, \exp(+A_0z)e_3, \cdots, \exp(+A_0z)e_n,$$
$$\exp(-A_0^Tz)e_1, \exp(-A_0^Tz)e_2, \exp(-A_0^Tz)e_3, \cdots, \exp(-A_0^Tz)e_n.$$

There exist solutions $\{Y_l^k(z) = \exp(+A_0z)e_l^k : 1 \leq l \leq n_k \text{ and } 1 \leq k \leq m\}$ to the system of ordinary differential equations $\frac{\partial\psi}{\partial z} - A_0\psi = 0$ with the limit

$$\lim_{|z|\to\infty} \frac{1}{z}\ln\|Y_l^k(z)\| = +\mu_k,$$

and solutions $\{Z_q^p(z) = \exp(-A_0^Tz)e_q^p : 1 \leq q \leq n_p \text{ and } 1 \leq p \leq m\}$ to the adjoint system of ordinary differential equations $\frac{\partial\psi}{\partial z} + A_0^T\psi = 0$ with the limit

$$\lim_{|z|\to\infty} \frac{1}{z}\ln\|Z_q^p(z)\| = -\mu_p.$$

If $(k,l) = (p,q)$, then $(Y_l^k(z), Z_q^p(z)) = 1$, for all $z \in R$. If $(k,l) \neq (p,q)$, then $(Y_l^k(z), Z_q^p(z)) = 0$, for all $z \in R$. Define m matrices by using the solutions of both systems

$$X^k(z,x) = \big(Y_1^k(z), \cdots, Y_{n_k}^k(z)\big)\big(Z_1^k(x), \cdots, Z_{n_k}^k(x)\big)^T$$
$$= \exp(+A_0z)\begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{n_k} & 0 \\ 0 & 0 & 0 \end{pmatrix}\exp(-A_0x)$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \exp[J_k(z-x)] & 0 \\ 0 & 0 & 0 \end{pmatrix} = \exp[\mu_k(z-x)]\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for all $1 \le k \le m$, where $* = I_{n_k}$ or $* = \exp[C_k(z-x)]$ is an $n_k \times n_k$ matrix, and $* = I_{n_k}$ if $z = x$. Then

$$X^k(z,x) = X^k(z-x,0),\ \exp[A_0(z-x)] = \sum_{k=1}^{m} X^k(z,x).$$

All the linearly independent solutions of the ordinary differential equations $\frac{\partial\psi}{\partial z} - A\psi = 0$ and $\frac{\partial\psi}{\partial z} + A^T\psi = 0$ are given by $TY_l^k(z)$ and $(T^T)^{-1}Z_q^p(z)$, respectively. Without loss of generality, denote these solutions also by $Y_l^k(z)$ and $Z_q^p(z)$. These solutions satisfy the same conditions as above, i.e.,

$$(Y_l^k(z), Z_q^p(z)) = 1 \ \text{ for } (k,l) = (p,q); \quad (Y_l^k(z), Z_q^p(z)) = 0 \ \text{ for } (k,l) \ne (p,q),$$

$$X^k(z,x) = X^k(z-x,0),\ \exp[A(z-x)] = \sum_{k=1}^{m} X^k(z,x).$$

Consider the perturbed differential equations

$$\frac{\partial\psi}{\partial z} - [A+B(z)]\psi = 0,\ \frac{\partial\psi}{\partial z} + [A+B(z)]^T\psi = 0.$$

There exist linearly independent solutions $\psi_l^k(z)$ and $\varphi_q^p(z)$ to these ordinary differential equations, such that for some constant $\rho > 0$, one has the asymptotic behaviors

$$\lim_{z\to+\infty} \frac{\psi_l^k(z) - Y_l^k(z)}{\exp[(+\mu_k - \rho)z]} = 0 \ \text{ for all } 1 \le l \le n_k, 1 \le k \le r,$$

$$\lim_{z\to-\infty} \frac{\psi_l^k(z) - Y_l^k(z)}{\exp[(+\mu_k + \rho)z]} = 0 \ \text{ for all } 1 \le l \le n_k, r < k \le m,$$

$$\lim_{z\to-\infty} \frac{\varphi_q^p(z) - Z_q^p(z)}{\exp[(-\mu_p + \rho)z]} = 0 \ \text{ for all } 1 \le q \le n_p, 1 \le p \le r,$$

$$\lim_{z\to+\infty} \frac{\varphi_q^p(z) - Z_q^p(z)}{\exp[(-\mu_p - \rho)z]} = 0 \ \text{ for all } 1 \le q \le n_p, r < p \le m.$$

It is clear that $\frac{d}{dz}(\psi_l^k(z), \varphi_q^p(z)) = 0$, hence $(\psi_l^k(z), \varphi_q^p(z))$ must be a constant. Without loss of generality, suppose that

$$(\psi_l^k(z), \varphi_q^p(z)) = 1, \ \text{ if } (k,l) = (p,q),$$
$$(\psi_l^k(z), \varphi_q^p(z)) = 0, \ \text{ if } (k,l) \ne (p,q).$$

Otherwise we can use Gram-Schmidt method to modify φ_q^p so that $\{\psi_l^k\}$ and $\{\varphi_q^p\}$ are mutually orthogonal. Define $Y^k(z,x) = \left(\psi_1^k(z), \cdots, \psi_{n_k}^k(z)\right)\left(\varphi_1^k(x), \cdots, \varphi_{n_k}^k(x)\right)^T$. We have

$$\frac{\partial}{\partial z} Y^k(z,x) - [A+B(z)]Y^k(z,x) = 0,$$
$$\frac{\partial}{\partial x} Y^k(z,x) + Y^k(z,x)[A+B(x)] = 0.$$

By orthogonality, we have $Y^k(z,z)Y^k(z,z) = Y^k(z,z)$, so $\frac{d}{dz}Y^k(z,z) = 0$. Therefore $Y^k(z,z)$ is a constant matrix. By the asymptotic analysis we see that $Y^k(z,z) = X^k(z,z)$, for all $z \in (-\infty,\infty)$. Hence $\sum_{k=1}^{m} Y^k(z,z) = I$. Now it is easy to see that

$$\Phi(z) = \sum_{k=1}^{r} \int_{-\infty}^{z} Y^k(z,x)f(x)dx - \sum_{k=r+1}^{m} \int_{z}^{\infty} Y^k(z,x)f(x)dx$$

is a bounded solution on $(-\infty, \infty)$ to the nonhomogeneous differential equation

$$\frac{\partial \psi}{\partial z} - [A + B(z)]\psi = f(z).$$

Any other solution ψ of the nonhomogeneous equation can be written as

$$\psi(z) = \Phi(z) + \sum_{k=1}^{m} \sum_{l=1}^{n_k} c_l^k \psi_l^k(z),$$

where c_l^k are constants. It is easy to show that ψ is bounded as $z \to \infty$ if and only if $c_l^k = 0$, for all $1 \leq l \leq n_k, r < k \leq m$. Therefore the solution ψ is bounded as $z \to +\infty$ if and only if $\lim\limits_{z \to +\infty} (\psi(z), \varphi(z)) = 0$, where $\varphi(z)$ is any bounded solution as $z \to \infty$ of the adjoint system $\frac{\partial \varphi}{\partial z} + [A + B(z)]^T \varphi = 0$. Similarly the solution ψ is bounded as $z \to -\infty$ if and only if $\lim\limits_{z \to -\infty} (\psi(z), \varphi(z)) = 0$, where $\varphi(z)$ is any bounded solution as $z \to -\infty$ of the adjoint system

$$\frac{\partial \varphi}{\partial z} + [A + B(z)]^T \varphi = 0.$$

Moreover, if the solution $\psi(z)$ of the nonhomogeneous equations $\psi_z - [A + B(z)]\psi = f(z)$ is unbounded, then $\psi(z) = \exp(\rho z) O(1)$ as $z \to +\infty$ or as $z \to -\infty$, for some positive constant $\rho > 0$.

§3. The Eigenvalue Problems

We will define a linear operator and then write out the eigenvalue problem. In moving coordinates, i.e. $z = x + vt$ and $U(z,t) = u(x,t) = u(z - vt, t)$, where $U = (U_1, U_2, \cdots, U_n)^T$, the nonlinear system of integral differential equations (1.1) can be written as

$$\frac{\partial U_i}{\partial t} + v\frac{\partial U_i}{\partial z} = f_i(U) + \alpha_i \int_{-\infty}^{\infty} K_i(z - y) H(U_i(y,t) - \theta_i) dy. \tag{3.1}$$

Clearly the traveling wave is a stationary solution of this system. The linearization of (2.1) about the wave solution ϕ is given by

$$\frac{\partial U_i}{\partial t} + v\frac{\partial U_i}{\partial z} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\phi) U_j + \alpha_i \sum_{j=1}^{m_i} \frac{K_i(z - z_{ij})}{|\phi_i'(z_{ij})|} U_i(z_{ij}, t). \tag{3.2}$$

Seeking for particular solutions of the form $U(z,t) = \exp(\lambda z)\psi(\lambda, z)$ of system (4.2) leads to the eigenvalue problem

$$\lambda \psi^i + v\frac{d\psi^i}{dz} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\phi) \psi^j + \alpha_i \sum_{j=1}^{m_i} \frac{K_i(z - z_{ij})}{|\phi_i'(z_{ij})|} \psi^i(z_{ij}), \tag{3.3}$$

where $\lambda \in C$ is the eigenvalue parameter and we allow some singular perturbation parameter $0 < \varepsilon \ll 1$ to implicitly appear in this problem. Moreover $z \in R$ is a real variable, $\psi = (\psi^1(\lambda, \varepsilon, z), \psi^2(\lambda, \varepsilon, z), \cdots, \psi^n(\lambda, \varepsilon, z))^T \in C^n$ is a complex vector-valued function of the variables λ, ε, z.

For our convenience we define the $n \times n$ matrix $A(\varepsilon, z) = -\left(\frac{\partial f_i}{\partial u_j}(\phi)\right)_{1 \leq i,j \leq n}$, which is independent of λ. Suppose that the real-valued matrix defined by $A(\varepsilon) = \lim\limits_{|z| \to \infty} A(\varepsilon, z)$ possesses n eigenvalues $\omega_i(\varepsilon)$ with positive real parts, the smallest eigenvalue being of order ε. Moreover suppose that there exists an unbounded, open, simply connected region $\Omega(\varepsilon)$ in the complex plane such that $\mathrm{Re}[\lambda + \omega_i(\varepsilon)] > 0$, for all $\lambda \in \Omega(\varepsilon)$. To study the above eigenvalues and eigenfunctions, we define a bounded linear differential operator $L(\varepsilon) : BC^1(R,C^n) \to BC^0(R,C^n)$ by

$$L(\varepsilon)\psi = -v(\varepsilon)\frac{\partial\psi}{\partial z} - A(\varepsilon,z)\psi + N(\lambda,\varepsilon,z),$$

where $N(\lambda,\varepsilon,z) = (N_1(\lambda,\varepsilon,z), N_2(\lambda,\varepsilon,z),\cdots,N_n(\lambda,\varepsilon,z))^T$ is a complex vector-valued function of λ,ε,z and each $N_i(\lambda,\varepsilon,z) = \alpha_i \sum_{j=1}^{m_i} \frac{K_i(z-z_{ij})}{|\phi_i'(z_{ij})|}\psi^i(\lambda,\varepsilon,z_{ij})$ for all $i = 1,2,\cdots,n$. The eigenvalue problem under consideration can then be written in a compact form $L(\varepsilon)\psi = \lambda\psi$. There is a subtle difference between this operator and regular differential operators. The explicit adjoint operator of regular operators are usually easy to find. But the adjoint operator of our operator is very hard to find explicitly. To deal with the parts corresponding to the nonlocal integral terms in problem (3.3), we arrange the complex numbers $\psi^i(\lambda,\varepsilon,z_{ij})$ in a scientific way. First we arrange each block $B_i(\lambda,\varepsilon) = (\psi^i(\lambda,\varepsilon,z_{i1}),\psi^i(\lambda,\varepsilon,z_{i2}),\cdots,\psi^i(\lambda,\varepsilon,z_{im_i})$ for all $i = 1,2,\cdots,n$. Recall that some $B_i(\lambda,\varepsilon)$ does not appear if $\alpha_i = 0$. Then we arrange

$$B(\lambda,\varepsilon) = (B_1(\lambda,\varepsilon),\cdots,B_n(\lambda,\varepsilon))^T = \psi^1(\lambda,\varepsilon,z_{11}),\cdots,\psi^n(\lambda,\varepsilon,z_{nm_n}))^T.$$

Clearly the vector $B(\lambda,\varepsilon) \in C^n$. Moreover, we define

$$g_{ij}(z) = \frac{\alpha_i}{|\phi_i'(z_{ij})|}K_i(z-z_{ij}),\ j = 1,2,\cdots,m_i,\ i = 1,2,\cdots,n.$$

Then each component of N is given by

$$N_i(\lambda,\varepsilon,z) = \sum_{j=1}^{m_i} g_{ij}(z)\psi^i(\lambda,\varepsilon,z_{ij}) = (0,\cdots,0,g_{i1}(z),g_{i2}(z),\cdots,g_{im_i}(z),0\cdots,0)\,B(\lambda,\varepsilon).$$

We then define an $n\times n$ matrix with each row being $(0,\cdots 0,g_{i1}(z),g_{i2}(z),\cdots,g_{im_i}(z),0\ \cdots,0)$,

$$G(z) = \begin{pmatrix} g_{11}(z) & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & g_{nm_n}(z) \end{pmatrix}.$$

The matrix G converges to the zero matrix exponentially fast as $z \to \pm\infty$, since each entry has a similar asymptotic behavior as $z \to \pm\infty$. The second part on the right hand side of (3.3) can be written as $G(z)B(\lambda,\varepsilon)$. Therefore the eigenvalue problem can also be expressed as $L(\varepsilon)\psi = -v(\varepsilon)\psi_z - A(\varepsilon,z)\psi + G(z)B(\lambda,\varepsilon) = \lambda\psi$. It is very difficult to directly find the solutions of the eigenvalue problem. However we can find the solutions of an intermediate problem, namely, we will first drop the terms involving $\psi(\lambda,\varepsilon,z_{ij})$. It is therefore necessary to introduce an intermediate operator $L_0(\varepsilon) : BC^1(R,C^n) \to BC^0(R,C^n)$ by

$$L_0(\varepsilon)\psi = -v(\varepsilon)\frac{\partial\psi}{\partial z} - A(\varepsilon,z)\psi.$$

If the complex number $\lambda \in \Omega(\varepsilon)$ is an eigenvalue of the operator $L(\varepsilon)$ or the operator $L_0(\varepsilon)$ associated with the eigenfunction $\psi(\lambda,\varepsilon,z)$, then $\overline{\lambda}$ is also an eigenvalue of $L(\varepsilon)$ or $L_0(\varepsilon)$ associated with the eigenfunction $\overline{\psi}(\lambda,\varepsilon,z)$, where $\overline{\lambda}$ stands for the conjugate of the complex number λ. However we will show that in some right half plane $\{\lambda$: Re$\lambda > -k\}$, where k is a positive constant, there is no eigenvalue of $L(\varepsilon)$ other than the neutral eigenvalue $\lambda = 0$.

The stability of the traveling wave solutions is determined completely by the structure of the spectrum of the operator $L(\varepsilon)$. The spectrum is divided into two parts: the essential spectrum, consisting of distinct smooth curves, and the normal spectrum, consisting of a few isolated eigenvalues with finite algebraic multiplicities. The essential spectrum consists of those complex numbers λ such that certain matrix $A(\lambda,\varepsilon)$ defined below has a purely imaginary eigenvalue $i\xi$. To guarantee the resting state $\phi=\phi_0$ of the traveling wave equations to be stable, the essential spectrum must be in some left half plane $\Omega(\varepsilon)^c$ and be uniformly bounded away from the imaginary axis. Then inside this region $\Omega(\varepsilon)$, all the spectrum of $L(\varepsilon)$ are eigenvalues with finite algebraic multiplicity. The normal spectrum is not an empty set because $\lambda=0$ is an eigenvalue of $L(\varepsilon)$ inside $\Omega(\varepsilon)$. If this neutral eigenvalue $\lambda=0$ is simple and if there exists no other eigenvalue of $L(\varepsilon)$ inside the region $\mathrm{Re}\lambda\geq 0$, then the traveling wave solution is exponentially stable. In general the normal spectrum is hard to analyse and need more advanced tools to detect. We will define and use some Evans function to achieve the stability result.

Let $m\geq 1$ be an integer. Define a Banach space $BC^m(R,C^n)=\{\psi:\psi$ is a complex vector-valued function defined on R, such that all its derivatives up to order m are bounded and uniformly continuous on $R\}$. Let us give the rigorous definitions for nonlinear and linear stabilities of the traveling waves (see also [1, 14, 18, 25]).

Definition. *The traveling wave solution $\phi=(\phi_1,\phi_2,\cdots,\phi_n)^T$ is exponentially stable relative to the full nonlinear system* (4.1)*, if there exist positive constants C,M and α such that for any solution $U=(U_1,U_2,\cdots,U_n)^T$ to* (4.1) *with the initial data $U_0=(U_{10},U_{20},\cdots,U_{n0})\in BC^1(R,R^n)$ and $\|U_0-\phi\|_\infty\leq C$, there exists a constant h with $|h|\leq M\|U_0-\phi\|_\infty$, such that $\|U(t)-\phi_h\|_\infty\leq M\|U_0-\phi\|_\infty\exp(-\alpha t)$ for all $t\geq 0$, where $\phi_h(z)=\phi(z+h)$. Let $\|\phi_z\|_\infty<\infty$. The first order derivative $\phi_z=(\phi_{1z},\phi_{2z},\cdots,\phi_{nz})$ of the traveling wave solution is exponentially stable relative to the linear system* (4.2) *if there exist positive constants M and α such that for any solution $V=(V_1,V_2,\cdots,V_n)^T$ to* (4.2) *with the initial data $\psi=(\psi^1,\psi^2,\cdots,\psi^n)^T\in BC^1(R,R^n)$, there exists a constant h with $|h|\leq M\|\psi\|_\infty$, such that*

$$\|V(t)-h\phi_z\|_\infty\leq M\|\psi\|_\infty\exp(-\alpha t)\quad \textit{for all } t\geq 0.$$

Remark. A necessary condition for exponential stability of traveling wave solutions is that the essential spectrum of the bounded linear differential operator $L(\varepsilon)$ defined below is located in a left half plane, uniformly bounded away from the imaginary axis.

To conclude this section, let us give the following propositions without proving them. Please see [25] for detailed proofs.

Proposition 3.1. *The linear stability of the traveling wave solution is equivalent to the nonlinear stability.*

Proposition 3.2. *If there exists a positive constant $\alpha>0$, such that all the spectrum, except for $\lambda=0$, of the operator $L(\varepsilon)$ satisfy the estimate* $\mathrm{Re}\,\lambda\leq-\alpha$, *and the eigenvalue $\lambda=0$ is algebraically simple, then the traveling wave solution is exponentially stable.*

The proofs of these results are similar to those given by Evans[11] and Alexander, Gardner and Jones[1]. With these results, to establish the asymptotic stability of the wave solutions, we only need to study the spectrum of the operator $L(\varepsilon)$.

§4. Intermediate Evans Function

We will define some analytic function to study the normal spectrum of the linear operator defined in Section 3. Before that we need to define some intermediate analytic function associated with the auxiliary eigenvalue problem $L_0(\varepsilon)\psi=\lambda\psi$. Rewrite the eigenvalue problem $L(\varepsilon)\psi=\lambda\psi$ as a first order linear nonhomogeneous system of differential equations

$$v(\varepsilon)\frac{\partial\psi}{\partial z}+[\lambda I+A(\varepsilon,z)]\psi=N(\lambda,\varepsilon,z). \tag{4.1}$$

We will also consider the following homogeneous auxiliary system

$$v(\varepsilon)\frac{\partial\psi}{\partial z}+[\lambda I+A(\varepsilon,z)]\psi=0, \tag{4.2}$$

corresponding to the intermediate eigenvalue problem $L_0(\varepsilon)=\lambda\psi$. The asymptotic system of both (4.1) and (4.2), as $|z|\to\infty$, is

$$v(\varepsilon)\frac{\partial\psi}{\partial z}+[\lambda I+A(\varepsilon)]\psi=0. \tag{4.3}$$

A careful observation tells us that each of the above matrices in (4.1), (4.2), (4.3) can be written as the sum of λI and some λ-independent matrix. This will simplify our calculation, the reason is that the eigenvectors of $A(\varepsilon)$ and thus $A(\lambda,\varepsilon)\equiv\lambda I+A(\varepsilon)$ are independent of λ and hence one can find certain relationship between the solutions for the case $\lambda\neq 0$ and the case $\lambda=0$. As a result, the intermediate Evans function would be independent of λ. Additionally, the eigenvalues of the matrix $A(\lambda,\varepsilon)$ are $\mu_i(\lambda,\varepsilon)\equiv\lambda+\omega_i(\varepsilon)$.

It is always easier to solve homogeneous equations than to solve nonhomogeneous equations. Let us now construct analytic solutions to the intermediate system (4.2). These solutions make up of the fundamental matrix of that system. Thus they are candidate functions to define the intermediate Evans functions. In the open, unbounded, simply connected region $\Omega(\varepsilon)$, the matrix $A(\lambda,\varepsilon)$ has n eigenvalues with positive real parts and no eigenvalue with negative real part. The eigenvectors corresponding to these eigenvalues are independent of λ. Alexander, Gardner and Jones[1] and Evans[14] gave a general method to construct solutions of the system (4.2). There are n linearly independent analytic solutions $Y_i(\lambda,\varepsilon,z)$ to the asymptotic system (4.3) with constant coefficients, such that they converge to zero exponentially fast as $z\to\infty$.

Lemma 4.1. *For each $i=1,2,\cdots,n$, there exists a unique complex vector-valued analytic function $\varphi_i=\varphi_i(\lambda,\varepsilon,z)$ and a positive constant $\delta_i=\delta_i(\lambda,\varepsilon)$, such that as $z\to+\infty$, there holds the asymptotic behavior*

$$\varphi_i(\lambda,\varepsilon,z)-Y_i(\lambda,\varepsilon,z)=\exp\Big[-\frac{\mu_i(\lambda,\varepsilon)}{v(\varepsilon)}z-\frac{\delta_i(\lambda,\varepsilon)}{v(\varepsilon)}z\Big]O(1).$$

It is easy to see $\Phi(\lambda,\varepsilon,z)=(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z))$ is a fundamental matrix of the homogeneous system

$$v(\varepsilon)\frac{\partial\psi}{\partial z}+[\lambda I+A(\varepsilon,z)]\psi=0.$$

Moreover, Φ is analytic in λ, for all fixed z and ε, and $\lim\limits_{z\to+\infty}\Phi(\lambda,\varepsilon,z)=0$, exponentially fast. In the same way, we can construct the solutions of the adjoint system of (4.2). Suppose that $S(\lambda,\varepsilon,z)=(S_1(\lambda,\varepsilon,z),S_2(\lambda,\varepsilon,z),\cdots,S_n(\lambda,\varepsilon,z))$ is a fundamental matrix of the adjoint system

$$v(\varepsilon)\frac{\partial\psi}{\partial z}=[\lambda I+A(\varepsilon,z)]^T\psi, \tag{4.4}$$

such that S is analytic in λ, for all fixed z and ε, and that $\lim\limits_{z\to-\infty}S(\lambda,\varepsilon,z)=0$, exponentially fast. Necessarily $S_i(\lambda,\varepsilon,z)^T\varphi_j(\lambda,\varepsilon,z)$ is independent of λ and z, for all $1\le i,j\le n$. We could assume that $S_i(\lambda,\varepsilon,z)^T\varphi_j(\lambda,\varepsilon,z)=0$ if $i\neq j$ and $S_i(\lambda,\varepsilon,z)^T\varphi_i(\lambda,\varepsilon,z)=D(\lambda,\varepsilon)$, for some appropriate analytic function $D(\lambda,\varepsilon)$. Given any fundamental matrices Φ of system (4.2) and S of system (4.4), we can always use the Gram-Schmidt method to normalize S, so that the above conditions hold. Suppose that there exists a nonzero complex analytic function $D(\lambda,\varepsilon)$, which we call the intermediate Evans function, such that

$$S(\lambda,\varepsilon,z)^T\Phi(\lambda,\varepsilon,z)=D(\lambda,\varepsilon)I.$$

An alternate way to handle $D(\lambda,\varepsilon)$ and S is the following. By using Φ we can directly construct the intermediate Evans function $D(\lambda,\varepsilon)$ and then find S. In fact we define the intermediate Evans function by

$$D(\lambda,\varepsilon)=\exp\Big[\frac{1}{v(\varepsilon)}\int_0^z n\lambda+\mathrm{Tr}A(\varepsilon,s)ds\Big]\det\left(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z)\right)$$

and then define a continuous linear functional

$$\begin{aligned}&\exp\Big[\frac{1}{v(\varepsilon)}\int_0^z n\lambda+\mathrm{Tr}A(\varepsilon,s)ds\Big]\\&\times\det(\varphi_1(\lambda,\varepsilon,z),\cdots,\varphi_{i-1}(\lambda,\varepsilon,z),f,\varphi_{i+1}(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z))\end{aligned}$$

on C^n. By Riesz representation theorem, there exists a unique function $S_i(\lambda,\varepsilon,z)$, such that for all point $f\in C^n$, the above linear functional $=(f,S_i(\lambda,\varepsilon,z))$. A simple calculation shows that

$$v(\varepsilon)(f,S_{iz})=v(\varepsilon)(f,S_i)_z=((\lambda I+A)f,S_i)=(f,(\lambda I+A)^TS_i).$$

Because f is an arbitrary vector in C^n, we must have $v(\varepsilon)S_{iz}=[\lambda I+A(\varepsilon,z)]^TS_i$, namely S_i is the solution of the adjoint system $v(\varepsilon)\psi_z=[\lambda I+A(\varepsilon,z)]^T\psi$, where $i=1,2,\cdots,n$. Moreover, we have $S_i^T(\lambda,\varepsilon,z)\varphi_i(\lambda,\varepsilon,z)=D(\lambda,\varepsilon)$ and $S_i^T(\lambda,\varepsilon,z)\varphi_j(\lambda,\varepsilon,z)=0$, for all $i\neq j$.

Remark. Since the eigenvalues satisfy $\mu_i(\lambda,\varepsilon)=\lambda+\omega_i(\varepsilon)$, and the eigenvector is independent of λ, by uniqueness we must have the equation $\varphi_i(\lambda,\varepsilon,z)=\varphi_i(0,\varepsilon,z)\exp[-\frac{\lambda}{v(\varepsilon)}z]$, for all $z\in R$ and $i=1,2,\cdots,n$. One can easily prove $\Phi(\lambda,\varepsilon,z)=\exp[-\frac{\lambda}{v(\varepsilon)}z]\Phi(0,\varepsilon,z)$ and $S(\lambda,\varepsilon,z)=\exp[\frac{\lambda}{v(\varepsilon)}z]S(0,\varepsilon,z)$, for all $\lambda\in\Omega(\varepsilon)$ and all $z\in R$. By virtue of these equations, one can verify that $D(\lambda,\varepsilon)=D(0,\varepsilon)\neq 0$, for all $\lambda\in\Omega(\varepsilon)$.

For convenience of later investigations, we need to refine the linearly independent solutions $\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z)$ of the intermediate system (4.2). Clearly

$$\begin{aligned}I(\lambda,\varepsilon,s,z)&=\Phi(\lambda,\varepsilon,z)S(\lambda,\varepsilon,s)^T\\&=\left(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z)\right)\left(S_1(\lambda,\varepsilon,s),S_2(\lambda,\varepsilon,s),\cdots,S_n(\lambda,\varepsilon,s)\right)^T\end{aligned}$$

is also a fundamental matrix of the intermediate system (4.2). In other words, the function

$$I_i(\lambda,\varepsilon,s,z)=\left(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z)\right)\begin{pmatrix}S_1^i(\lambda,\varepsilon,s)\\S_2^i(\lambda,\varepsilon,s)\\\cdots\\S_n^i(\lambda,\varepsilon,s)\end{pmatrix}$$

is a solution of (4.2) and $I_1(\lambda,\varepsilon,s,z),I_2(\lambda,\varepsilon,s,z),\cdots,I_n(\lambda,\varepsilon,s,z)$ are linearly independent. These solutions are analytic in λ, for all fixed ε and z. Obviously $I(\lambda,\varepsilon,z,z)=D(\lambda,\varepsilon)I$. We can use the new fundamental matrix $I(\lambda,\varepsilon,s,z)$ to define intermediate Evans function parameterized by z as follows

$$\begin{aligned}D(\lambda,\varepsilon,s)&=\exp\Big[\frac{1}{v(\varepsilon)}\int_0^z n\lambda+\mathrm{Tr}A(\varepsilon,\tau)d\tau\Big]\det I(\lambda,\varepsilon,s,z)\\&=\exp\Big[\frac{1}{v(\varepsilon)}\int_0^z n\lambda+\mathrm{Tr}A(\varepsilon,\tau)d\tau\Big]\det S(\lambda,\varepsilon,s)\det\Phi(\lambda,\varepsilon,z)\\&=D(\lambda,\varepsilon)\det S(\lambda,\varepsilon,s).\end{aligned}$$

This is also independent of z and is analytic in λ. The asymptotic behavior of $I(\lambda,\varepsilon,s,z)$ as

$z \to \pm\infty$ is very similar to that of $\Phi(\lambda, \varepsilon, z)$. On the other hand, the asymptotic behavior of $I(\lambda, \varepsilon, s, z)$ as $s \to \pm\infty$ is very similar to that of $S(\lambda, \varepsilon, s)$.

§5. The Evans Function for the Eigenvalue Problem (3.3)

We now make use of the intermediate Evans function $D(\lambda, \varepsilon)$ to find solutions of the original eigenvalue problem. First these solutions must be well defined. Each of them must be unique, complex analytic. In addition, the asymptotic behavior as $|z| \to \infty$, as $|\lambda| \to +\infty$ and as $\varepsilon \to 0$ must be clearly studied. By using the method of variation of parameter to solve the nonhomogeneous differential equations, we can find linearly independent, complex analytic solutions $\psi_i(\lambda, \varepsilon, z)$, where $i = 1, 2, \cdots n$, such that $\lim\limits_{z \to +\infty} \psi_i(\lambda, \varepsilon, z) = 0$.

All other complex analytic solutions can be written as the linear combination of these analytic solutions. The details are given below.

For any smooth scalar function $c(\lambda, \varepsilon, z)$ and $i = 1, 2, \cdots, n$, $c(\lambda, \varepsilon, z)\varphi_i(\lambda, \varepsilon, z)$ will not be a solution of (3.3). One must use the method of variation of parameter, namely, employing the fundamental matrix $(\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z), \cdots, \varphi_n(\lambda, \varepsilon, z))$ to construct a solution of (3.3). Suppose that

$$\begin{aligned}\psi(\lambda, \varepsilon, z) &= c_1(\lambda, \varepsilon, z)\varphi_1(\lambda, \varepsilon, z) + c_2(\lambda, \varepsilon, z)\varphi_2(\lambda, \varepsilon, z) + \cdots + c_n(\lambda, \varepsilon, z)\varphi_n(\lambda, \varepsilon, z) \\ &= (\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z), \cdots, \varphi_n(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon, z) \\ c_2(\lambda, \varepsilon, z) \\ \cdots \\ c_n(\lambda, \varepsilon, z) \end{pmatrix}\end{aligned}$$

is a solution of the nonhomogeneous system (3.3). By product rule, we have

$$\begin{aligned}& v(\varepsilon)(\varphi_1, \varphi_2, \cdots, \varphi_n)\frac{\partial}{\partial z}\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} + v(\varepsilon)\frac{\partial}{\partial z}(\varphi_1, \varphi_2, \cdots, \varphi_n)\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} \\ & + [\lambda I + A(\varepsilon, z)](\varphi_1, \varphi_2, \cdots, \varphi_n)\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = N(\lambda, \varepsilon, z).\end{aligned}$$

Notice that

$$v(\varepsilon)\frac{\partial}{\partial z}(\varphi_1, \varphi_2, \cdots, \varphi_n) + [\lambda I + A(\varepsilon, z)](\varphi_1, \varphi_2, \cdots, \varphi_n) = 0.$$

Therefore we get

$$v(\varepsilon)(\varphi_1, \varphi_2, \cdots, \varphi_n)\frac{\partial}{\partial z}\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = N(\lambda, \varepsilon, z).$$

This is a first order linear system of differential equations. We need to solve this system for $(c_1(\lambda, \varepsilon, z), c_2(\lambda, \varepsilon, z), \cdots, c_n(\lambda, \varepsilon, z))$. By using the properties of the intermediate Evans function $D(\lambda, \varepsilon) = S(\lambda, \varepsilon, z)^T \Phi(\lambda, \varepsilon, z)$, it is easy to get

$$\frac{\partial}{\partial z}\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = \frac{1}{v(\varepsilon)D(\lambda, \varepsilon)}\left(S_1(\lambda, \varepsilon, z), S_2(\lambda, \varepsilon, z), \cdots, S_n(\lambda, \varepsilon, z)\right)^T N(\lambda, \varepsilon, z).$$

Solving these equations yields

$$\begin{pmatrix} c_1(\lambda,\varepsilon,z) \\ c_2(\lambda,\varepsilon,z) \\ \cdots \\ c_n(\lambda,\varepsilon,z) \end{pmatrix} = \begin{pmatrix} c_1(\lambda,\varepsilon) \\ c_2(\lambda,\varepsilon) \\ \cdots \\ c_n(\lambda,\varepsilon) \end{pmatrix}$$
$$+ \frac{1}{v(\varepsilon)D(\lambda,\varepsilon)} \int_{-\infty}^{z} \left(S_1(\lambda,\varepsilon,s), S_2(\lambda,\varepsilon,s), \cdots, S_n(\lambda,\varepsilon,s)\right)^T N(\lambda,\varepsilon,s)ds,$$

where $(c_1(\lambda,\varepsilon), c_2(\lambda,\varepsilon), \cdots, c_n(\lambda,\varepsilon))$ is a complex vector-valued function of λ and ε to be determined later. Therefore we obtain the possibly analytic solutions $\psi(\lambda,\varepsilon,z)$.

Lemma 5.1. *Suppose that $\varphi_1, \varphi_2, \cdots, \varphi_n$ are the solutions of the linear homogeneous equations. Then the solutions of the eigenvalue problem* (3.3) *is given by*

$$\begin{aligned}
\psi(\lambda,\varepsilon,z) &= (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z)) \begin{pmatrix} c_1(\lambda,\varepsilon,z) \\ c_2(\lambda,\varepsilon,z) \\ \cdots \\ c_n(\lambda,\varepsilon,z) \end{pmatrix} \\
&= (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z)) \begin{pmatrix} c_1(\lambda,\varepsilon) \\ c_2(\lambda,\varepsilon) \\ \cdots \\ c_n(\lambda,\varepsilon) \end{pmatrix} \\
&\quad + \frac{1}{v(\varepsilon)D(\lambda,\varepsilon)} (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z)) \\
&\quad \times \int_{-\infty}^{z} \left(S_1(\lambda,\varepsilon,s), S_2(\lambda,\varepsilon,s), \cdots, S_n(\lambda,\varepsilon,s)\right)^T N(\lambda,\varepsilon,s)ds.
\end{aligned}$$

If $\psi^i(\lambda,\varepsilon,z_{ij}), j = 1,2,\cdots,m_i, i = 1,2,\cdots,n$, and $(c_1(\lambda,\varepsilon), c_2(\lambda,\varepsilon), \cdots, c_n(\lambda,\varepsilon))$ are all complex analytic functions in λ for all fixed ε, then $\psi(\lambda,\varepsilon,z)$ is also analytic in λ, for all fixed z and ε. Let us investigate whether or not the solution $\psi(\lambda,\varepsilon,z)$ is bounded on $(-\infty,\infty)$. Clearly we have the asymptotic behaviors if $K_1, K_2, \cdots, K_n$ converge to zero sufficiently fast as $|z| \to \infty$. This can be attained by assuming ρ_i to be large enough, for $i = 1,2,\cdots,n$.

Lemma 5.2. *Under the above assumptions, there hold the following limits*

$$\lim_{z\to\pm\infty} (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z))$$
$$\times \int_{-\infty}^{z} \left(S_1(\lambda,\varepsilon,s), S_2(\lambda,\varepsilon,s), \cdots, S_n(\lambda,\varepsilon,s)\right)^T N(\lambda,\varepsilon,s)ds = 0,$$

$$\lim_{z\to+\infty} (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z)) \begin{pmatrix} c_1(\lambda,\varepsilon) \\ c_2(\lambda,\varepsilon) \\ \cdots \\ c_n(\lambda,\varepsilon) \end{pmatrix} = 0,$$

$$\lim_{z\to-\infty} \left\| (\varphi_1(\lambda,\varepsilon,z), \varphi_2(\lambda,\varepsilon,z), \cdots, \varphi_n(\lambda,\varepsilon,z)) \begin{pmatrix} c_1(\lambda,\varepsilon) \\ c_2(\lambda,\varepsilon) \\ \cdots \\ c_n(\lambda,\varepsilon) \end{pmatrix} \right\| = +\infty,$$

if $(c_1(\lambda,\varepsilon), c_2(\lambda,\varepsilon), \cdots, c_n(\lambda,\varepsilon)) \neq (0,0,\cdots,0)$.

It is therefore clear to see that

$$\lim_{z\to+\infty}(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z))\begin{pmatrix}c_1(\lambda,\varepsilon,z)\\c_2(\lambda,\varepsilon,z)\\\cdots\\c_n(\lambda,\varepsilon,z)\end{pmatrix}=0,$$

$$\lim_{z\to-\infty}\left\|(\varphi_1(\lambda,\varepsilon,z),\varphi_2(\lambda,\varepsilon,z),\cdots,\varphi_n(\lambda,\varepsilon,z))\begin{pmatrix}c_1(\lambda,\varepsilon,z)\\c_2(\lambda,\varepsilon,z)\\\cdots\\c_n(\lambda,\varepsilon,z)\end{pmatrix}\right\|=+\infty,$$

if $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))\neq(0,0,\cdots,0)$. On the other hand the last limit is equal to zero if $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))=(0,0,\cdots,0)$. Therefore the vector-valued function

$$\psi(\lambda,\varepsilon,z)=c_1(\lambda,\varepsilon,z)\varphi_1(\lambda,\varepsilon,z)+c_2(\lambda,\varepsilon,z)\varphi_2(\lambda,\varepsilon,z)+\cdots+c_n(\lambda,\varepsilon,z)\varphi_n(\lambda,\varepsilon,z)$$

is a good candidate of eigenfunctions of the eigenvalue problem (3.3). The complex number $\lambda\in\Omega(\varepsilon)$ is an eigenvalue of $L(\varepsilon)$ if and only if

$$(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))=(0,0,\cdots,0).$$

Motivated by the work of Alexander, Gardner and Jones[1], Evans[14], Pego and Weinstein[18], we would like to relate the eigenvalues of the operator $L(\varepsilon)$ to zeroes of some analytic function. If one can define such a function and investigate its zeroes, then the necessary information of wave stability would be obtained. Because of the presence of the nonlocal integral terms, the Evans function for the equation (3.3) must be defined in an unusual way. We need n linearly independent, complex analytic solutions which approach zero exponentially fast as $z\to+\infty$. The above vector $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))$ is not arbitrary, it will be determined by the compatibility condition of $\psi(\lambda,\varepsilon,z)$ at $z=z_{ij},j=1,2,\cdots,m_i,i=1,2,\cdots,n$. It is by solving n linearly independent analytic vectors $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))$ that we can find linearly independent, complex analytic solutions $\psi_i(\lambda,\varepsilon,z),i=1,2,\cdots,n$. To solve the vectors $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))$, we start from the following k-component equations

$$\begin{aligned}\psi^k(\lambda,\varepsilon,z)&=(\varphi_1^k(\lambda,\varepsilon,z),\varphi_2^k(\lambda,\varepsilon,z),\cdots,\varphi_n^k(\lambda,\varepsilon,z))\begin{pmatrix}c_1(\lambda,\varepsilon)\\c_2(\lambda,\varepsilon)\\\cdots\\c_n(\lambda,\varepsilon)\end{pmatrix}\\&\quad+\frac{1}{v(\varepsilon)D(\lambda,\varepsilon)}(\varphi_1^k(\lambda,\varepsilon,z),\varphi_2^k(\lambda,\varepsilon,z),\cdots,\varphi_n^k(\lambda,\varepsilon,z))\\&\quad\times\int_{-\infty}^{z}(S_1(\lambda,\varepsilon,s),S_2(\lambda,\varepsilon,s),\cdots,S_n(\lambda,\varepsilon,s))^T N(\lambda,\varepsilon,s)ds.\end{aligned}$$

Substituting z_{kl} for z, we get

$$\begin{aligned}\psi^k(\lambda,\varepsilon,z_{kl})&=\left(\varphi_1^k(\lambda,\varepsilon,z_{kl}),\varphi_2^k(\lambda,\varepsilon,z_{kl}),\cdots,\varphi_n^k(\lambda,\varepsilon,z_{kl})\right)\begin{pmatrix}c_1(\lambda,\varepsilon)\\c_2(\lambda,\varepsilon)\\\cdots\\c_n(\lambda,\varepsilon)\end{pmatrix}\\&\quad+\frac{1}{v(\varepsilon)D(\lambda,\varepsilon)}\left(\varphi_1^k(\lambda,\varepsilon,z_{kl}),\varphi_2^k(\lambda,\varepsilon,z_{kl}),\cdots,\varphi_n^k(\lambda,\varepsilon,z_{kl})\right)\\&\quad\times\int_{-\infty}^{z_{kl}}(S_1(\lambda,\varepsilon,z),S_2(\lambda,\varepsilon,z),\cdots,S_n(\lambda,\varepsilon,z))^T N(\lambda,\varepsilon,z)dz,\end{aligned}$$

where $l = 1, 2, \cdots, m_k, k = 1, 2, \cdots, n$. We need a compact form of equations. Arranging the complex numbers $\psi^k(\lambda, \varepsilon, z_{kl})$ in the same way as we did for $B(\lambda, \varepsilon)$, we obtain

$$B(\lambda, \varepsilon) = P(\lambda, \varepsilon) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \\ \cdots \\ c_n(\lambda, \varepsilon) \end{pmatrix} + Q(\lambda, \varepsilon)B(\lambda, \varepsilon),$$

where $P(\lambda, \varepsilon)$ is an $n \times n$ analytic matrix in λ and each row is

$$\varphi_1^k(\lambda, \varepsilon, z_{kl}), \varphi_2^k(\lambda, \varepsilon, z_{kl}), \cdots, \varphi_n^k(\lambda, \varepsilon, z_{kl}),\ 1 \le l \le m_k,\ 1 \le k \le n.$$

Similarly $Q(\lambda, \varepsilon)$ is an $n \times n$ analytic matrix in λ and each row is

$$\frac{1}{v(\varepsilon)D(\lambda, \varepsilon)} \left(\varphi_1^k(\lambda, \varepsilon, z_{kl}), \varphi_2^k(\lambda, \varepsilon, z_{kl}), \cdots, \varphi_n^k(\lambda, \varepsilon, z_{kl})\right)$$
$$\times \int_{-\infty}^{z_{kl}} (S_1(\lambda, \varepsilon, z), S_2(\lambda, \varepsilon, z), \cdots, S_n(\lambda, \varepsilon, z))^T G(z)dz.$$

Notice that

$$\left(\varphi_1^k(\lambda, \varepsilon, z_{kl}), \varphi_2^k(\lambda, \varepsilon, z_{kl}), \cdots, \varphi_n^k(\lambda, \varepsilon, z_{kl})\right) (S_1(\lambda, \varepsilon, z), S_2(\lambda, \varepsilon, z), \cdots, S_n(\lambda, \varepsilon, z))^T$$

is a solution of the adjoint system (4.4). Hence it converges to zero exponentially fast as $s \to -\infty$, and the above integral is absolutely convergent. Define another $n \times n$ analytic matrix in λ by

$$M(\lambda, \varepsilon) = I - Q(\lambda, \varepsilon). \tag{5.1}$$

Thus in matrix notation we get

$$P(\lambda, \varepsilon) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \\ \cdots \\ c_n(\lambda, \varepsilon) \end{pmatrix} = M(\lambda, \varepsilon)B(\lambda, \varepsilon). \tag{5.2}$$

Since $\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z), \cdots, \varphi_n(\lambda, \varepsilon, z)$ grow at different rate, the determinant $\det P(\lambda, \varepsilon) \neq 0$, a more rigorous proof can be established by using projectivized equations for each individual system. One can easily solve the above equations to obtain the vector $(c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon), \cdots, c_n(\lambda, \varepsilon))$, provided the prescribed complex vectors $B(\lambda, \varepsilon)$ are given. Because we are investigating analytic solutions, we would choose $B(\lambda, \varepsilon)$ to be analytic functions of λ and smooth functions of ε. Hence $(c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon), \cdots, c_n(\lambda, \varepsilon))$ would be also analytic. This in turn shows that $\psi(\lambda, \varepsilon, z)$ is complex analytic in λ, for fixed $z \in R$ and $\varepsilon > 0$. Letting

$$B(\lambda, \varepsilon) = (1, 0, \cdots, 0)^T, (0, 1, 0, \cdots, 0)^T, \cdots, (0, \cdots, 0, 1)^T,$$

respectively, we obtain n linearly independent, complex analytic solutions $\psi_i(\lambda, \varepsilon, z)$. Any other choice of $B(\lambda, \varepsilon)$ would also give a well defined solution of (3.3). Actually all the solutions of (3.3) with $\lim\limits_{z \to +\infty} \psi(\lambda, \varepsilon, z) = 0$ can be written in the form $\psi(\lambda, \varepsilon, z) = c_1(\lambda, \varepsilon, z)\varphi_1(\lambda, \varepsilon, z) + c_2(\lambda, \varepsilon, z)\varphi_2(\lambda, \varepsilon, z) + \cdots + c_n(\lambda, \varepsilon, z)\varphi_n(\lambda, \varepsilon, z)$. By the way, any nontrivial solution of the original eigenvalue problem can not vanish at $z = z_{ij}, j = 1, 2, \cdots, m_i, i = 1, 2, \cdots, n$ simultaneously, i.e. $B(\lambda, \varepsilon) \neq 0$. We now define a complex analytic function, called the Evans function, by

$$E(\lambda, \varepsilon) = \det M(\lambda, \varepsilon). \tag{5.3}$$

This function is well defined. Since $M(\lambda, \varepsilon)$ is analytic in λ for fixed $\varepsilon > 0$ and is real if λ is real, the Evans function $E(\lambda, \varepsilon)$ has the same properties. If $\lambda \in \Omega(\varepsilon)$ is such that

$E(\lambda,\varepsilon)=0$, then there must be some nontrivial complex vector $B(\lambda,\varepsilon)$, analytic in λ and smooth in ε, such that

$$P(\lambda,\varepsilon)\begin{pmatrix} c_1(\lambda,\varepsilon)\\ c_2(\lambda,\varepsilon)\\ \cdots \\ c_n(\lambda,\varepsilon)\end{pmatrix} = M(\lambda,\varepsilon)B(\lambda,\varepsilon)=0,$$

hence $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))=(0,0,\cdots,0)$ and λ is an eigenvalue. On the other hand, if λ is an eigenvalue, then $(c_1(\lambda,\varepsilon),c_2(\lambda,\varepsilon),\cdots,c_n(\lambda,\varepsilon))=(0,0,\cdots,0)$ and

$$M(\lambda,\varepsilon)B(\lambda,\varepsilon)=0.$$

This implies that $E(\lambda,\varepsilon)=0$, since $B(\lambda,\varepsilon)\neq 0$. The above analysis shows that $\lambda\in\Omega(\varepsilon)$ is an eigenvalue of $L(\varepsilon)$ if and only if $E(\lambda,\varepsilon)=0$.

Lemma 5.3. *The complex number $\lambda\in\Omega(\varepsilon)$ is an eigenvalue of the operator $L(\varepsilon)$ if and only if it is a zero of the Evans function*: $E(\lambda,\varepsilon)=0$.

Obviously by projectivized equations, the rank of $M(\lambda,\varepsilon)=n-1$ if $E(\lambda,\varepsilon)=0$; and the rank of $M(\lambda,\varepsilon)=n$ if $E(\lambda,\varepsilon)\neq 0$. Define the null spaces $N(\lambda,\varepsilon)=\{v\in C^n : M(\lambda,\varepsilon)v=0\}$. Thus the dimension $\dim N(\lambda,\varepsilon)=1$ if $E(\lambda,\varepsilon)=0$; and $\dim N(\lambda,\varepsilon)=0$ if $E(\lambda,\varepsilon)\neq 0$. Consequently we have the geometric multiplicity or the dimension $\dim N(L(\varepsilon)-\lambda I)=1$ if $\lambda\in\Omega(\varepsilon)$ is an eigenvalue and $\dim N(L(\varepsilon)-\lambda I)=0$ if $\lambda\in\Omega(\varepsilon)$ is not an eigenvalue. In other words, for each fixed $\lambda\in\Omega(\varepsilon)$, there can only be at most one eigenfunction corresponding to it. Actually we get the following more accurate result.

Lemma 5.4. *Suppose that $\lambda_0\in\Omega(\varepsilon)$ is an eigenvalue of the operator $L(\varepsilon)$. Then the algebraic multiplicity of λ_0 as an eigenvalue is equal to the order of the zero λ_0 of the Evans function $E(\lambda,\varepsilon)$.*

Proof. Suppose that $\lambda_0\in\Omega(\varepsilon)$ is an eigenvalue of $L(\varepsilon)$ and that

$$\psi(\lambda_0,\varepsilon,z),\frac{\partial\psi}{\partial\lambda}(\lambda_0,\varepsilon,z),\frac{\partial^2\psi}{\partial\lambda^2}(\lambda_0,\varepsilon,z),\cdots,\frac{\partial^m\psi}{\partial\lambda^m}(\lambda_0,\varepsilon,z)$$

are bounded on R, while $\frac{\partial^{m+1}\psi}{\partial\lambda^{m+1}}(\lambda_0,\varepsilon,z)$ is not bounded on R, for some integer $m\geq 1$. Then it is not difficult to show that

$$E(\lambda_0,\varepsilon)=\frac{\partial E}{\partial\lambda}(\lambda_0,\varepsilon)=\frac{\partial^2 E}{\partial\lambda^2}(\lambda_0,\varepsilon)=\cdots=\frac{\partial^m E}{\partial\lambda^m}(\lambda_0,\varepsilon)=0\neq\frac{\partial^{m+1}E}{\partial\lambda^{m+1}}(\lambda_0,\varepsilon).$$

Namely the order of the zero λ_0 of the Evans function $E(\lambda,\varepsilon)$ is equal to $m+1$. The proof of this lemma is very long but not very relevant. The detail is given in a later paper. We give the outline here. First we define $m+1$ continuous linear functionals $a_k(\lambda_0,\varepsilon,\psi)$ from the Banach space $X=BC(R,C^n)$ to a subspace of X spanned by the above vector-valued bounded functions. Then we define a projection operator

$$S_{\lambda_0}: X\to \text{ span }\Big\{\psi(\lambda_0,\varepsilon,z),\frac{\partial}{\partial\lambda}\psi(\lambda_0,\varepsilon,z),\cdots,\frac{\partial^m}{\partial\lambda^m}\psi(\lambda_0,\varepsilon,z)\Big\},$$

by $S_{\lambda_0}\psi=\sum\limits_{k=0}^{m}a_k(\lambda_0,\varepsilon,\psi)\frac{\partial^k}{\partial\lambda^k}\psi(\lambda_0,\varepsilon,z)$.

Then using functional analysis we can prove that for all $\psi\in X$, there holds

$$\frac{1}{2\pi i}\int_\Gamma[(z+\lambda_0)I-L(\varepsilon)]^{-1}\psi dz=S_{\lambda_0}\psi.$$

Finally for all $\psi\in X$, since $\psi=S_{\lambda_0}\psi+(I-S_{\lambda_0})\psi$, we have the result

$$\frac{1}{2\pi}\int_\Gamma[(z+\lambda_0)I-L(\varepsilon)]^{-1}\psi dz=S_{\lambda_0}\psi=\sum_{k=0}^{m}a_k(\lambda_0,\varepsilon,\psi)\frac{\partial^k}{\partial\lambda^k}\psi(\lambda_0,\varepsilon,z).$$

This implies that if $\lambda_0 \in \sigma(L(\varepsilon))\bigcap\Omega$ is a zero of order $m+1>0$ of $E(\lambda,\varepsilon)$, then $\dim S_{\lambda_0}X = m+1$, where

$$S_{\lambda_0} = \frac{1}{2\pi i}\int_\Gamma [(z+\lambda_0)I - L(\varepsilon)]^{-1}dz,$$

where Γ is a sufficiently small circle around the origin.

Lemma 5.5. *$E(\lambda,\varepsilon)$ is real valued when λ is a real number.*

The proof is very easy and we omit it. To find the eigenvalues of the operator $L(\varepsilon)$ in the domain $\Omega(\varepsilon)$, we need to investigate the properties of the Evans function. This requires us to study the analytic functions $P(\lambda,\varepsilon)$ and $Q(\lambda,\varepsilon)$ and find their asymptotic behaviors as $|\lambda|\to\infty$.

Let us investigate the asymtotic behavior of $E(\lambda,\varepsilon)$ as $|\lambda|\to\infty$.

Proposition 5.1. *Let the bounded smooth function f satisfy $f' \in L^1(a,b)$, where (a,b) is a finite interval or $a=-\infty$ or $b=\infty$. Then we have the limit*

$$\lim_{|\xi|\to\infty}\int_a^b \exp(ix\xi)f(x)dx = 0. \tag{5.4}$$

Proof. By integration by parts, one can complete the proof.

By Lebesgue's dominated convergence theorem and Proposition 5.1, we have the asymptotic behavior as $|\lambda|\to\infty$.

Lemma 5.6. *There hold the limits*

$$\lim_{\lambda\in\Omega(\varepsilon),|\lambda|\to\infty} Q(\lambda,\varepsilon) = 0, \quad \lim_{\lambda\in\Omega(\varepsilon),|\lambda|\to\infty} E(\lambda,\varepsilon) = 1, \tag{5.5}$$

uniformly with respect to ε.

Proof. By Lebesgue's dominated convergence theorem one can prove this lemma easily.

Thus there exists a positive constant M, independent of λ,ε and z, such that every $\lambda\in\Omega(\varepsilon)$ with $|\lambda|>M$ is not an eigenvalue of the operator $L(\varepsilon)$. One must search for eigenvalues inside the circle $|\lambda|=M$. If there is a vertical line situated at $\text{Re}\lambda = -k$ for some positive constant k, such that all the nontrivial eigenvalues of $L(\varepsilon)$ are located to the left of this vertical line and $\lambda=0$ is a simple eigenvalue, then the wave solutions are asymptotically stable.

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