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INSTANTANEOUS SHRINKING AND LOCALIZATION OF FUNCTIONS IN $\Upsilon_\lambda(m,p,q,N)$ AND THEIR APPLICATIONS**

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Abstract

The aim of this paper is to discuss the instantaneous shrinking and localization of the support of functions in $\Upsilon_\lambda(m,p,q,N)$ and their applications to some nonlinear parabolic equations including the porous medium equation $u_t = \Delta u^m - u^q,\ m > 0,\ q > 0$ and the p-Laplace equation $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q,\ p > 1,\ q > 0$. In particular, as an application of the results, the necessary and sufficient condition for the solutions of the above p-Laplace equation with nonnegative finite Borel measures as initial conditions to have the instantaneous shrinking property of the support is obtained. This is an answer to an open problem posed by R. Kersner and A. Shishkov.

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§1. Introduction

As well known, the solutions of semilinear parabolic equations $u_t = \Delta u - u^q,\ 0 < q < 1$ have the ISS (the instantaneous shrinking of the support) property (see [1–5]).

Such property is defined as follows.

Definition 1.1. *A function $u \in L^\infty_{\mathrm{loc}}(Q)$ is said to have the ISS property, if for any $\tau > 0$, there exists a positive number $R = R(\tau)$ such that*

$$u(x,t) = 0$$

for a. e. $(x,t) \in (\mathbb{R}^N \setminus B_R) \times (\tau, +\infty)$, where $\mathbb{R}^N$ is the N-dimensional Euclidean space, and $Q \equiv \mathbb{R}^N \times (0, +\infty)$.

Up to the present, there have been many papers which are devoted to the generations to other kinds of equations (see [6–11]). One of the typical examples of them is the porous medium equation

$$u_t = \Delta u^m - u^q, \qquad m > 0, \quad q > 0 \tag{1.1}$$

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(see [6, 8, 9]). Another typical example is the p-Laplace equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q, \qquad p > 1, \quad q > 0 \tag{1.2}$$

(see [10]).

We easily check that the solutions of (1.1) and (1.2) belong to $\mathrm{Y}_\lambda(m, p, q, N)$ (see Section 10) which is defined as follows.

Definition 1.2. *Let $m > 0$, $p > 1$, $q > 0$ and $\lambda \geq 1$. A nonnegative function $u \in Y_\lambda(m, p, q, N)$ if and only if $u \in L^\infty(0, +\infty; L^\lambda(\mathbb{R}^N))$ satisfies the following conditions* (a)–(c):

(a) *For any $\delta \in (0, 1)$, we have*

$$u \in L^\infty(\delta, +\infty; L^\infty(\mathbb{R}^N)), \quad |\nabla u| \in L^p_{\text{loc}}(0, +\infty; L^p_{\text{loc}}(\mathbb{R}^N)).$$

(b) *There exists a positive number Λ_1 such that, for all $\alpha \in (0, +\infty)$ and all $\phi \in C^\infty(0, +\infty;\, C_0^\infty\ (\mathbb{R}^N))$ with $\phi \geq 0$, we have*

$$\begin{aligned}
&\operatorname*{ess\ sup}_{s<t<T} \int_{\mathbb{R}^N} \phi^p(x,t) u^{1+\alpha}(x,t)dx + \alpha_p \sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\phi u^{[m(p-1)+\alpha]/p}\}|^p dxdt \\
&\leq \Lambda_1 \Big\{ \iint_{Q_{s,T}} [\phi^{p-1}|\phi_t| u^{\alpha+1} + \alpha^{1-p}(1+\alpha)|\nabla\phi|^p u^{m(p-1)+\alpha} - (\alpha+1)\phi^p u^{q+\alpha}] dxdt \\
&\quad + \int_{\mathbb{R}^N} \phi^p(x,s) u^{\alpha+1}(x,s) dx \Big\}
\end{aligned}$$

for a. e. $s \in (0, T)$ with $0 < T < +\infty$, where $\alpha_p = \alpha(1+\alpha)[m(p-1)+\alpha]^{-p}$, $Q_{s,T}(u > \delta) = \{(x,t) \in Q_{s,T} : u(x,t) > \delta\}$, and $Q_{s,T} = \mathbb{R}^N \times (s, T)$ for $0 < s < T < +\infty$.

(c) *There exists a positive number Λ_2 such that, for all $\beta \in (0, \min\{1; m(p-1); q\})$ and all $\phi \in C^\infty(0, +\infty;\, C_0^\infty\ (\mathbb{R}^N))$ with $\phi \geq 0$, we have*

$$\begin{aligned}
&\operatorname*{ess\ sup}_{s<t<T} \int_{\mathbb{R}^N} \phi^p(x,t) u^{1-\beta}(x,t)dx + \beta_p \sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\phi u^{[m(p-1)-\beta]/p}\}|^p dxdt \\
&\leq \Lambda_2 \Big\{ \iint_{Q_{s,T}} [\phi^{p-1}|\phi_t| u^{1-\beta} + \beta^{1-p}(1-\beta)|\nabla\phi|^p u^{m(p-1)-\beta} + (1-\beta)\phi^p u^{q-\beta}] dxdt \\
&\quad + \int_{\mathbb{R}^N} \phi^p(x,T) u^{1-\beta}(x,T) dx \Big\}
\end{aligned}$$

for a. e. $T \in (s, +\infty)$ with $s > 0$, where $\beta_p = \beta(1-\beta)[m(p-1)-\beta]^{-p}$.

Our main results are the following theorems and their applications.

Theorem 1.1. *Assume that $u \in Y_\lambda(m, p, q, N)$, $\theta > 0$ and $0 < q < 1$. If $q < m(p-1)$, then the function u has the ISS property, where $\theta = m(p-1) - 1 + p\lambda/N$.*

Theorem 1.2. *Assume that $u \in Y_\lambda(m, p, q, N)$, $\theta > 0$ and $0 < q < 1$. If $q \geq m(p-1)$, then the function u has no the ISS property.*

Remark 1.1. If $u \in \mathrm{Y}_\lambda(m, p, q, N)$ with $\theta > 0$ and $0 < q < 1$, then $q < m(p-1)$ is the necessary and sufficient condition for the function u to have the ISS property.

The above conclusions can be applied to some nonlinear partial differential equations including (1.1) and (1.2). For example, let us see Remark 1.2 as follows.

Remark 1.2. Assume that u is a solution of the Cauchy problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q, & p > 1, \quad 1 > q > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Then we have the following conclusions (i) and (ii).

(i) If u_0 is a non-zero nonnegative finite Borel measure in $\mathbb{R}^N$, then $u \in \mathrm{Y}_1(1, p, q, N)$ (see Proposition 10.1). If $p - 2 + p/N > 0$ and $0 < q < 1$, then Theorem 1.1 and Theorem

1.2 imply that the solution u of the Cauchy problem (1.3) has the ISS property if and only if $q < p-1$. This is a very interesting answer to an open problem posed by R. Kersner and A. Shishkov in [10].

(ii) If $u_0 \in L^\lambda(\mathbb{R}^N)(1 \le \lambda < +\infty)$ is a non-zero nonnegative function, then $u \in \mathrm{Y}_\lambda(1,p,q,N)$ (see Proposition 10.2). If $p-2+p\lambda/N > 0$ and $0 < q < 1$, then, by Theorem 1.1 and Theorem 1.2, the solution u of the Cauchy problem (1.3) has the ISS property if and only if $q < p-1$. Clearly, Theorem 3 proved by R. Kersner and A. Shishkov in [10] is extended.

Remark 1.3. Similarly to Remark 1.2, Theorem 1.1 and Theorem 1.2 also can be applied to the equation (1.1). Here we omit the details.

All results as above hold only for $0 < q < 1$. In fact, in the case $q \ge 1$, we have the following Theorems.

Theorem 1.3. *Let $u \in Y_\lambda(m,p,q,N)$. If $\min\{m(p-1);q\} \ge 1$, then*

$$|\operatorname{supp} u(\cdot,s)| \le |\operatorname{supp} u(\cdot,t)| \qquad \textit{for a. e.} \quad s,t \quad \textit{with} \quad 0<s<t.$$

Remark 1.4. Theorem 1.3 implies that, if $q \ge 1$, then the function $u \in \mathrm{Y}_\lambda(m,p,q,N)$ has no the ISS property in general.

But, for $q \ge 1$, we obtain an interesting phenomenon called the LOC (Localization) property which is defined as follows.

Definition 1.3. *A function $u \in L^\infty_{\mathrm{loc}}(Q)$ is said to have the LOC property, if there exists a positive number L such that $u(x,t)=0$ for a. e. $(x,t) \in (\mathbb{R}^N \setminus B_L) \times (0,+\infty)$.*

In addition, we have

Theorem 1.4. *Assume that $\theta > 0$, $q \ge 1$, $u \in Y_\lambda(m,p,q,N)$, and*

$$u(x,t) \to 0 \quad \textit{a. e. in} \quad \mathbb{R}^N \setminus B_{R_0} \quad \textit{as} \quad t \to 0^+ \tag{1.4}$$

for some positive constant R_0. If $q < m(p-1)$, then the function u has the LOC property.

Theorem 1.5. *Assume that $\theta > 0$ and $u \in Y_\lambda(m,p,q,N)$ is a non-zero nonnegative function. If $q > m(p-1) > 1$, then the function u has no the LOC property.*

Remark 1.5. The conclusions in Theorem 1.4 and Theorem 1.5 are optimal.

Such LOC property for some nonlinear parabolic equations including (1.1) and (1.2) has obtained by R. Kersner in [12] and Yuan Hongjun in [13, 14].

Remark 1.6. Similarly to Remark 1.2, as an application of Theorem 1.4 and Theorem 1.5, one can in fact extend some results in [12–14]. We omit the details here.

In order to prove Theorem 1.1 and Theorem 1.4 we need the following Theorem 1.6 and Theorem 1.7.

Theorem 1.6. *Assume that $u \in Y_\lambda(m,p,q,N)$. If $\theta > 0$, then*

$$u(x,t) \le CM_\lambda^{p/N\theta}\left(\frac{1}{t}\right)^{1/\theta} \quad \textit{for a. e.} \quad (x,t) \in Q.$$

Here and thoughout this paper, C stands for a positive constant depending only on m, p, q, N, λ, Λ_1 and Λ_2; and γ stands for a positive constant depending only on m, p, q, λ, Λ_1, Λ_2 and M_λ; and $M_\lambda = \operatorname{ess}\sup\limits_{0<t<+\infty} \int_{\mathbb{R}^N} u^\lambda(x,t)dx$.

Theorem 1.7. *Assume that $u \in Y_\lambda(m,p,q,N)$ and $\theta > 0$. If $0 < q < min\{1; m(p-1)\}$, then there exists a time T_0 such that*

$$u(x,t) = 0 \quad \textit{for a. e.} \quad (x,t) \in \mathbb{R}^N \times (T_0,+\infty).$$

Remark 1.7. Theorem 1.7 implies that $u \in \mathrm{Y}_\lambda(m,p,q,N)$ has the extinction property, providied that $0 < q < \min\{1; m(p-1)\}$. Similarly to Remark 1.2, Theorem 1.7 can be applied to some nonlinear parabolic equations including (1.1) and (1.2). The details are omitted here.

Remark 1.8. Our method in this paper can be applied to study other problem including finite propagation of perturbations for nonlinear parabolic equations. The details are also omitted here.

Remark 1.9. The conclusion in Theorem 1.5 seems to be true even if $q = m(p-1)$. But we are not able to prove it yet. This is an interesting problem.

The proofs of Theorem 1.6 and Theorem 1.7 are completed in Sections 3–4, respectively. In the process of proving Theorem 1.6 and Theorem 1.7 we need some fundamental lemmas in Section 2. Using Theorem 1.6 and Theorem 1.7 we shall prove Theorem 1.1 in Section 5. The proofs of Theorem 1.2–Theorem 1.5 are given in Sections 6–9, respectively. In the last Section 10, we shall prove that the solutions of (1.1) and (1.2) belong to $\mathrm{Y}_\lambda(m,p,q,N)$.

§2. Fundamental Lemmas

Lemma 2.1. *If $h : \mathbb{R}^l \mapsto [0,+\infty)$ is a nonnegative bounded function on $[a_1,b_1]\times[a_2,b_2]\times \cdots \times [a_l,b_l]$ such that*

$$h(\rho_1,\rho_2,\cdots,\rho_l) \le ah(R_1,R_2,\cdots,R_l) + \sum_{i=1}^{l}\Big(\frac{A_i}{(R_i-\rho_i)^{\alpha_i}} + B_i\Big) \tag{2.1}$$

for all ρ_i, R_i with $a_i \le \rho_i < R_i \le b_i$ $(i = 1,2,\cdots,l)$, where a, α_i $(i = 1,2,\cdots,l)$, A_i $(i = 1,2,\cdots,l)$ and B_i $(i = 1,2,\cdots,l)$ are positive constants, and $a < 1$, then

$$h(\rho_1,\rho_2,\cdots,\rho_l) \le C\sum_{i=1}^{l}\Big(\frac{A_i}{(R_i-\rho_i)^{\alpha_i}} + B_i\Big) \tag{2.2}$$

for all ρ_i, R_i with $a_i \le \rho_i < R_i \le b_i$ $(i = 1,2,\cdots,l)$, where C is a positive constant depending only on α_i $(i = 1,2,\cdots,l)$ and a.

Remark 2.1. The proof is similar to that given for the case $l = 1$ in [15]. Therefore we omit it.

Lemma 2.2. *Let y_n $(n = 0,1,2,\cdots)$ be a sequence of real numbers satisfying the following inequalities $0 \le y_{n+1} \le cb^n y_n^{1+\sigma}$ for $n = 0,1,2,\cdots$, where $c > 0$, $\sigma > 0$ and $b > 1$. Then*

$$y_n \le c^{[(1+\sigma)^n-1]/\sigma} b^{[(1+\sigma)^n-1-n\sigma]/\sigma^2} y_0^{(1+\sigma)^n}$$

for $n = 0,1,2,\cdots$. In particular, we have the following conclusions.

(i) *The following inequality holds:*

$$\overline{\lim_{n\to+\infty}} y_n^{1/(1+\sigma)^n} \le c^{1/\sigma} b^{1/\sigma^2} y_0.$$

(ii) *If $y_0 < c^{-1/\sigma} b^{-1/\sigma^2}$, then $\lim\limits_{n\to+\infty} y_n = 0$.*

The proof can be found in [16].

Lemma 2.3. *Assume that $p \ge 1$, $\sigma \ge 1$, $u \in W_0^{1,p}(\Omega)$, and Ω is a bounded and smooth domain in $\mathbb{R}^N$. Then we have*

(i) *If $p < N$, then $||u||_{L^{Np/(N-p)}(\Omega)} \le C_1||\nabla u||_{L^p(\Omega)}$ where C_1 is a positive constant depending only on p and N.*

(ii) *If $p \ge N$, then $||u||_{L^\gamma(\Omega)} \le C_2||\nabla u||^{\Theta}_{L^p(\Omega)}||u||^{1-\Theta}_{L^\sigma(\Omega)}$ for $\gamma > \sigma$, where $C_2 = \max\{\gamma(N-1)/N; 1 + (p-1)\sigma/N\}^{\Theta}$, and*

$$\frac{1}{\gamma} = \frac{1}{\sigma} - \Theta\left(\frac{1}{\sigma} - \frac{1}{p} + \frac{1}{N}\right).$$

The proof can be found in [16].

Lemma 2.4. *Assume that* $u \in L^\infty(0,T;L^1(\Omega))$, $v \in L^\infty(0,T;L^\sigma(\Omega))$ $(\sigma \in [1,\sigma_0))$, $\phi \in C_0^1(\Omega_T)$ *are nonnegative functions,* Ω *is a bounded domain in* $\mathbb{R}^N$, $\Omega_T = \Omega \times (0,T)$. *If* $v_\delta \equiv \max\{v;\delta\} \in L^p(0,T;W^{1,p}(\Omega))$ $(p>1)$ *for every* $\delta \in (0,1)$ *satisfying the following conditions:*

$$S \equiv \sup_{0<\delta<1} \iint_{\Omega_T(v>\delta)} |\nabla(\phi v)|^p dxdt < +\infty,$$

then the following conclusions hold.

(i) *If* $p<N$, *then*

$$\iint_{\Omega_T} u^{p/N}(\phi v)^p dxdt \le C_3 S\Big(\operatorname{ess}\sup_{0<t<T}\int_\Omega u(x,t)dx\Big)^{p/N},$$

where C_3 *is positive constant depending only on* p *and* N.

(ii) *If* $p \ge N$, *then*

$$\iint_{\Omega_T} u^{1-p/\Theta\gamma}(\phi v)^{p/\Theta} dxdt$$

$$\le C_4 S\Big(\operatorname{ess}\sup_{0<t<T}\int_\Omega u(x,t)dx\Big)^{1-p/\Theta\gamma}\Big(\operatorname{ess}\sup_{0<t<T}\int_\Omega (\phi v(x,t))^\sigma dx\Big)^{p(1-\Theta)/\sigma\Theta}$$

for all $\gamma > \sigma$ *with*

$$\frac{1}{\gamma} = \frac{1}{\sigma} - \Theta\Big(\frac{1}{\sigma} - \frac{1}{p} + \frac{1}{N}\Big), \tag{2.4}$$

where C_4 *is a positive constant depending only on* p, N *and* σ_0.

The proof is omitted here.

§3. Proof of Theorem 1.6

Let $0<s<\tau<T<+\infty$. For $n=1,2,\cdots$, denote $T_n = \tau - \frac{\tau-s}{2^{n-1}}$, $I_n = (T_n, T)$, and $h_n \in C^\infty(0,+\infty)$ such that

$$\begin{cases} h_n = 0 \quad \text{in} \quad (0,T_n); \qquad h_n = 1 \quad \text{in } I_{n+1}, \\ 0 \le h_n \le 1, \quad |h_n'| \le \frac{2^n C}{\tau - s} \qquad \text{in} \quad (0,+\infty), \end{cases} \tag{3.1}$$

and $\xi_R \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \xi_R = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{2R}; \qquad \xi_R = 1 \quad B_R, \\ 0 \le \xi_R \le 1, \quad |\nabla \xi_R| \le \frac{C}{R}, \quad \text{in} \quad \mathbb{R}^N. \end{cases} \tag{3.2}$$

Choosing $\phi(x,t) = \xi_R(x)h_n(t)$ in (b) of Definition 1.2 and using (3.1) and (3.2) we compute

$$\operatorname{ess}\sup_{s<t<T}\int_{\mathbb{R}^N} (\xi_R h_n)^p u^{1+\alpha}(x,t)dx + \alpha_p \sup_{0<\delta<1}\iint_{Q_{s,T}(u>\delta)} |\nabla\{\xi_R h_n u^{[m(p-1)+\alpha]/p}\}|^p dxdt$$

$$\le \frac{C2^n}{\tau - s}\iint_{\Omega_n} u^{\alpha+1}dxdt + C\alpha R^{-p}\int_s^T\int_{B_{2R}} u^{m(p-1)+\alpha}dxdt \tag{3.3}$$

for all $\alpha \ge \max\{1; \lambda - m(p-1)\}$, where $\Omega_n = \mathbb{R}^N \times (T_n, T)$. This implies that

$$\operatorname{ess}\sup_{s<t<T}\int_{\mathbb{R}^N} (\xi_R h_n)^p u^{1+\alpha}(x,t)dx \le \frac{C2^n}{\tau-s}\iint_{\Omega_n} u^{\alpha+1}dxdt + C\alpha T R^{-p} M_\lambda ||u||_{L^\infty(Q_{s,T})}^{m(p-1)+\alpha-\lambda}.$$

Letting $R \to +\infty$, we get

$$\operatorname{ess}\sup_{s<t<T}\int_{\mathbb{R}^N} h_n^p u^{1+\alpha}(x,t)dx \le \frac{C2^n}{\tau-s}\iint_{\Omega_n} u^{\alpha+1}dxdt. \tag{3.4}$$

Denote

$$\alpha_0 = \max\{1; \lambda - m(p-1); \lambda - 1; 2|m(p-1)-1|/(p-1)\}. \tag{3.5}$$

Then the following inequalities hold:

$$\iint_{\Omega_{n+1}} u^{p(\alpha+1)/N+\alpha+m(p-1)} dxdt \le \alpha^{p-2}\Big(\frac{C2^n}{\tau - s}\iint_{\Omega_n} u^{\alpha+1} dxdt\Big)^{1+p/N} \tag{3.6}$$

for all $\alpha \ge \alpha_0$.

We prove only the case $p < N$, and the proof in the case $p \ge N$ is similar and omitted.

In the case $p < N$, using (i) in Lemma 2.4 and (3.3) we compute

$$\begin{aligned}&\iint_{Q_{s,T}} [(\xi_R h_n)^p u^{\alpha+1}]^{p/N} [\xi_R h_n u^{(m(p-1)+\alpha)/p}]^p dxdt\\ &\le \alpha^{p-2}\left(\frac{C2^n}{\tau - s}\iint_{\Omega_n} u^{\alpha+1} dxdt + C\alpha T R^{-p} M_\lambda ||u||^{m(p-1)+\alpha-\lambda}_{L^\infty(Q_{s,T})}\right)^{1+p/N}\end{aligned}$$

for all $\alpha \ge \alpha_0$. Letting $R \to \infty$, we obtain (3.6) in the case $p < N$. Therefore, the inequalities (3.6) hold for all $\alpha \ge \alpha_0$.

Let l be a fixed positive integer such that

$$\kappa^l - 1 - N|m(p-1)-1|/p \ge \alpha_0,$$

where $\kappa = 1 + p/N$. Then, choosing $\alpha = \kappa^n - 1 - N[m(p-1)-1]/p$ in (3.6) we get

$$\iint_{\Omega_{n+1}} u^{\kappa^{n+1} - N[m(p-1)-1]/p} dxdt \le C[2^\kappa \kappa^{p-2}]^n \left(\frac{1}{\tau - s}\iint_{\Omega_n} u^{\kappa^n - N[m(p-1)-1]/p} dxdt\right)^\kappa$$

for all $n \ge l$. This implies that

$$\begin{aligned}&\Big(\iint_{\Omega_{n+l}} u^{\kappa^{n+l} - N[m(p-1)-1]/p} dxdt\Big)^{1/\kappa^{n+l}}\\ &\le \Big(\frac{C}{\tau - s}\Big)^{a_n} (2^\kappa \kappa^{p-2})^{b_n} \Big(\iint_{\Omega_l} u^{\kappa^l - N[m(p-1)-1]/p} dxdt\Big)^{1/\kappa^l},\end{aligned}$$

where $a_n = \sum\limits_{j=0}^{n-1} \big(\frac{1}{\kappa}\big)^{j+l}$, $\quad b_n = \sum\limits_{j=0}^{n-1} \frac{j+l}{\kappa^{j+l}}$. Letting $n \to +\infty$ we have

$$||u||_{L^\infty(Q_{\tau,T})} \le \Big(\frac{C}{(\tau - s)^{N\kappa/p}}\iint_{Q_{s,T}} u^{\kappa^l - N[m(p-1)-1]/p} dxdt\Big)^{1/\kappa^l}. \tag{3.7}$$

Using Young's inequality we obatin

$$||u||_{L^\infty(Q_{\tau,T})} \le \frac{1}{2}||u||_{L^\infty(Q_{s,T})} + C\Big(\frac{1}{(\tau - s)^{N\kappa/p}}\iint_{Q_{s,T}} u^\lambda dxdt\Big)^{p/N\theta}.$$

Applying Lemma 2.1 we conclude that

$$||u||_{L^\infty(Q_{\tau,T})} \le C\Big(\frac{TM_\lambda}{(\tau - s)^{N\kappa/p}}\Big)^{p/N\theta}$$

for all $0 < s < \tau < T$. Thus the proof of Theorem 1.6 is completed.

§4. Proof of Theorem 1.7

Let us denote the following functions

$$\begin{cases} f(R,s) = ||u||_{L^\infty(B_R(y)\times(T-s,T))}, \quad F(R,s) = \int_{T-s}^T \int_{B_R(y)} u^\lambda(x,t) dxdt,\\ g(\tau) = \big(\frac{1}{\tau}\big)^{1/\theta}, \quad G(R,\rho,t,s,\tau) = \frac{g^{m(p-1)-k}(\tau)}{(R-\rho)^p} + \frac{g^{1-k}(\tau)}{t-s}, \end{cases} \tag{4.1}$$

where $0 < \rho < R < +\infty$, $0 < s < t < T < +\infty$, $0 < \tau < +\infty$, $k = \min\{m(p-1); 1\}$, $B_R(y) = \{x \in \mathbb{R}^N : |x-y| < R\}$ and $y \in \mathbb{R}^N$. Then we have

Lemma 4.1. *Let $u \in Y_\lambda(m,p,q,N)$, $0 < \rho_1 < \rho_2 < +\infty$ and $0 < \tau < s_1 < s_2 < T < +\infty$ with $T - s_2 \geq \tau$. If $q < k$, then*

$$F(\rho_1, s_1) \leq \gamma G(\rho_2, \rho_1, s_2, s_1, \tau) f^{k-q}(\rho_2, s_2) F(\rho_2, s_2).$$

Proof. For $0 < \rho_1 < \rho_2$ and $0 < \tau < s_1 < s_2 < T$ with $T - s_2 \geq \tau$, denote two functions $H \in C^\infty(0,T)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} H(t) = 1 \quad \forall t \in (T-s_1, T), \quad H(t) = 0, \quad \forall t \in (0,T) \setminus (T-s_2, T); \\ 0 \leq H(t) \leq 1, \quad |H'(t)| \leq \frac{C}{s_2 - s_1}, \qquad \forall t \in [0,T]; \end{cases} \tag{4.2}$$

$$\begin{cases} \psi(x) = 1 \quad \forall x \in B_{\rho_1}(y), \quad \psi(x) = 0, \quad \forall x \in \mathbb{R}^N \setminus B_{\rho_2}(y); \\ 0 \leq \psi(x) \leq 1, \quad |\nabla \psi(x)| \leq \frac{C}{\rho_2 - \rho_1}, \qquad \forall x \in \mathbb{R}^N. \end{cases} \tag{4.3}$$

Taking $\phi(x,t) = \psi(x)H(t)$ in (b) of Definition 1.2 we get

$$\begin{aligned} &\operatorname{ess} \sup_{T-s_2<t<T} \int_{\mathbb{R}^N} (\psi H)^p(x,t) u^{1+\alpha}(x,t)dx \\ &+ \alpha_p \sup_{0<\delta<1} \iint_{Q_{T-s_2,T}(u>\delta)} |\nabla\{(\psi H)u^{[m(p-1)+\alpha]/p}\}|^p dxdt \\ \leq{}& \Lambda_1 \Big\{ \iint_{Q_{T-s_2,T}} [(\psi H)^{p-1}|\psi H_t|u^{\alpha+1} + \alpha^{1-p}(1+\alpha)|\nabla(\psi H)|^p u^{m(p-1)+\alpha} \\ &- (\alpha+1)(\psi H)^p u^{q+\alpha}]dxdt + \int_{\mathbb{R}^N} (\psi H)^p(x, T-s_2)u^{\alpha+1}(x, T-s_2)dx \Big\}. \end{aligned}$$

Letting $\alpha = \lambda - q > 0$, we have

$$\iint_{Q_{T-s_2,T}} (\psi H)^p u^\lambda dxdt \leq C \iint_{Q_{T-s_2,T}} [\psi^p |H_t| u^{\lambda+1-q} + H^p |\nabla\psi|^p u^{m(p-1)+1-q}]dxdt. \tag{4.4}$$

On the other hand, it follows from Theorem 1.6 and (4.1) that

$$f(\rho_2, s_2) \leq \Big(\frac{\gamma}{T - s_2}\Big)^{1/\theta} \leq \gamma g(\tau). \tag{4.5}$$

Using (4.1) and (4.5) we compute

$$\iint_{Q_{T-s_2,T}} \psi^p |H_t| u^{\lambda+1-q} dxdt \leq \frac{\gamma}{s_2 - s_1} \cdot g^{1-k}(\tau) f^{k-q}(\rho_2, s_2) F(\rho_2, s_2), \tag{4.6}$$

$$\iint_{Q_{T-s_2,T}} H^p |\nabla\psi|^p u^{\lambda+m(p-1)-q} dxdt \leq \frac{\gamma}{(\rho_2 - \rho_1)^p} \cdot g^{m(p-1)-k}(\tau) f^{k-q}(\rho_2, s_2) F(\rho_2, s_2). \tag{4.7}$$

Combining (4.6) and (4.7) with (4.4) we obtain the conclusion of Lemma 4.1. Thus the proof is completed.

Lemma 4.2. *Let $u \in Y_\lambda(m,p,q,N)$, $0 < \rho_2 < \rho_3 < +\infty$ and $0 < \tau < s_2 < s_3 < T < +\infty$ with $T - s_3 \geq \tau$. If $q < k$, then*

$$f(\rho_2, s_2) \leq \gamma G^{N\kappa/p\varpi}(\rho_3, \rho_2, s_3, s_2, \tau) F^{1/\varpi}(\rho_3, s_3)$$

for some positive constant γ, where $\varpi = \lambda + 1 - k + N[m(p-1) - k]/p$.

Proof. For $0 < \rho_2 < \rho_3$ and $0 < \tau < s_2 < s_3 < T$ with $T - s_3 \geq \tau$, and

$$T_n = T - s_2 - \frac{s_3 - s_2}{2^n}, \quad T_n^* = \frac{T_n + T_{n+1}}{2}, \quad n = 1, 2, \cdots,$$

$$R_n = \rho_2 + \frac{\rho_3 - \rho_2}{2^n}, \quad R_n^* = \frac{R_n + R_{n+1}}{2}, \quad n = 1, 2, \cdots,$$

by Theorem 1.6, we have

$$||u||_{L^\infty(\mathbb{R}^N\times(T_n,+\infty))}\le \gamma g(\tau),\quad n=1,2,\cdots. \tag{4.8}$$

Denote $\Omega_n=B_{R_n}(y)\times(T_n,T)$, $\Omega_n^*=B_{R_n^*}(y)\times(T_n^*,T)$, $y\in\mathbb{R}^N$; and a number of functions $H_n\in C^\infty[T_n,T]$, $H_n^*\in C^\infty[T_n^*,T]$, $\psi_n\in C_0^\infty(\mathbb{R}^N)$ and $\psi_n^*\in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} H_n(t)=1 \quad \forall t\in(T_{n+1},T), \quad H_n(t)=0, \quad \forall t\in(0,T)\setminus(T_n^*,T);\\ 0\le H_n(t)\le 1, \quad |H_n'(t)|\le \frac{C2^n}{s_3-s_2}, \qquad \forall t\in[0,T];\end{cases} \tag{4.9}$$

$$\begin{cases} H_n^*(t)=1 \quad \forall t\in(T_n^*,T), \quad H_n^*(t)=0, \quad \forall t\in(0,T)\setminus(T_n,T);\\ 0\le H_n^*(t)\le 1, \quad |(H_n^*)'(t)|\le \frac{C2^n}{s_3-s_2}, \qquad \forall t\in[0,T];\end{cases} \tag{4.10}$$

$$\begin{cases} \psi_n(x)=1, \quad \forall x\in B_{R_{n+1}}(y), \quad \psi_n(x)=0, \quad \forall x\in\mathbb{R}^N\setminus B_{R_n^*}(y);\\ 0\le \psi_n(x)\le 1, \quad |\nabla\psi_n(x)|\le \frac{C2^n}{\rho_3-\rho_2}, \qquad \forall x\in\mathbb{R}^N;\end{cases} \tag{4.11}$$

$$\begin{cases} \psi_n^*(x)=1, \quad \forall x\in B_{R_n^*}(y), \quad \psi_n^*(x)=0, \quad \forall x\in\mathbb{R}^N\setminus B_{R_n}(y);\\ 0\le \psi_n^*(x)\le 1, \quad |\nabla\psi_n^*(x)|\le \frac{C2^n}{\rho_3-\rho_2}, \qquad \forall x\in\mathbb{R}^N.\end{cases} \tag{4.12}$$

Choosing $\phi(x,t)=\psi_n(x)H_n(t)$ in (b) of Definition 1.2 and using (4.8), (4.9) and (4.11), we get

$$\begin{aligned}
&\operatorname{ess}\sup_{T_n^*<t<T}\int_{\mathbb{R}^N}(\psi_nH_n)^p u^{1+\alpha}(x,t)dx\\
&\quad+\alpha_p\sup_{0<\delta<1}\iint_{Q_{T_n^*,T}(u>\delta)}|\nabla\{\psi_RH_nu^{[m(p-1)+\alpha]/p}\}|^pdxdt\\
&\le\Lambda_1\Big\{\iint_{Q_{T_n^*,T}}[\psi_n^pH_n^{p-1}|H_n'|u^{\alpha+1}+\alpha^{1-p}(1+\alpha)H_n^p|\nabla\psi_n|^pu^{m(p-1)+\alpha}\\
&\quad-(\alpha+1)\psi_n^pH_n^pu^{q+\alpha}]dxdt+\int_{\mathbb{R}^N}\psi_n^pH_n^pu^{\alpha+1}(x,T_n^*)dx\Big\}\\
&\le\frac{C2^n}{s_3-s_2}\iint_{\Omega_n^*}u^{\alpha+1}dxdt+\frac{C2^{pn}\alpha}{(\rho_3-\rho_2)^p}\iint_{\Omega_n^*}u^{m(p-1)+\alpha}dxdt\\
&\le\gamma2^{(p+1)n}\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n^*}u^{\alpha+k}dxdt
\end{aligned}$$

for all $\alpha\ge\max\{1;\lambda-m(p-1)\}$. This implies that

$$\begin{aligned}
&\operatorname{ess}\sup_{T_n^*<t<T}\int_{\mathbb{R}^N}(\psi_nH_n)^p u^{1+\alpha}(x,t)dx\\
&\quad+\alpha_p\sup_{0<\delta<1}\iint_{Q_{T_n^*,T}(u>\delta)}|\nabla\{\psi_nH_nu^{[m(p-1)+\alpha]/p}\}|^pdxdt\\
&\le\gamma2^{(p+1)n}\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n^*}u^{\alpha+k}dxdt.
\end{aligned}\tag{4.13}$$

Similarly, we also have

$$\begin{aligned}
&\operatorname{ess}\sup_{T_n<t<T}\int_{\mathbb{R}^N}(\psi_n^*H_n^*)^p u^{1+\alpha}(x,t)dx\\
&\quad+\alpha_p\sup_{0<\delta<1}\iint_{Q_{T_n,T}(u>\delta)}|\nabla\{\psi_n^*H_n^*u^{[m(p-1)+\alpha]/p}\}|^pdxdt\\
&\le\gamma2^{(p+1)n}\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n}u^{\alpha+k}dxdt,
\end{aligned}$$

which implies that

$$\operatorname{ess}\sup_{T_n<t<T}\int_{\mathbb{R}^N}(\psi_n^*H_n^*)^p u^{1+\alpha}(x,t)dx\le\gamma 2^{(p+1)n}G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n}u^{\alpha+k}dxdt \tag{4.14}$$

for all $\alpha\ge\max\{1;\lambda-m(p-1)\}$.

Let us prove that the following inequality holds:

$$\begin{aligned}&\iint_{\Omega_{n+1}}u^{p(\alpha+1)/N+\alpha+m(p-1)}dxdt\\&\le\gamma\alpha^{p-2}\Big(2^{n(p+1)}\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n}u^{\alpha+k}dxdt\Big)^{\kappa}\end{aligned}\tag{4.15}$$

for all $\alpha\ge\alpha_0$, where $\kappa=1+p/N$, α_0 is defined by (3.5).

We consider only the case $p\ge N$. The proof for the case $p<N$ is similar, and is omitted here.

In the case $p\ge N$, we denote $\sigma=p(\alpha+1)/[m(p-1)+\alpha]$ and then have

$$1\le(p+1)/2\le\sigma\le p+1,\quad\forall\alpha\ge\alpha_0.$$

For γ and Θ defined by (2.4), applying (ii) of Lemma 2.4 and (4.13) with (4.14), we compute

$$\begin{aligned}&\iint_{Q_{T_*^n,T}}[(\psi_nH_n)^pu^{\alpha+1}]^{1-p/\Theta\gamma}[\psi_nH_nu^{(m(p-1)+\alpha)/p}]^{p/\Theta}dxdt\\
&\le C\Big(\operatorname{ess}\sup_{T_n^*<t<T}\int_{\mathbb{R}^N}(\psi_nH_n)^pu^{\alpha+1}dx\Big)^{1-p/\Theta\gamma}\\
&\quad\cdot\Big(\operatorname{ess}\sup_{T_n^*<t<T}\int_{\mathbb{R}^N}(\psi_nH_nu^{[m(p-1)+\alpha]/p})^{\sigma}dx\Big)^{p(1-\Theta)/\sigma\Theta}\\
&\quad\cdot\sup_{0<\delta<1}\iint_{Q_{T_n^*,T}(0<\delta<1)}|\nabla(\psi_nH_nu^{\alpha+m(p-1)})|^pdxdt\\
&\le\gamma\alpha^{p-2}\Big(\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n}u^{\alpha+k}dxdt\Big)^{2-p/\Theta\gamma}\\
&\quad\cdot\Big\{\operatorname{ess}\sup_{T_n<t<T}\iint_{\Omega_n}(\psi_n^*H_n^*)^pu^{\alpha+1}dxdt\Big\}^{p(1-\Theta)/\sigma\Theta}\\
&\le\gamma\alpha^{p-2}\Big(2^{(p+1)n}\alpha G(\rho_3,\rho_2,s_3,s_2,\tau)\iint_{\Omega_n}u^{\alpha+k}dxdt\Big)^{1+p/N}\end{aligned}$$

for all $\alpha\ge\alpha_0$. Therefore, (4.15) holds in the case $p\ge N$. Thus (4.15) is proved for all $\alpha\ge\alpha_0$.

Let l be a fixed positive integer such that $\kappa^l+k-1+N[k-m(p-1)]/p\ge\alpha_0$, where $\kappa=1+p/N$. Then, choosing $\alpha=\kappa^n+k-1+N[k-m(p-1)]$ in (4.11) we get

$$\begin{aligned}&\Big(\iint_{\Omega_{n+l}}u^{\kappa^{n+l}-\varpi+\lambda}dxdt\Big)^{1/\kappa^{n+l}}\\&\le(\gamma G(\rho_3,\rho_2,s_3,s_2,\tau))^{a_n}(2^{(p+1)\kappa}\kappa^{1+p/N})^{b_n}\Big(\iint_{\Omega_l}u^{\kappa^l-\varpi+\lambda}dxdt\Big)^{1/\kappa^l},\end{aligned}$$

where

$$a_n=\sum_{j=0}^{n-1}\Big(\frac{1}{\kappa}\Big)^{j+l},\quad b_n=\sum_{j=0}^{n-1}\frac{j+l}{\kappa^{j+l}}.$$

Letting $n \to +\infty$ we have

$$||u||_{L^\infty(B_{\rho_2}(y)\times(T-s_2,T))} \le \gamma\Big(G^{N\kappa/p}(\rho_3,\rho_2,s_3,s_2,\tau)\int_{T-s_2}^{T}\int_{B_{\rho_3}(y)} u^{\kappa^l-\varpi+\lambda}dxdt\Big)^{1/\kappa^l}.$$

Using Young's inequality and (4.1), we obatin

$$\begin{aligned} f(\rho_2,s_2) &\le \gamma f^{(\kappa^l-\varpi)/\kappa^l}(\rho_3,s_3)\Big(G^{N\kappa/p}(\rho_3,\rho_2,s_3,s_2,\tau)F(\rho_3,s_3)\Big)^{1/\kappa^l} \\ &\le \frac{1}{2}f(\rho_2,s_2)+\gamma G^{N\kappa/p\varpi}(\rho_3,\rho_2,s_3,s_2,\tau)F^{1/\varpi}(\rho_3,s_3) \end{aligned}$$

for all $0<s_2<s_3<T$ with $T-s_3\ge\tau$ and $0<\rho_2<\rho_3$. Applying Lemma 2.1 we conclude that

$$f(\rho_2,s_2) \le \gamma G^{N\kappa/p\varpi}(\rho_3,\rho_2,s_3,s_2,\tau)F^{1/\varpi}(\rho_3,s_3).$$

Thus the proof of Lemma 4.2 is completed.

Lemma 4.3. *Let $u\in Y_\lambda(m,p,q,N)$. If*

$$\frac{1}{T^{1+\beta_0}}\int_{T/2}^{T}\int_{B_{R(T)}} u^\lambda(x,t)dxdt \le b_0 \tag{4.16}$$

for some $T\in(0,+\infty)$, then

$$u(x,t)=0 \tag{4.17}$$

for a. e. $(x,t)\in B_{R(T)/2}\times(3T/4,T)$, where $R(T)=T^{\lambda/N\theta}$, and

$$\beta_0=\frac{\beta_1-\beta_2}{\beta_2},\quad \beta_1=\Big(1+\frac{1-k}{\theta}\Big)\Big(1+\frac{N\kappa(k-q)}{p\varpi}\Big),\quad \beta_2=\frac{k-q}{\varpi}, \tag{4.18}$$

and b_0 is some positive constant depending only on m, p, λ, q, N, Λ_1 and Λ_2.

Proof. Let $0<\rho_1<\rho_2<\rho_3<+\infty$ and $0<\tau<s_1<s_2<s_3<T<+\infty$ with $T-s_3\ge\tau$. Applying Lemma 4.1 we get

$$F(\rho_1,s_1)\le\gamma G(\rho_2,\rho_1,s_2,s_1,\tau)f^{k-q}(\rho_2,s_2)F(\rho_2,s_2). \tag{4.19}$$

Using Lemma 4.2 we have

$$f(\rho_2,s_2)\le\gamma G^{N\kappa/p\varpi}(\rho_3,\rho_2,s_3,s_2,\tau)F^{1/\varpi}(\rho_3,s_3). \tag{4.20}$$

Combining (4.19) and (4.20) we obtain

$$F(\rho_1,s_1)\le\gamma G(\rho_2,\rho_1,s_2,s_1,\tau)G^{N\kappa(k-q)/p\varpi}(\rho_3,\rho_2,s_3,s_2,\tau)F^{(k-q)/\varpi}(\rho_3,s_3)F(\rho_2,s_2). \tag{4.21}$$

For $0<\rho<R<+\infty$, $0<\tau<s<T<+\infty$ with $T-s\ge\tau$, taking

$$\rho_1=\rho,\quad \rho_2=\frac{\rho+R}{2},\quad \rho_3=R;\qquad s_1=\tau,\quad s_2=\frac{\tau+s}{2},\quad s_3=s$$

in (4.21) we get

$$\begin{aligned} F(\rho,\tau) &\le \gamma G\Big(\frac{\rho+R}{2},\rho,\frac{\tau+s}{2},\tau,\tau\Big)\cdot G^{N\kappa\beta_2/p}\Big(R,\frac{\rho+R}{2},s,\frac{\tau+s}{2},\tau\Big) \\ &\quad\cdot F^{\beta_2}(R,s)F\Big(\frac{\rho+R}{2},\frac{\tau+s}{2}\Big). \end{aligned} \tag{4.22}$$

From (4.1), it follows that

$$\begin{aligned} &G\Big(\frac{\rho+R}{2},\rho,\frac{\tau+s}{2},\tau,\tau\Big)=G\Big(R,\frac{\rho+R}{2},s,\frac{\tau+s}{2},\tau\Big) \\ &=\frac{2^pg^{1-k}(\tau)}{(R-\rho)^p}+\frac{2g^{m(p-1)-1}(\tau)}{s-\tau}\le 2^pG(R,\rho,s,\tau,\tau). \end{aligned} \tag{4.23}$$

In addition, by (4.1), we also have

$$F\left(\frac{\rho+R}{2},\frac{\tau+s}{2}\right)\le F(R,s). \tag{4.24}$$

Combining (4.23) and (4.24) with (4.22) we conclude that

$$F(\rho,\tau)\le\gamma G^{1+N\kappa\beta_2/p}(R,\rho,s,\tau,\tau)F^{1+\beta_2}(R,s) \tag{4.25}$$

for $0<\rho<R<+\infty$ and $0<\tau<s<T<+\infty$ with $T-s\ge\tau$.

For $n=1,2,\cdots$, denote

$$R_n=R(T)+\frac{R(T)}{2^n},\quad T_n=\frac{T}{4}+\frac{T}{4^n},\qquad n=1,2,\cdots,$$

where $R(T)=T^{r/(N\Theta+pr)}$. Choosing $R=R_n$, $\rho=R_{n+1}$, $s=T_n$ and $\tau=T_{n+1}$ in (4.25), we have

$$F(R_{n+1},T_{n+1})\le\gamma G^{1+N\kappa\beta_2/p}(R_n,R_{n+1},T_n,T_{n+1},T_{n+1})F^{1+\beta_2}(R_n,T_n) \tag{4.26}$$

for $n=1,2,\cdots$. It follows from (4.1) that

$$\begin{aligned}&G(R_n,R_{n+1},T_n,T_{n+1},T_{n+1})\\&\le\frac{g^{1-k}(T_{n+1})}{T_n-T_{n+1}}+\frac{g^{m(p-1)-k}(T_{n+1})}{(R_n-R_{n+1})^p}\le\gamma4^{(p+1)n}\left(\frac{1}{T}\right)^{1+N(1-k)/(N\Theta+pr)}.\end{aligned}$$

Therefore, by (4.26), we get $A_{n+1}\le\gamma b^nT^{-\beta_1}A_n^{1+\beta_2}$, $n=1,2,\cdots$, where $A_n=F(R_n,T_n)$, and $b=4^{(p+1)(1+N\kappa\beta_2/p)}$. By Lemma 2.2, if

$$(\gamma T^{-\beta_1})^{1/\beta_2}b^{1/\beta_2^2}A_0<1, \tag{4.27}$$

then

$$\lim_{n\to+\infty}A_n=0. \tag{4.28}$$

Thus the proof of Lemma 4.3 is proved.

Proof of Theorem 1.7. It follows from Lemma 4.3 that, if

$$\frac{1}{T^{1+\beta_0}}\int_{T/2}^{T}\int_{B_{R(T)}}u^\lambda(x,t)dxdt\le b_0, \tag{4.29}$$

then

$$u(x,t)=0 \tag{4.30}$$

for a. e. $(x,t)\in B_{R(T)/2}\times(3T/4,T)$, where $R(T)=T^{\lambda/N\theta}$. Choosing T_0 such that $T_0=(M_\lambda/2b_0)^{1/\beta_0}$we have (4.29) for all $T\in(T_0,+\infty)$ and all $y\in\mathbb{R}^N$ and then obtain (4.30) for all $T\in(T_0,+\infty)$ and all $y\in\mathbb{R}^N$. This implies that $u(x,t)=0$ for a. e. $(x,t)\in\mathbb{R}^N\times(T_0,+\infty)$. Thus the proof of Theorem 1.7 is completed.

§5. Proof of Theorem 1.1

From Theorem 1.7, it follows that $u(x,t)=0$ for a. e. $(x,t)\in\mathbb{R}^N\times(T_0,+\infty)$. By (a) in Definition 1.2, we get

$$\int_0^{+\infty}\int_{\mathbb{R}^N}u^\lambda(x,t)dxdt<+\infty. \tag{5.1}$$

For any $T\in(0,T_0)$, by (5.1), there exists a positive constant $L=L(T)$ such that

$$\int_0^{+\infty}\int_{\mathbb{R}^N\setminus B_L}u^\lambda(x,t)dxdt<b_0T^{1+\beta_0}. \tag{5.2}$$

Let $y \in \mathbb{R}^N \setminus B_{L+R(T)}$. Then we have $B_{R(T)}(y) \subset \mathbb{R}^N \setminus B_L$. Using (5.2) we obtain (4.16). By Lemma 4.3, we get $u(x,t) = 0$ for a. e. $(x,t) \in B_{R(T)/2}(y) \times (3T/4, T)$ with $y \in \mathbb{R}^N \setminus B_{L+R(T)}$. Thus Theorem 1.1 is proved.

§6. Proof of Theorem 1.2

Let $u \in \Upsilon_\lambda(m,p,q,N)$ be a nonneagtive function which has the ISS property. Then, for any $\tau > 0$, there exists a positive constant $L = L(\tau)$ such that

$$\mathrm{supp} u(\cdot,t) \subset B_L \tag{6.1}$$

for all $t \in (\tau, +\infty)$. Take ξ_R as (3.2) and choose $\phi(x,t) = \xi_R(x)$ in (c) of Definition 1.2 to have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s<t<T} \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,t)dx \\
&+ \frac{\beta(1-\beta)}{(m(p-1)-\beta)^p} \sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\xi_R u^{(m(p-1)-\beta)/p}\}|^p dxdt \\
\le\, & C\Big\{ \iint_{Q_{s,T}} [\beta^{1-p}(1-\beta)|\nabla\xi_R|^p u^{m(p-1)-\beta} + (1-\beta)\xi_R^p u^{q-\beta}]dxdt \\
&+ \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,T)dx \Big\}
\end{aligned}$$

for $T \in (s, +\infty)$ with $s \ge \tau > 0$. This implies that

$$\begin{aligned}
&\sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\xi_R u^{(m(p-1)-\beta)/p}\}|^p dxdt \\
\le\, & \frac{C(p-1-\beta)^p}{\beta(1-\beta)} \Big\{ \iint_{Q_{s,T}} (1-\beta)\xi_R^p u^{q-\beta} dxdt + \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,T)dx \Big\}
\end{aligned}$$

for all $R > 2L$ and a. e. $T \in (s, +\infty)$ with $s \ge \tau > 0$. Using Lemma 2.4 we compute

$$\iint_{Q_{s,T}} \xi_R^p u^{m(p-1)-\beta} dxdt \le \frac{C(m(p-1)-\beta)^p |B_L|(T-s)(||u||^{q-\beta}_{L^\infty(Q_{s,T})} + ||u||^{1-\beta}_{L^\infty(Q_{s,T})})}{\beta(1-\beta)}.$$

For $1 > q \ge m(p-1) > 0$, letting $\beta \uparrow m(p-1)$, by (6.1), we conclude that $|Q_{s,T}(u>0)| = 0$ $(\forall T > s > \tau)$. Therefore taking $\tau \to 0^+$ and $s \to 0^+$ we get Theorem 1.2.

§7. Proof of Theorem 1.3

Let $u \in \Upsilon_\lambda(m,p,q,N)$ be a nonneagtive function which has the ISS property. Then, for any $\tau > 0$, there exists a positive constant $L = L(\tau)$ such that

$$\mathrm{supp} u(\cdot,t) \subset B_L, \quad \forall t \in (\tau, +\infty). \tag{7.1}$$

Take ξ_R as (3.2) and choose $\phi(x,t) = \xi_R(x)$ in (c) of Definition 1.2 to have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s<t<T} \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,t)dx \\
&+ \frac{\beta(1-\beta)}{(m(p-1)-\beta)^p} \sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\xi_R u^{(m(p-1)-\beta)/p}\}|^p dxdt \\
\le\, & C\Big\{ \iint_{Q_{s,T}} [\beta^{1-p}(1-\beta)|\nabla\xi_R|^p u^{m(p-1)-\beta} + (1-\beta)\xi_R^p u^{q-\beta}]dxdt \\
&+ \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,T)dx \Big\}
\end{aligned}$$

for a. e. $T \in (s, +\infty)$ with $s \geq \tau > 0$. This implies that

$$\operatorname*{ess\,sup}_{s<t<T} \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,t)dx \leq C\Big\{ \iint_{Q_{s,T}} (1-\beta)\xi_R^p u^{q-\beta} dxdt + \int_{\mathbb{R}^N} \xi_R^p(x) u^{1-\beta}(x,T)dx \Big\}$$

for all $R > 2L$. For $\max\{m(p-1); q\} \geq 1$, letting $\beta \uparrow 1$, by (7.1), we conclude that

$$|\mathrm{supp} u(\cdot, s)| \leq |\mathrm{supp} u(\cdot, T)|$$

for a. e. s, T with $s < T$. Thus, we get Theorem 1.3.

§8. Proof of Theorem 1.4

In order to prove Theorem 1.4 we need following results.

Lemma 8.1. *Assume that $u \in Y_\lambda(m, p, q, N)$, $m(p-1) > 1$ and $q > 0$. If (1.4) holds, then*

$$||u||_{L^\infty(B_\rho(y)\times(0,T))} \leq \Big(\frac{C}{(R-\rho)^{N+p}} \int_0^T \int_{B_R(y)} u^{m(p-1)+\lambda-1} dxdt \Big)^{1/\lambda}$$

for all $T > 0$ and all $y \in \mathbb{R}^N \setminus B_{R+R_0}$ with $R > \rho > 0$, where $\theta = m(p-1) - 1 + p\lambda/N$.

Proof. For $R > \rho > 0$, let $y \in \mathbb{R}^N \setminus B_{R_0+R}$, denote $R_n = \rho + \frac{R-\rho}{2^n}$, $n = 1, 2, \cdots$, and $\xi_n \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \xi_n = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_n}(y); \quad \xi_n = 1 \quad B_{R_{n+1}}(y), \\ 0 \leq \xi_n \leq 1, \quad |\nabla \xi_n| \leq \frac{C2^n}{R-\rho}, \quad \text{in} \quad \mathbb{R}^N. \end{cases} \tag{8.1}$$

Choosing $\phi(x,t) = \xi_n(x)$ in (b) of Definition 1.2 we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{s<t<T} \int_{\mathbb{R}^N} (\xi_n)^p u^{1+\alpha}(x,t)dx + \alpha_p \sup_{0<\delta<1} \iint_{Q_{s,T}(u>\delta)} |\nabla\{\xi_n u^{[m(p-1)+\alpha]/p}\}|^p dxdt \\ &\leq \Lambda_1 \Big\{ \iint_{Q_{s,T}} [\alpha^{1-p}(1+\alpha)|\nabla \xi_n|^p u^{m(p-1)+\alpha} + \int_{\mathbb{R}^N} \xi_n^p u^{\alpha+1}(x,s)dx \Big\} \end{aligned}$$

for a. e. $s \in (0, T)$ with $T > 0$. Using (1.4) and (8.1) and letting $s \to 0^+$, we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<t<T} \int_{\mathbb{R}^N} (\xi_n)^p u^{1+\alpha}(x,t)dx + \alpha_p \sup_{0<\delta<1} \iint_{Q_{0,T}(u>\delta)} |\nabla\{\xi_n u^{[m(p-1)+\alpha]/p}\}|^p dxdt \\ &\leq \frac{C\alpha}{(R-\rho)^p} \iint_{\Omega_n} u^{m(p-1)+\alpha} dxdt \end{aligned} \tag{8.2}$$

for all $\alpha \geq \max\{\lambda; 1; \lambda - m(p-1)\}$, where $\Omega_n = B_{R_n}(y) \times (0, T)$.

Similarly to (3.6), by (8.2), we have

$$\iint_{\Omega_{n+1}} u^{p(\alpha+1)/N+\alpha+m(p-1)} dxdt \leq \Big(\frac{C2^{pn}\alpha^p}{(R-\rho)^p} \iint_{\Omega_n} u^{\alpha+m(p-1)} dxdt \Big)^{1+p/N} \tag{8.3}$$

for all $\alpha \geq \alpha_0$, and $n = 1, 2, \cdots$.

Let l be a fixed positive integer such that $\kappa^l - 1 \geq \alpha_0$, where $\kappa = 1 + p/N$, and α_0 is defined by (3.5). Then, choosing $\alpha = \kappa^n - 1$ in (8.3) we get

$$\iint_{\Omega_{n+1}} u^{\kappa^{n+1}-1+m(p-1)} dxdt \leq C[2^{p\kappa}\kappa^{p\kappa}]^n \Big(\frac{1}{(R-\rho)^p} \iint_{\Omega_n} u^{\kappa^n-1+m(p-1)} dxdt \Big)^\kappa$$

for all $n \geq l$. This implies that

$$\begin{aligned} &\Big(\iint_{\Omega_{n+l}} u^{\kappa^{n+l}-1+m(p-1)} dxdt \Big)^{1/\kappa^{n+l}} \\ &\leq \Big(\frac{C}{(R-\rho)^p} \Big)^{a_n} (2^{p\kappa}\kappa^{p\kappa})^{b_n} \Big(\iint_{\Omega_l} u^{\kappa^l-1+m(p-1)} dxdt \Big)^{1/\kappa^l}, \end{aligned}$$

where

$$a_n=\sum_{j=0}^{n-1}\Big(\frac{1}{\kappa}\Big)^{j+l},\quad b_n=\sum_{j=0}^{n-1}\frac{j+l}{\kappa^{j+l}}.$$

Letting $n\to+\infty$ we have

$$||u||_{L^\infty(B_\rho(y)\times(0,T))}\le C\Big(\frac{1}{(R-\rho)^{N\kappa}}\iint_{B_R(y)\times(0,T)}u^{\kappa^l-1+m(p-1)}dxdt\Big)^{1/\kappa^l}.$$

Using Young's inequality we obatin

$$||u||_{L^\infty(B_\rho(y)\times(0,T))}\le\frac{1}{2}||u||_{L^\infty(B_R(y)\times(0,T))}+\Big(\frac{C}{(R-\rho)^{N\kappa}}\int_0^T\int_{B_R(y)}u^{\lambda+m(p-1)-1}dxdt\Big)^{1/\lambda}.$$

Applying Lemma 2.1 we conclude that

$$||u||_{L^\infty(B_\rho(y)\times(0,T))}\le\Big(\frac{C}{(R-\rho)^{N\kappa}}\int_0^T\int_{B_R(y)}u^{\lambda+m(p-1)-1}dxdt\Big)^{1/\lambda}.$$

Thus the proof of Lemma 8.1 is completed.

Lemma 8.2. *Assume that $u\in Y_\lambda(m,p,q,N)$, $m(p-1)>q\ge1$. Then we have*

$$\iint_Q u^{q+\lambda-1}dxdt\le CM_\lambda.$$

Proof. Let $R>0$ and $0<s<T<+\infty$ and denote $\xi_R\in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases}\xi_R=0\quad\text{in}\quad\mathbb{R}^N\setminus B_{2R};\qquad\xi_R=1\quad B_R,\\ 0\le\xi_R\le1,\quad|\nabla\xi_R|\le\frac{C}{R},\quad\text{in}\quad\mathbb{R}^N.\end{cases}\tag{8.4}$$

Choosing $\phi(x,t)=\xi_R(x)$ in (b) of Definition 1.2 we get

$$\begin{aligned}&\operatorname{ess}\sup_{s<t<T}\int_{\mathbb{R}^N}\xi_R^pu^{1+\alpha}(x,t)dx+\alpha_p\sup_{0<\delta<1}\iint_{Q_{s,T}(u>\delta)}|\nabla\{\xi_Ru^{[m(p-1)+\alpha]/p}\}|^pdxdt\\ &\le\Lambda_1\Big\{\iint_{Q_{s,T}}[\alpha^{1-p}(1+\alpha)h_n|\nabla\xi_R|^pu^{m(p-1)+\alpha}-(\alpha+1)\xi_R^ph_n^pu^{q+\alpha}]dxdt\\ &\quad+\int_{\mathbb{R}^N}\xi_R^ph_n^pu^{\alpha+1}(x,s)dx\Big\}\end{aligned}$$

for a. e. $s\in(0,T)$ with $T>0$. By (8.4), we get

$$\iint_{Q_{s,T}}\xi_R^pu^{q+\alpha}dxdt\le C\alpha^{1-p}(1+\alpha)R^{-p}\iint_{Q_{s,T}}u^{m(p-1)+\alpha}dxdt+C\int_{\mathbb{R}^N}\xi_R^pu^{\alpha+1}(x,s)dx$$

for $\alpha>\lambda-1\ge0$. Letting $R\to+\infty$, by $m(p-1)\ge1$, we have

$$\iint_{Q_{s,T}}u^{q+\alpha}dxdt\le C\int_{\mathbb{R}^N}u^{\alpha+1}(x,s)dx$$

for all $\alpha>\lambda-1\ge0$. Letting $\alpha\downarrow(\lambda-1)$ we get

$$\iint_{Q_{s,T}}u^{q+\lambda-1}dxdt\le C\int_{\mathbb{R}^N}u^\lambda(x,s)dx$$

for a. e. $s\in(0,T)$ with $0<T<+\infty$. Taking $s\to0^+$ and $T\to+\infty$ we get

$$\iint_Q u^{q+\lambda-1}dxdt\le CM_\lambda.$$

Thus the proof is completed.

Proposition 8.1. *Assume that $u \in Y_\lambda(m,p,q,N)$, $1 \le q < m(p-1)$. If (1.4) holds, then there exists a positive constant L_1 depending only on $m,p,q,\lambda,N,\Lambda_1$, Λ_2 and M_λ such that*

$$u(x,t)=0 \qquad for \quad a.\ e. \quad (x,t)\in(\mathbb{R}^N\setminus B_{L_1})\times(0,1].$$

Proof. Applying Lemma 8.1 we get

$$||u||_{L^\infty(B_\rho(y)\times(0,T))} \le \left(\frac{C}{(R-\rho)^{N+p}}\int_0^T\int_{B_R(y)} u^{m(p-1)-1+\lambda}dxdt\right)^{1/\lambda} \tag{8.5}$$

for all $T>0$ and all $y\in\mathbb{R}^N\setminus B_{R+R_0}$ with $R>\rho>0$.

Denote $R_n=\frac{R}{2}+\frac{R}{2^n}$, $n=1,2,\cdots$. Then (8.5) implies that

$$||u||_{L^\infty(B_{R_{n+1}}(y)\times(0,T))} \le \Big(\frac{C2^{(p+N)n}}{R^{N+p}}\int_0^T\int_{B_{R_n}(y)} u^{m(p-1)-1+\lambda}dxdt\Big)^{1/\lambda},$$

which implies that $A_{n+1}\le C2^{(p+N)n}R^{-p/\lambda}A_{n+1}^{1+\sigma}$, where $\sigma=[m(p-1)-1]/\lambda>0$ and $A_n=||u||_{L^\infty(B_{R_n}(y)\times(0,T))}$, $0<T\le1$. By Lemma 2.2, if

$$A_0<(CR^{-p/\lambda})^{-1/\sigma}2^{-(p+N)/\sigma^2}, \tag{8.6}$$

then we have

$$\lim_{n\to+\infty}A_n=||u||_{L^\infty(B_{R/2}(y)\times(0,T))}=0. \tag{8.7}$$

On the other hand, using Theorem 1.6 and (8.5), we compute

$$||u||_{L^\infty(B_R(y)\times(0,T))}\le\Big\{\frac{C}{(2R-R)^{N+p}}\int_0^T\int_{B_{2R}(y)}u^{m(p-1)-1+\lambda}dxdt\Big\}^{1/\lambda}\le\frac{\gamma}{R^{N+p}}$$

for all $0<T\le1$ and all $y\in\mathbb{R}^N\setminus B_{2R+R_0}$. This implies

$$||u||_{L^\infty(B_R(y)\times(0,T))}\le\gamma R^{-N-p}. \tag{8.8}$$

Choose $R=R_*>1$ such that

$$\gamma R_*^{-N-p}\le2^{-1}\cdot(CR_*^{-p/\lambda})^{-1/\sigma}2^{-p/\sigma^2}. \tag{8.9}$$

From (8.8) and (8.9), it follows that (8.6) holds for all $R\ge R_*$ with $y\in\mathbb{R}^N\setminus B_{2R+R_0}$, and then (8.7) holds for all $R\ge R_*$ with $y\in\mathbb{R}^N\setminus B_{2R+R_0}$. Therefore, we conclude that $u(x,t)=0$ for all $y\in\mathbb{R}^N\setminus B_{R_0+2R_*}$. Thus the proof of Proposition 8.1 is completed.

Let us denote the following functions

$$I(R;y)=||u||_{L^\infty(B_R(y)\times(1,T+1))};\qquad J(R;y)=\int_1^{T+1}\int_{B_R(y)}u^{m(p-1)+\lambda-1}dxdt, \tag{8.10}$$

where $T>0$ and $R>\rho>0$, and $y\in\mathbb{R}^N$. Then we have

Lemma 8.3. *Assume that $u\in Y_\lambda(m,p,q,N)$, and $1\le q<m(p-1)$. If*

$$u(x,t)=0 \tag{8.11}$$

for a. e. $(x,t)\in(\mathbb{R}^N\setminus B_{L_1})\times(0,1]$ for some positive constant L_1, then

$$J(\rho_1;y)\le\frac{\gamma}{(\rho_2-\rho_1)^p}I^{m(p-1)-q}(\rho_2;y)J(\rho_2;y)$$

for all $y\in\mathbb{R}^N\setminus B_{\rho_2+L_1}$ with $\rho_2>\rho_1>0$.

Proof. For $0<\rho_1<\rho_2$ with $y\in B_{2R+L_1}$, denote a function $\psi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{cases}\psi(x)=1\quad\forall x\in B_{\rho_1}(y),\quad \psi(x)=0,\quad\forall x\in\mathbb{R}^N\setminus B_{\rho_2}(y);\\ 0\le\psi(x)\le1,\quad |\nabla\psi(x)|\le\frac{C}{\rho_2-\rho_1},\qquad\forall x\in\mathbb{R}^N.\end{cases} \tag{8.12}$$

Taking $\phi(x,t)=\psi(x)$ in (b) of Definition 1.2 we get

$$\begin{aligned}&\operatorname{ess}\sup_{s<t<T+1}\int_{\mathbb{R}^N}\psi^p(x,t)u^{1+\alpha}(x,t)dx+\alpha_p\sup_{0<\delta<1}\iint_{Q_{s,T}(u>\delta)}|\nabla\{\psi u^{[m(p-1)+\alpha]/p}\}|^p dxdt\\ \le{}&\Lambda_1\Big\{\iint_{Q_{s,T+1}}[\alpha^{1-p}(1+\alpha)|\nabla\psi|^p u^{m(p-1)+\alpha}-(\alpha+1)\psi^p u^{q+\alpha}]dxdt\\ &+\int_{\mathbb{R}^N}\psi(x)u^{\alpha+1}(x,s)dx\Big\}\end{aligned}$$

for a. e. $s>1$. Letting $s\downarrow 1$ and $\alpha=m(p-1)-q+\lambda-1>0$, by (8.10) and (8.11), we have

$$\iint_{Q_{1,T+1}}\psi^p u^{m(p-1)+\lambda-1}dxdt\le\frac{C}{(\rho_2-\rho_1)^p}\int_1^{T+1}\int_{B_{\rho_2}(y)}u^{2m(p-1)+\lambda-1-q}dxdt. \tag{8.13}$$

On the other hand, by (8.10), we get

$$\int_1^{T+1}\int_{B_{\rho_2}(y)}u^{2m(p-1)+\lambda-1-q}dxdt\le I^{m(p-1)-q}(\rho_2;y)J(\rho_2;y). \tag{8.14}$$

Combining (8.13) and (8.14) we obtain the conclusion of Lemma 8.3. Thus Lemma 8.3 is proved.

Lemma 8.4. *Assume that $u\in Y_\lambda(m,p,q,N)$, $m(p-1)>1$ and $q>0$. If (8.11) holds, then*

$$I(\rho_2;y)\le\frac{\gamma}{(\rho_3-\rho_2)^{(N+p)/\lambda}}J^{1/\lambda}(\rho_3;y)$$

for all $y\in\mathbb{R}^N\setminus B_{2\rho_3+L_1}$ with $\rho_3>\rho_2>0$.

The proof is similar to that of Lemma 8.1. Therefore, we omit it.

Proposition 8.2. *Assume that $u\in Y_\lambda(m,p,q,N)$, and $1\le q<m(p-1)$. If (8.13) holds, then*

$$J(\rho;y)\le\frac{\gamma}{(R-\rho)^{\beta_3}}J^{1+\beta_4}(\rho_2;y)$$

for all $y\in\mathbb{R}^N\setminus B_{R+L_1}$ with $R>\rho>0$, where

$$\beta_3=p+\frac{(N+p)(m(p-1)-q)}{\lambda},\qquad \beta_4=\frac{m(p-1)-q}{\lambda}.$$

Proof. For $R>\rho>0$ and $y\in\mathbb{R}^N\setminus B_{R+L_1}$, using Lemma 8.3 we get

$$J(\rho;y)\le\frac{\gamma}{(R-\rho)^p}I^{m(p-1)-q}\Big(\frac{R+\rho}{2};y\Big)J(R;y). \tag{8.15}$$

In addition, it follows from Lemma 8.4 that

$$I\Big(\frac{R+\rho}{2};y\Big)\le\frac{\gamma}{(R-\rho)^{(N+p)/\lambda}}J^{1/\lambda}(R;y). \tag{8.16}$$

Combining (8.16) with (8.15) we get the conclusion of Proposition 8.2. Thus Proposition 8.2 is proved.

Proof of Theorem 1.4. Using Proposition 8.1 we get

$$u(x,t)=0 \tag{8.17}$$

for a. e. $(x,t)\in(\mathbb{R}^N\setminus B_{L_1})\times(0,1]$, where L_1 is a positive constant depending only on m,p,q,λ,Λ_1 and Λ_2 and M_λ.

For $R>0$ and $y\in\mathbb{R}^N\setminus B_{2R+L_1}$, using Proposition 8.2 we have

$$J(\rho;y)\le\frac{\gamma}{(R-\rho)^{\beta_3}}J^{1+\beta_4}(\rho_2;y). \tag{8.18}$$

Let us define $R_n=\frac{R}{2}+\frac{R}{2^{n+1}},\quad n=0,1,2,\cdots$. Then, from (8.18), we have $A_{n+1}\le\frac{2^{n\beta_3}\gamma}{R^{\beta_3}}A_n^{1+\beta_4}$ for $n=1,2,\cdots$, where $A_n=J(R_n;y)$. By Lemma 2.2 , if

$$A_0<(\gamma R^{-\beta_3})^{-1/\beta_4}2^{-\beta_3/\beta_4^2}, \tag{8.19}$$

then

$$\lim_{n\to+\infty}A_n=\lim_{n\to+\infty}J(R_n;y)=\int_1^{T+1}\int_{B_{R/2}(y)}u^{m(p-1)+\lambda-1}dxdt=0. \tag{8.20}$$

On the other hand, using Theorem 1.6 and Lemma 8.2 we compute

$$A_0=\int_1^{T+1}\int_{B_R(y)}u^{m(p-1)+\lambda-1}dxdt\le||u||^{m(p-1)-q}_{L^\infty(Q_{1,T+1})}\iint_Q u^q dxdt\le\gamma_0 \tag{8.21}$$

for some positive constant γ_0. Choose $R=R_*$ such that

$$\gamma_0<(\gamma R_*^{-\beta_3})^{-1/\beta_4}2^{-\beta_3/\beta_4^2}. \tag{8.22}$$

Therefore, from (8.21) and (8.22) we obtain that (8.19) holds for all $R\ge R_*$, and then (8.20) holds for all $R\ge R_*$ with $y\in\mathbb{R}^N\setminus B_{2R+B_{L_1}}$. This implies that $u(y,t)=0$ for a. e. $(y,t)\in(\mathbb{R}^N\setminus B_{2R_*+L_1})\times(1,T+1)$. Thus Theorem 1.4 is completed.

§9. Proof of Theorem 1.5

Let $u\in\Upsilon_\lambda(m,p,q,N)$ be a nonneagtive function which has the LOC property. Then there exists a positive constant L such that

$$\operatorname{supp}u(\cdot,t)\subset B_L\qquad\text{for}\quad\text{a. e.}\quad t\in(0,+\infty). \tag{9.1}$$

In order to prove Theorem 1.5 we need the following lemmas.

Lemma 9.1. *Suppose that $u\in Y_\lambda(m,p,q,N)$ and (9.1) holds. If $m(p-1)>1$, then $u(x,t)\le\frac{\gamma_1}{t^{m(p-1)-1}}$ for a. e. $(x,t)\in Q$, where γ_1 is a positive constant depending only on $m,p,q,N,\lambda,\Lambda_1,\Lambda_2$ and L.*

Proof. For $0<s<\tau<T$, using (3.7) and applying the Young inequality we compute

$$||u||_{L^\infty(Q_{\tau,T})}\le\frac12||u||_{L^\infty(Q_{\tau,T})}+\Big(\frac{C}{(\tau-s)^{N\kappa/p}}\iint_{Q_{s,T}}dxdt\Big)^{p/N[m(p-1)-1]}.$$

By Lemma 2.1 we conclude that

$$||u||_{L^\infty(Q_{\tau,T})}\le\Big(\frac{C}{(\tau-s)^{N\kappa/p}}\iint_{Q_{s,T}}dxdt\Big)^{p/N[m(p-1)-1]}$$

for all $0<s<\tau<T$. This implies that

$$||u||_{L^\infty(Q_{T/2,T})}\le\frac{C}{T^{1/[m(p-1)-1]}}.$$

Thus the proof of Lemma 9.1 is completed.

Lemma 9.2. *Suppose that $u\in Y_\lambda(m,p,q,N)$ and (9.1) holds. If $q>m(p-1)>1$, then $u(x,t)=0$ for a. e. $(x,t)\in\mathbb{R}^N\times(T_1,+\infty)$, where T_1 is some time depending only on $m,p,q,N,\lambda,\Lambda_1,\Lambda_2$ and L.*

Proof. Let β be a positive number such that

$$\beta=\max\Big\{\frac12;1-\frac{N[q-m(p-1)]}{p}\Big\} \tag{9.2}$$

and take ξ_R as (3.2) and $\phi(x,t)=\xi_R(x)$ in (c) of Definition 1.2 to have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s<t<T}\int_{\mathbb{R}^N}\xi_R^p(x)u^{1-\beta}(x,t)dx \\
&\quad+\frac{\beta(1-\beta)}{(m(p-1)-\beta)^p}\sup_{0<\delta<1}\iint_{Q_{s,T}(u>\delta)}|\nabla\{\xi_R u^{(m(p-1)-\beta)/p}\}|^p dxdt \\
&\le C\Big\{\iint_{Q_{s,T}}[\beta^{1-p}(1-\beta)|\nabla\xi_R|^p u^{m(p-1)-\beta}+(1-\beta)\xi_R^p u^{q-\beta}]dxdt \\
&\quad+\int_{\mathbb{R}^N}\xi_R^p(x,T)u^{1-\beta}(x,T)dx\Big\}
\end{aligned}$$

for $s\in(\tau,T)$ with $\tau<T<+\infty$. This implies that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s<t<T}\int_{\mathbb{R}^N}\xi_R^p(x)u^{1-\beta}(x,t)dx+\sup_{0<\delta<1}\iint_{Q_{s,T}(u>\delta)}|\nabla\{\xi_R u^{(m(p-1)-\beta)/p}\}|^p dxdt \\
&\le C\left\{\iint_{Q_{s,T}}(1-\beta)\xi_R^p u^{q-\beta}dxdt+\int_{\mathbb{R}^N}\xi_R^p(x)u^{1-\beta}(x,T)dx\right\}
\end{aligned}$$

for all $R>2L$. For $q>m(p-1)$, using Lemma 2.2, we conclude that

$$\begin{aligned}
&\iint_{Q_{s,T}(L)}u^{m(p-1)-\beta+p(1-\beta)/N}dxdt \\
&\le C\left\{\iint_{Q_{s,T}(L)}u^{q-\beta}dxdt+\int_{B_L}u^{1-\beta}(x,T)dx\right\}^{1+p/N}, \qquad (9.3)
\end{aligned}$$

where $0<s<T$ and $Q_{s,T}(L)=B_L\times(s,T)$.

Denote

$$\sigma=m(p-1)-\beta+\frac{p(1-\beta)}{N},\qquad \nu=q-\beta-\sigma\ge 0,$$

where β is defined by (9.2). Then, by (9.3), we get

$$\iint_{Q_{s,T}(L)}u^\sigma dxdt\le C\Big\{\iint_{Q_{s,T}(L)}u^{\sigma+\nu}dxdt\Big\}^\kappa+C\Big\{\int_{B_L}u^{1-\beta}(x,T)dx\Big\}^\kappa, \qquad (9.4)$$

where $\kappa=1+p/N$. Using Lemma 9.1 we compute

$$u(x,t)\le\gamma_2 \qquad (9.5)$$

for a.e. $(x,t)\in Q_{1,+\infty}$, and

$$\begin{aligned}
\int_{B_L}u^{1-\beta}(x,T)dx&\le\int_{B_L}\Big(\frac{\gamma_1}{t^{1/[m(p-1)-1]}}\Big)^{1-\beta}dx \\
&\le\gamma_3 T^{-(1-\beta)/[m(p-1)-1]} \qquad (9.6)
\end{aligned}$$

for a. e. $T\in(0,+\infty)$; and

$$\begin{aligned}
\iint_{Q_{s,T}(L)}u^{\sigma+\nu}dxdt&\le\iint_{Q_{s,T}(L)}\Big(\frac{\gamma_1}{t^{m(p-1)-1}}\Big)^{\sigma+\nu}dxdt \\
&\le\gamma_4 s^{(\beta-1)/[m(p-1)-1]} \qquad (9.7)
\end{aligned}$$

for $0<s<T$, where γ_2,γ_3 and γ_4 are positive constants depending not on s,T. Combining (9.5) with (9.7) we get

$$C\Big\{\iint_{Q_{s,T}(L)}u^{\sigma+\nu}dxdt\Big\}^\kappa\le\frac12\iint_{Q_{s,T}(L)}u^\sigma dxdt$$

for all s, T with $T_1 < s < T$, where T_1 is some positive number. From (9.4), it follows that

$$\frac{1}{2}\iint_{Q_{s,T}(L)} u^\sigma dxdt \le C\Big\{\int_{B_L} u^{1-\beta}(x,T)dx\Big\}^\kappa$$

for all $T > s > T_1$. Using (9.6) and letting $T \to +\infty$ we have

$$\frac{1}{2}\iint_{Q_{s,+\infty}(L)} u^\sigma dxdt = 0$$

for all $s \in (T_1, +\infty)$. Thus Lemma 9.2 is proved.

Proof of Theorem 1.5. By Lemma 9.2, we get $u(x,t) = 0$ for a. e. $(x,t) \in \mathbb{R}^N \times (T_1, +\infty)$. This implies that $|\operatorname{supp} u(\cdot,t)| = 0$ for a. e. $t \in (T_1, +\infty)$. Using Theorem 1.3 we conclude that $|\operatorname{supp} u(\cdot,t)| = 0$ for a. e. $t \in (0, T_1)$. Thus we get $u(x,t) = 0$ for a. e. $(x,t) \in Q$. This contradicts the assumptions of Theorem 1.5. Thus the proof of Theorem 1.5 is completed.

§10. The Applications to Nonlinear Parabolic Equations

In this section we shall prove that the solutions of (1.1) and (1.2) belong to Y_λ (m,p,q,N) with some m, p, q, N in general.

Let u_0 be a non-zero nonnegative function satisfying one of the following conditions:

(H1) u_0 is a finite Borel measure in $\mathbb{R}^N$;

(H2) $u_0 \in L^\lambda(\mathbb{R}^N)$ for some $1 \le \lambda < +\infty$.

We shall consider the following problems.

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q, & p > 1, \quad q > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{A}$$

$$\begin{cases} u_t = \Delta u^m - u^q, & m > 0, \quad q > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{B}$$

First, we consider the Cauchy problem (A) in the cases (H1) and (H2).

(i) In the case (H1).

By the existence and non-existence proved by H. Brezis and A. Friedman in [17] and by Zhao Junning in [18], if $p - 2 + p/N > 0$ and $q < p - 1 + p/N$, then the Cauchy problem (A) has a nonnegative solution $u \in C(Q) \cap L^\infty(0,\infty; L^1(\mathbb{R}^N))$ satisfying

$$\begin{aligned} &\int_{\mathbb{R}^N} \Phi(x,t)u(x,t)dx - \int_{\mathbb{R}^N} \Phi(x,s)u(x,s)dx \\ &= \iint_{Q_{s,t}} [u\Phi_t - |\nabla u|^{p-2}\nabla u \nabla\Phi - \Phi u^q] dxd\tau \end{aligned} \tag{10.1}$$

for all s, t with $0 < s < t$, and for all $\Phi \in L^p(0,+\infty; W_0^{1,p}(\mathbb{R}^N))$ with $\Phi_t \in L^2_{\text{loc}}(Q)$.

In particular, we have

Proposition 10.1. *Let u be a solution of* (A), *and $p-2+p/N > 0$ and $q < p-1+p/N$. If* (H1) *holds, then $u \in Y_1(1,p,q,N)$.*

The proof is similar to that given in [19]. Therefore we omit it.

(ii) In the case (H2).

By the existence proved by E. Di Benedetto in [19], if $p - 2 + p\lambda/N > 0$ and $q < p - 1 + p\lambda/N$, then the Cauchy problem (A) has a nonnegative solution $u \in C(Q) \cap L^\infty(0,\infty; L^\lambda(\mathbb{R}^N))$ satisfying (10.1).

Similarly to Proposition 10.1, we also have

Proposition 10.2. *Let u be a solution of* (A)*, and $p-2+p\lambda/N>0$ and $q<p-1+p\lambda/N$. If* (H1) *holds, then $u\in Y_\lambda(1,p,q,N)$.*

In addition we consider the Cauchy problem (B) in the cases (H1) and (H2).

Similarly to Propositions 10.1–10.2, we have

Proposition 10.3. *Let u be a solution of* (B)*, and $m-1+2/N>0$ and $q<m+2/N$. If* (H1) *holds, then $u\in Y_1(m,2,q,N)$.*

Proposition 10.4. *Let u be a solution of* (B)*, and $m-1+2\lambda/N>0$ and $q<m+2\lambda/N$. If* (H2) *holds, then $u\in Y_\lambda(m,2,q,N)$.*

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