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HARMONIC FUNCTIONS ON PRODUCT OF CERTAIN KÄHLER MANIFOLDS WITH A POLE**

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Abstract

It is proved that there is no nonconstant harmonic function of finite energy on product of certain Kählerian manifolds with a pole.

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In the paper [1], Xin Y. L. proved that any finite energy harmonic map from classical bounded symmetric domains (except $\Re_{IV}(2) = H^2 \times H^2$) into any complete Riemannian manifold has to be constant. And he asked if there exists a harmonic map of finite energy from $\Re_{IV}(2) = H^2 \times H^2$ into a Riemannian manifold. In this short note, we will partially answer this question. Actually, we will prove

Theorem. *Let M_1^m, M_2^n be complete noncompact Kähler manifolds of complex dimensions m and n respectively. Assume all of the geodesic spheres in M_1 (resp. M_2) centered at some point $x_1 \in M_1$, (resp. $x_2 \in M_2$) are convex. Then any finite energy harmonic function on $M_1 \times M_2$ has to be constant.*

As a consequence, one has

Corollary. *There exists no nonconstant harmonic function of finite energy on $H^2 \times H^2$.*

The idea of our proof is completely different from Xin's. His proof mainly makes use of an integral identity and an explicit computation of Hessian of the distance function on classical bounded symmetric domains. Unfortunately, this does not work for the present case. The key point to our proof is an observation saying that any finite energy harmonic function on complete Kähler manifolds is pluriharmonic. Actually, this is not new. For more general harmonic maps case, it goes back to the famous argument of Sampson[2]. Sampson only considered the case where the domain manifolds are compact Kählerian. In fact, his argument can be easily extended to L^2-case[3]. Here, we only need to consider the function case. For sake of completeness, we will give a complete argument of the function case.

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Lemma 1. *Let M be a complete Kähler manifold. Assume u is a harmonic function of finite energy on M. Then u is pluriharmonic, i.e., $\partial\overline{\partial}u = 0$.*

Proof. Since u is harmonic, du is a harmonic 1-form; so, the local Hodge theorem shows ∂u is a harmonic $(1,0)$-form. The finiteness of energy of u also guarantees that ∂u is in L^2-space.

In the following, we make a general discussion: If α is a harmonic L^2-form, then α is both closed and coclosed. Obviously, if one applies this to the above form ∂u, then one has $\partial\overline{\partial}u = 0$, i.e., u is pluriharmonic.

Take a C^∞ cutoff function η as follows:

$$\begin{aligned} \eta(x) &= 1, \quad x \in B_{x_0}(r); \\ 0 \le \eta(x) &\le 1, \quad x \in B_{x_0}(2r) \setminus B_{x_0}(r); \\ \eta(x) &= 0, \quad x \in M \setminus B_{x_0}(2r); \\ |d\eta|(x) &\le \frac{2}{r}, \quad x \in M. \end{aligned}$$

Then, one has

$$\begin{aligned} 0 &= \langle\langle \Delta\alpha, \eta^2\alpha \rangle\rangle \\ &= \langle\langle d\alpha, d(\eta^2\alpha)\rangle\rangle + \langle\langle d^\star\alpha, d^\star(\eta^2\alpha)\rangle\rangle \\ &= \|\eta d\alpha\|_{L^2}^2 + 2\langle\langle d\alpha, \eta d\eta \wedge \alpha\rangle\rangle \\ &\quad + \langle\langle d\star\alpha, d(\eta^2\star\alpha)\rangle\rangle \\ &= \|\eta d\alpha\|_{L^2}^2 + 2\langle\langle d\alpha, \eta d\eta\wedge\alpha\rangle\rangle + \|\eta d\star\alpha\|_{L^2}^2 \\ &\quad + 2\langle\langle d\star\alpha, \eta d\eta \wedge (\star\alpha)\rangle\rangle \\ &\ge \|\eta d\alpha\|_{L^2}^2 + \|\eta d^\star\alpha\|_{L^2}^2 - 2\|\eta d\alpha\|_{L^2}\|d\eta\wedge\alpha\|_{L^2} \\ &\quad - 2\|\eta d^\star\alpha\|_{L^2}\|d\eta\wedge\star\alpha\|_{L^2} \\ &\ge \|\eta d\alpha\|_{L^2}^2 + \|\eta d^\star\alpha\|_{L^2}^2 - \frac{c}{r}(\|\eta d\alpha\|_{L^2}^2 + \|\eta d^\star\alpha\|_{L^2}^2)^{\frac{1}{2}}. \end{aligned}$$

Here, c is a positive constant depending only on L^2-norm of α. It is not difficult to see that, for r sufficiently large,

$$\|\eta d\alpha\|_{L^2}^2 + \|\eta d^\star\alpha\|_{L^2}^2 = 0,$$

i.e., α is both closed and coclosed.

In order to prove the theorem , we need the following lemma.

Lemma 2. *Let M be a complete noncompact manifold* (*not necessarily Kählerian*) *and its all spheres $\partial B_{x_0}(R)$ be convex with respect to some points $x_0 \in M$. Assume that f is a nonnegative, subharmonic and L^1 function, then f has to be zero.*

Proof. Consider the integral $\int_{\partial B_{x_0}(R)} f$ as a function of R, and compute its derivatives.

Then, using Stokes' formula, one has

$$\begin{aligned}\frac{d}{dR}\int\limits_{\partial B_{x_0}(R)} f &= \lim_{\Delta R\to 0}\frac{1}{\Delta R}\Big[\int\limits_{\partial B_{x_0}(R+\Delta R)} f-\int\limits_{\partial B_{x_0}(R)} f\Big]\\ &= \lim_{\Delta R\to 0}\frac{1}{\Delta R}\int\limits_{B_{x_0}(R+\Delta R)\setminus B_{x_0}(R)} \operatorname{div}(f\nabla r)\\ &= \lim_{\Delta R\to 0}\frac{1}{\Delta R}\int\limits_{B_{x_0}(R+\Delta R)\setminus B_{x_0}(R)} \Big(\frac{\partial f}{\partial r}+f\Delta r\Big)\\ &= \int\limits_{\partial B_{x_0}(R)}\Big(\frac{\partial f}{\partial r}+f\Delta r\Big).\end{aligned}$$

Here, r is the distance function to x_0. Again, since all $\partial B_{x_0}(R)$ are convex, equivalently, this says $\Delta r\geq 0$. So, again using Stokes' formula, one has

$$\frac{d}{dR}\int\limits_{\partial B_{x_0}(R)} f\geq\int\limits_{\partial B_{x_0}(R)}\frac{\partial f}{\partial r}=\int\limits_{B_{x_0}(R)}\Delta f\geq 0.$$

Namely, $\int_{\partial B_{x_0}(R)} f$ is a monotone increasing function with respect to R. But, f's L^1-ness forces $\int_{\partial B_{x_0}(R)} f$ to be identically a zero function, and so is f.

Proof of Theorem. Let $z=(z_1,z_2,\cdots,z_m)$ and $w=(w_1,w_2,\cdots,w_n)$ be local coordinates of M_1 and M_2. Then, the condition of energy finiteness of u is

$$\int_{M_1}\Big(\int_{M_2}\Big(\sum|\partial_{z_i}u|^2+\sum|\partial_{w_k}u|^2\Big)\star 1\Big)\star 1<+\infty.$$

Here, $\star 1$ represents the volume elements of M_1 and M_2 respectively. By Lemma 1, one has

$$\partial_{z_i}\overline{\partial}_{z_j}u=\partial_{z_i}\overline{\partial}_{w_k}u=\overline{\partial}_{z_i}\partial_{w_k}u=\partial_{w_k}\overline{\partial}_{w_l}u=0\quad\text{for } 1\leq i,j\leq m,\quad 1\leq k,l\leq m,$$

By means of symmetry of the variables z,w, we only consider the integral

$$\int_{M_1}\Big(\int_{M_2}\sum|\partial_{w_k}u|^2(z,w)\star 1\Big)\star 1<\infty.$$

By Fubini's Theorem, one has, for almost all $z\in M_1$,

$$\int_{M_2}\sum|\partial_{w_k}u|^2(z,w)\star 1<+\infty.$$

Actually, since u is smooth, we can assume that the above fact is valid for all $z\in M_1$ and the above integral is smooth with respect to the variable $z\in M_1$. (If necessary, one can change the above integral domain M_2 into any compact subset of M_2. It is easy to see that the following computation and the reasonings still are valid.) Set

$$f(z)=\int_{M_2}\sum|\partial_{w_k}u|^2(z,w)\star 1.$$

Obviously,

$$\int_{M_1}f(z)\star 1<+\infty,$$

i.e., $f\in L^1$. Compute $\Delta_z f(z)$ (since our computation is pointwise and M_1 is Kählerian, without loss of generality, one can assume $\Delta_z = 4\sum \partial_{z_i}\overline{\partial}_{z_i}$)

$$\begin{aligned}
&\sum \partial_{z_i}\overline{\partial}_{z_i} f(z)\\
&= \int_{M_2} \sum \partial_{z_i}\overline{\partial}_{z_i} \sum |\partial_{w_k} u|^2(z,w) \star 1\\
&= \int_{M_2} \sum (\partial_{z_i}(\overline{\partial}_{z_i}\partial_{w_k} u \overline{\partial}_{w_k} u) + \partial_{z_i}(\partial_{w_k} u \overline{\partial}_{z_i}\overline{\partial}_{w_k} u)) \star 1\\
&= \int_{M_2} \sum |\partial_{z_i}\partial_{w_k} u|^2 \star 1 \geq 0.
\end{aligned}$$

Here, we use u's pluriharmonicity and independence of z and w. Thus, f, as a function on M_1, satisfies the conditions of Lemma 2. So, f is a zero function, i.e.,

$$\partial_{w_k} u = 0.$$

The same reasoning derives

$$\partial_{z_i} u = 0.$$

So, u is constant.

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