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EXISTENCE AND UNIQUENESS OF STRONG PERIODIC SOLUTION OF THE EVOLUTION SYSTEM DESCRIBING GEOPHYSICAL FLOW—PART I: IN BOUNDED DOMAINS**

ZHAO Chunshan*   LI Kaitai*

Abstract

The existence and uniqueness of a strong periodic solution of the evolution system describing geophysical flow in bounded domains of $R^N(N=3,4)$ are proven if external forces are periodic in time and sufficiently small.

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§1. Introduction

Discovered by R. Hide[3], the following evolution system has been used to describe the geophysical flow:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u - \frac{1}{\rho\mu}(B\cdot\nabla)B + \frac{1}{2\rho\mu}\nabla(|B|^2) + \frac{1}{\rho}\nabla p = f(x,t), \\ \frac{\partial B}{\partial t} - \lambda\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u + \frac{1}{\mu}\nabla q = g(x,t), \\ \nabla\cdot u = 0, \qquad \nabla\cdot B = 0, \end{cases} \tag{E.S}$$

where $u = u(x_1,\cdots,x_N,t) = (u_1,\cdots,u_N)$ and $B = B(x_1,\cdots,x_N,t) = (B_1,\cdots,B_N)$ are the velocity vectors of Eulerian flow and magnetic fields respectively. $p(x,t)$ and $q(x,t)$ are pressures. $f(x,t)$ and $g(x,t)$ are volume forces. ρ and ν are the constants of density and viscosity of the flow respectively, μ is the constant of magnetic permeability and $\lambda = \frac{\eta}{\mu}$ with electrical resistivity η.

In this paper,we will investigate the existence and uniqueness of a strong periodic solution of the system (E.S) under boundary conditions

$$u(x,t) = 0, \quad B(x,t) = 0 \quad \text{for} \quad (x,t)\in\partial\Omega\times R, \tag{1.1}$$

where Ω is a bounded domain of $R^N(N=3,4)$ with smooth boundary $\partial\Omega$. The problems we consider are as follows: Suppose that the external forces $f(x,t)$ and $g(x,t)$ are periodic in time with some period ω, then we try to prove the existence and uniqueness of a strong

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*School of Science, Xi'an Jiaotong University, Xi'an 710049, China. **E-mail:** ktli@xjtu.edu.cn

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periodic solution (u, B) of the system (E.S) under the boundary conditions (1.1) with the same period ω, i.e.,

$$u(x,t+\omega) = u(x,t), \quad B(x,t+\omega) = B(x,t) \quad \text{for} \quad (x,t) \in \Omega \times R. \tag{1.2}$$

At first, let us recollect some known results related to the initial value problems of (E.S) together with initial data

$$u(x,0) = u_0(x), \quad B(x,0) = B_0(x), \quad x \in \Omega. \tag{1.3}$$

The global existence and uniqueness of a strong solution of the system (E.S) together with (1.1) and (1.3) in 2D bounded domains and the local existence and uniqueness of strong solutions in 3D bounded domain were obtained in [6,7]. The global existence and uniqueness of a strong solution to the initial and boundary value problems of (E.S) in 3D bounded or unbounded domains were proven under the assumptions that both initial values and external volume forces are sufficiently small or ν, λ are sufficiently large (see [7,10] for details). Strong periodic solutions of Navier-Stokes equations with sufficient small external forces were shown in [4,5,8]. Our work was motivated by the above works, and we will show the unique existence of a strong periodic solution of the system (E.S) in bounded domain Ω of R^N for $N = 3, 4$ under some assumptions on external forces $f(x,t)$ and $g(x,t)$. Clearly, if q =constant and $g(x,t) = 0$, the system (E.S) are the well-known MHD equations, so our results can be applied to the MHD equations nearly without any modifications.

This paper is arranged as follows: We presente our main results and some preliminaries in Section 2. The approximate solutions and various estimates of them are given in Section 3. In Section 4, the proofs of our main results are presented.

§2. Preliminaries and Main Results

Firstly, let us introduce some functional spaces and notations as follows: Throughout this paper, $L^2(\Omega)$, $H^m(\Omega)$ and $H_0^m(\Omega)$ are usual Sobolev spaces[1] of vector-valued functions. The inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let $\hat{C}_0^\infty(\Omega) = \{\phi \in C_0^\infty(\Omega), \nabla \cdot \phi = 0\}$ and H, V be the closures of $\hat{C}_0^\infty(\Omega)$ in $L^2(\Omega)$ and $H^1(\Omega)$ respectively.

Let $P : L^2(\Omega) \to H$ be the Leray projecting operator[9]. Then $A = -P\Delta$ is the well-known Stokes operator with domain $D(A) = H^2(\Omega) \bigcap V$. Since A is a positive operator with compact inverse, there exist eigenvalues $\{\lambda_j\}$ and eigenvectors $\{w_j\}$ $(j = 1, 2, \cdots)$ satisfying

$$Aw_j = \lambda_j w_j, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty.$$

Then the periodicity problem of the system (E.S) together with (1.1)–(1.2) can be formulated as follows:

$$\begin{aligned} u_t + \nu Au + P(u\cdot\nabla)u - \frac{1}{\rho\mu}P(B\cdot\nabla)B &= Pf, \\ B_t + \lambda AB + P(u\cdot\nabla)B - P(B\cdot\nabla)u &= Pg, \\ u(t+\omega) = u(t), \qquad B(t+\omega) &= B(t). \end{aligned} \tag{2.1}$$

Next, we introduce some functional spaces consisting of ω-periodic functions. Let X be a Banach space, $C^k(\omega; X)$ denotes the set of X-valued ω-periodic functions on R with continuous derivatives up to order k. The norm in $C^k(\omega; X)$ is defined as $\|f\|_{C^k(\omega;X)} = \sup\limits_{0\le t\le\omega} \Big\{ \sum\limits_{i=0}^{k} \|D_t^i f\|_X \Big\}$. Define $L^r(\omega; X)$ $(1 \le r \le \infty)$ as the set of ω-periodic X-valued measurable functions on R such that

$$\begin{aligned} \|f\|_{L^r(\omega;X)} &= \Big(\int_0^\omega \|f\|_X^r dt \Big)^{\frac{1}{r}} < \infty, \quad \text{if} \quad 1 \le r < \infty, \\ \|f\|_{L^\infty(\omega;X)} &= \sup_{0\le t\le\omega} \|f(t)\|_X < \infty. \end{aligned}$$

Let $W^{k,r}(\omega;X)$ denote the set of functions which belong to $L^r(\omega;X)$ together with their partial derivatives with respect to t up to order k. Particularly, $H^k(\omega;X)=W^{k,2}(\omega;X)$ if X is a Hilbert space.

Before stating our main results, we present a proposition which will be used occasionally later.

Proposition 2.1.[2] *If* $N\ge 2$ *and* $0\le\delta<\frac{1}{2}+\frac{N}{4}$, *the following estimate holds with a constant* $c_1=c_1(\delta,\theta_1,\theta_2)$:

$$\|A^{-\delta}P(u\cdot\nabla)v\|\le c_1\|A^{\theta_1}u\|\|A^{\theta_2}v\|,\ \forall u\in D(A^{\theta_1}),v\in D(A^{\theta_2}),\tag{2.2}$$

with $\delta+\theta_1+\theta_2\ge\frac{N}{4}+\frac{1}{2}$, $\theta_2+\delta>\frac{1}{2}$ *and* $\theta_1,\theta_2>0$.

We need the following inequality due to Giga and Miyakawa[2]:

$$\|u\|_{L^r(\Omega)}\le c_2\|A^\gamma u\|,\quad\text{if}\quad\frac{1}{r}\ge\frac{1}{2}-\frac{2\gamma}{N},\gamma\ge 0.$$

Particularly, if $r=N$, then

$$\|u\|_{L^N(\Omega)}\le c_3\|A^\gamma u\|\quad\text{with}\quad\gamma=\frac{N}{4}-\frac{1}{2}\ge 0.$$

Now our main results read as follows.

Theorem 2.1. *For any* $f(x,t),g(x,t)\in H^1(\omega;H)$, *there exists a positive constant* $K_0=K_0(N)$ *such that if* $M=\max\Big(\sup\limits_{0\le t\le\omega}\|f\|_{L^{\frac{N}{2}}(\Omega)},\sup\limits_{0\le t\le\omega}\|g\|_{L^{\frac{N}{2}}(\Omega)}\Big)\le K_0$, *the problem* (2.1) *has a strong* ω*-periodic solution* $(u(x,t),B(x,t))$ *satisfying*

$$u(x,t),B(x,t)\in H^2(\omega;H)\cap H^1(\omega;D(A))\cap L^\infty(\omega;D(A))\cap W^{1,\infty}(\omega;V).$$

Theorem 2.2. *If* M *defined in above theorem is sufficiently small, the strong periodic solution obtained in Theorem* 2.1 *of the problem* (2.1) *is unique.*

§3. Approximate Solutions and Estimates of Them

Firstly, we will show the existence of approximate solutions of the problem (2.1) under conditions in Theorem 2.1. We consider the system of ODE as follows:

$$\begin{aligned}(u_{nt}+\nu Au_n+P(u_n\cdot\nabla)u_n-\frac{1}{\rho\mu}P(B_n\cdot\nabla)B_n,w_i)&=(f,w_i)\quad(i=1,\cdots,n),\\(B_{nt}+\lambda AB_n+P(u_n\cdot\nabla)B_n-P(B_n\cdot\nabla)u_n,w_i)&=(g,w_i)\quad(i=1,\cdots,n),\\u_n(t+\omega)=u_n(t),\qquad B_n(t+\omega)&=B_n(t),\end{aligned}\tag{3.1}$$

where $u_n=\sum\limits_{i=1}^n c_{in}(t)w_i,\ B_n(t)=\sum\limits_{i=1}^n d_{in}(t)w_i$. Let W_n be span$\{w_1,w_2,\cdots,w_n\}$. It is well known that for any $v_n(t)=\sum\limits_{i=1}^n b_{in}(t)w_i,\ b_n(t)=\sum\limits_{i=1}^n r_{in}(t)w_i\in C^1(\omega;W_n)$, there exists a unique ω-periodic solution $(u_n(t),B_n(t))\in C^1(\omega;W_n)\times C^1(\omega;W_n)$ of the following linear problem

$$\begin{aligned}(u_{nt}+\nu Au_n,w_i)&=(f-P(v_n\cdot\nabla)v_n+\frac{1}{\rho\mu}(b_n\cdot\nabla)b_n,w_i)\quad(i=1,\cdots,n),\\(B_{nt}+\lambda AB_n,w_i)&=(g-P(v_n\cdot\nabla)b_n+(b_n\cdot\nabla)v_n,w_i)\quad(i=1,\cdots,n).\end{aligned}\tag{3.2}$$

Moreover, it is clear that the mapping $F:(v_n,b_n)\to(u_n,B_n)$ defined by (3.2) is continuous and compact in $C^1(\omega;W_n)\times C^1(\omega;W_n)$. Thereby, we shall prove the existence of periodic solutions of the problem (3.1) by applying the well-known Leray-Schauder fixed point theorem. To apply this theorem, it is sufficient to show the following uniform boundedness

$$\sup_{0\le t\le\omega}(\|u_n(t)\|^2+\frac{1}{\rho\mu}\|B_n(t)\|^2)\le c_4\tag{3.3}$$

holds with respect to κ $(0 \le \kappa \le 1)$ for all possible solutions of the following ODE

$$(u_{nt} + \nu A u_n, w_i) = (f - \kappa P(u_n \cdot \nabla)u_n + \frac{\kappa}{\rho\mu} P(B_n \cdot \nabla)B_n, w_i) \quad (i = 1, \cdots, n), \tag{3.4}$$

$$(B_{nt} + \lambda A B_n, w_i) = (g - \kappa P(u_n \cdot \nabla)B_n + \kappa P(B_n\ cdot\nabla)u_n, w_i) \quad (i = 1, \cdots, n). \tag{3.5}$$

In fact, multiplying (3.4) by $c_{in}(t)$ and (3.5) by $\frac{1}{\rho\mu} d_{in}(t)$ and then summing up over i, we obtain

$$(u_{nt}, u_n) + \nu(Au_n, u_n) = (f - \kappa P(u_n \cdot \nabla)u_n, u_n) + \frac{\kappa}{\rho\mu}(P(B_n \cdot \nabla)B_n, u_n),$$

$$\frac{1}{\rho\mu}(B_{nt}, B_n) + \frac{\lambda}{\rho\mu}(AB_n, B_n) = \frac{1}{\rho\mu}(g - \kappa P(u_n \cdot \nabla)B_n, B_n) + \frac{\kappa}{\rho\mu}(P(B_n \cdot \nabla)u_n, B_n).$$

By summing above two equalities together and noticing that

$$(P(u \cdot \nabla)v, w) = -(P(u \cdot \nabla)w, v), \quad \forall u, v, w \in V,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_n\|^2 + \frac{1}{\rho\mu}\|B_n\|^2) + \nu\|\nabla u\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_n\|^2 \\
&\le \|f\|_{L^{\frac{2N}{N+2}}}\|u_n\|_{L^{\frac{2N}{N-2}}} + \frac{1}{\rho\mu}\|g\|_{L^{\frac{2N}{N+2}}}\|B_n\|_{L^{\frac{2N}{N-2}}} \\
&\le c_3 c(N,\Omega)(\|f\|_{L^{\frac{N}{2}}}\|\nabla u_n\| + \frac{1}{\rho\mu}\|g\|_{L^{\frac{N}{2}}}\|\nabla B_n\|),
\end{aligned}$$

where $c(N, \Omega) = |\Omega|^{\frac{N-2}{2N}}$ and $|\Omega|$ is the volume of Ω. Then we obtain

$$\frac{d}{dt}\Big(\|u_n\|^2 + \frac{1}{\rho\mu}\|B_n\|^2\Big) + \nu\|\nabla u\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_n\|^2 \le c_3^2 c(N,\Omega)^2 M^2 \Big(\frac{1}{\nu} + \frac{1}{\rho\mu\lambda}\Big). \tag{3.6}$$

Furthermore, considering the periodicity of u_n, B_n and integrating (3.6) over $[0, \omega]$, we have

$$\nu \int_0^\omega \|\nabla u_n(t)\|^2 dt + \frac{\lambda}{\rho\mu}\int_0^\omega \|\nabla B_n(t)\|^2 dt \le c_3^2 c^2(N,\Omega) M^2 \Big(\frac{1}{\nu} + \frac{1}{\rho\mu\lambda}\Big)\omega. \tag{3.7}$$

According to the following inequality[9]

$$\|A^\alpha v\| \le \lambda_1^{\alpha-\beta}\|A^\beta v\|, \quad \forall v \in D(A^\beta) \quad (0 \le \alpha \le \beta)$$

and from (3.7), it follows that there exists $t^* \in [0, \omega]$ such that

$$\nu\|u_n(t^*)\|^2 + \frac{\lambda}{\rho\mu}\|B_n(t^*)\|^2 \le \lambda_1^{-1} c_3^2 c^2(N,\Omega) M^2 \Big(\frac{1}{\nu} + \frac{1}{\rho\mu\lambda}\Big).$$

By integrating (3.6) again from t^* to $t + \omega$ $(\forall t \in [0, \omega])$, we obtain

$$\sup_{0\le t\le\omega}(\|u_n(t)\|^2 + \frac{1}{\rho\mu}\|B_n(t)\|^2) \le c_3^2 c^2(N,\Omega) M^2 (2\omega + \beta^{-1}\lambda_1^{-1})\Big(\frac{1}{\nu} + \frac{1}{\rho\mu\lambda}\Big) \tag{3.8}$$

with $\beta = \min(\nu, \lambda)$.

Since the right-hand side of (3.8) is independent of κ, we have proven the existence of solution $(u_n, B_n) \in C^1(\omega; W_n) \times C^1(\omega; W_n)$ according to the Leray-Schauder fixed point theorem.

In the following, we will prove the uniform boundedness of $\|A^\gamma u_n(t)\|$ and $\|A^\gamma B_n(t)\|$ $(\gamma = \frac{N}{4} - \frac{1}{2})$ with respect to n. Firstly, we note that we can choose the bases $\{w_i\}$ $(i = 1, 2, \cdots)$ such that the eigenvectors $\{w_i\}$ $(i = 1, 2, \cdots)$ are also eigenvectors of $A^\gamma (0 \le \gamma \le 1)$ and that $Aw_i = \lambda_i w_i$, $A^\gamma w_i = \lambda_i^\gamma w_i$ $(i = 1, 2, \cdots)$.

Lemma 3.1. *Let $(u_n(t), B_n(t))$ be solutions of the problem (3.1). If $M \le K_0 = \min(K^{-2}, 1)$ with*

$$K = \beta^{-\frac{1}{2}}\lambda_1^{-\gamma}\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big)\alpha_1 + c_1(\beta^{-1}+\alpha_2^{-1})\beta^{-\frac{1}{2}}\lambda_1^{\gamma-\frac{1}{2}}c_3c(N,\Omega)\sqrt{\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}},$$

where $\gamma = \frac{N}{4}-\frac{1}{2}$, $\alpha_1 = \max\Big\{\Big(1+\frac{1}{2}\sqrt{\frac{1}{\rho\mu}}\Big(1+\frac{\lambda}{\nu\sqrt{\rho\mu}}\Big)\Big), \sqrt{\frac{1}{\rho\mu}}\Big(1+\frac{1}{2}\Big(\nu+\frac{\lambda}{\sqrt{\rho\mu}}\Big)\frac{\rho\mu}{\lambda}\Big)\Big\}$ and $\alpha_2 = \min\Big(\nu, \frac{\lambda}{\rho\mu}\Big)$, then the following estimate holds:

$$\|A^\gamma u_n(t)\|^2 + \frac{1}{\rho\mu}\|A^\gamma B_n(t)\|^2 \le \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big), \quad \forall t\in(-\infty,\infty).$$

Proof. Multiplying the first system of equations in (3.1) with $\lambda_i^{2\gamma}c_{in}(t)w_i$ and the second system of equations in (3.1) with $\frac{1}{\rho\mu}\lambda_i^{2\gamma}d_{in}(t)w_i$, and then summing up over i, we obtain

$$(u_{nt}+\nu Au_n, A^{2\gamma}u_n) = (f-P(u_n\cdot\nabla)u_n, A^{2\gamma}u_n) + \frac{1}{\rho\mu}(P(B_n\cdot\nabla)B_n, A^{2\gamma}u_n),$$

$$\frac{1}{\rho\mu}(B_{nt}+\lambda AB_n, A^{2\gamma}B_n) = \frac{1}{\rho\mu}(g-P(u_n\cdot\nabla)B_n, A^{2\gamma}B_n) + \frac{1}{\rho\mu}(P(B_n\cdot\nabla)u_n, A^{2\gamma}B_n).$$

Summing above two equalities together and noticing Proposition 2.1 yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|A^\gamma u_n\|^2+\frac{1}{\rho\mu}\|A^\gamma B_n\|^2\Big)+\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2+\frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2\\
&\le c_3c(N,\Omega)\beta^{-1}M\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|+\frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|\Big)\\
&\quad + c_1(\beta^{-1}\|A^\gamma u_n\|+(\rho\mu\alpha_2)^{-1}\|A^\gamma B_n\|)\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2+\frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2\Big).
\end{aligned}\tag{3.9}$$

But

$$\begin{aligned}
\nu\|A^\gamma u_n(t^*)\|^2+\frac{\lambda}{\rho\mu}\|A^\gamma B_n(t^*)\|^2 &\le \lambda_1^{2\gamma-1}(\nu\|\nabla u_n(t^*)\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_n(t^*)\|^2)\\
&\le \lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M^2\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big).
\end{aligned}$$

Hence, we get

$$\|A^\gamma u_n(t^*)\|^2+\frac{1}{\rho\mu}\|A^\gamma B_n(t^*)\|^2 \le \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big),$$

by assuming $M<1$. Then we can set

$$\begin{aligned}
T^* = \sup\Big\{T\Big|&\|A^\gamma u_n(t)\|^2+\frac{1}{\rho\mu}\|A^\gamma B_n(t)\|^2\\
&\le \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big),\quad \forall t\in[t^*,T)\Big\}.
\end{aligned}$$

Next we will show $T^*=\infty$. In fact, if $T^*(t^*<T^*)$ is finite, it should follow that

$$\|A^\gamma u_n(T^*)\|^2+\frac{1}{\rho\mu}\|A^\gamma B_n(T^*)\|^2 = \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big).\tag{3.10}$$

Therefore, for such a value $t=T^*$, the estimates of the right-hand side of (3.9) are

$$\begin{aligned}
&c_3c(N,\Omega)\beta^{-1}M\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|+\frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|\Big)\\
&\le \beta^{-\frac{1}{2}}\lambda_1^{-\gamma}\Big(\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}\Big)^{-1}M^{\frac{1}{2}}\alpha_1\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2+\frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}u_n\|^2\Big),
\end{aligned}$$

$$c_1(\beta^{-1}\|A^\gamma u_n\| + (\rho\mu\alpha_2)^{-1}\|A^\gamma B_n\|)\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2 + \frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2\Big)$$
$$\le c_1(\beta^{-1}+\alpha_2^{-1})\beta^{-\frac12}\lambda_1^{\gamma-\frac12}c_3c(N,\Omega)\sqrt{\frac{1}{\nu}+\frac{1}{\rho\mu\lambda}}M^{\frac12}\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2 + \frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2\Big),$$

where (3.10) has been used.

Then we obtain

$$\frac12\frac{d}{dt}\Big(\|A^\gamma u_n\|^2 + \frac{1}{\rho\mu}\|A^\gamma B_n\|^2\Big) + \nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2 + \frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2$$
$$\le KM^{\frac12}\Big(\nu\|A^{\frac{1+2\gamma}{2}}u_n\|^2 + \frac{\lambda}{\rho\mu}\|A^{\frac{1+2\gamma}{2}}B_n\|^2\Big),$$

where K was stated in Lemma 3.1.

According to the assumptions in this lemma, it follows that

$$\frac{d}{dt}\Big(\|A^\gamma u_n\|^2 + \frac{1}{\rho\mu}\|A^\gamma B_n\|^2\Big) < 0, \quad \text{at} \quad t = T^*.$$

Thus, in a neighborhood of $t = T^*$, it follows that

$$\|A^\gamma u_n(t)\|^2 + \frac{1}{\rho\mu}\|A^\gamma B_n(t)\|^2 \le \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac1\nu + \frac{1}{\rho\mu\lambda}\Big) \text{ for any } t\in[T^*, T^*+\delta),$$

which implies $T^* = \infty$.

Combining it with the periodicity of (u_n, B_n) yields

$$\|A^\gamma u_n(t)\|^2 + \frac{1}{\rho\mu}\|A^\gamma B_n(t)\|^2 \le \beta^{-1}\lambda_1^{2\gamma-1}c_3^2c^2(N,\Omega)M\Big(\frac1\nu + \frac{1}{\rho\mu\lambda}\Big), \quad \forall t\in(-\infty,\infty).$$

Then the proof of this lemma is completed.

To prove the convergence of approximate solutions, we need give estimates of higher order derivatives of approximate solutions. Firstly, from Lemma 3.1, we know

$$\sup_t\|A^\gamma u_n\|, \sup_t\|A^\gamma B_n(t)\| \le c(M),$$

where $\gamma = \frac{N}{4} - \frac12$ and $c(M)$ denotes a constant depending on M and independent of n. So we can make $c(M) < c$ for any positive constant c if M is small enough.

Lemma 3.2. *Let $(u_n(t), B_n(t))$ be the solutions of the problem (3.1) and*

$$M_0 = \max\Big\{\Big(\int_0^\omega \|f\|^2dt\Big)^{\frac12}, \Big(\int_0^\omega \|g\|^2dt\Big)^{\frac12}\Big\},$$
$$M_1 = \max\Big\{\Big(\int_0^\omega \|f_t\|^2dt\Big)^{\frac12}, \Big(\int_0^\omega \|g_t\|^2dt\Big)^{\frac12}\Big\}.$$

Then we have

$$\sup_t\|\nabla u_n(t)\|, \sup_t\|\nabla B_n(t)\| \le c(M_0, M), \quad \sup_t\|u_{nt}(t)\|, \sup_t\|B_{nt}(t)\| \le c(M_0, M_1, M),$$

$$\sup_t\|Au_n(t)\|, \sup_t\|AB_n(t)\| \le c(M_0, M_1, M),$$
$$\sup_t\|\nabla u_{nt}(t)\|, \sup_t\|\nabla B_{nt}(t)\| \le c(M_0, M_1, M),$$
$$\int_0^\omega \|Au_{nt}(t)\|^2dt, \int_0^\omega \|AB_{nt}(t)\|^2dt \le c(M_0, M_1, M),$$
$$\int_0^\omega \|u_{ntt}(t)\|^2dt, \int_0^\omega \|B_{ntt}(t)\|^2dt \le c(M_0, M_1, M).$$

Proof. From (3.1), we obtain

$$(u_{nt} + \nu Au_n, Au_n) = (f - P(u_n \cdot \nabla)u_n, Au_n) + \frac{1}{\rho\mu}(P(B_n \cdot \nabla)B_n, Au_n),$$

$$\frac{1}{\rho\mu}(B_{nt} + \lambda AB_n, AB_n) = \frac{1}{\rho\mu}(g - P(u_n \cdot \nabla)B_n, AB_n) + \frac{1}{\rho\mu}(P(B_n \cdot \nabla)u_n, AB_n).$$

Summing these two equalities together yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u_n\|^2 + \frac{1}{\rho\mu}\|\nabla B_n\|^2\Big) + \nu\|Au_n\|^2 + \frac{\lambda}{\rho\mu}\|AB_n\|^2\\ &\le \|f\|\|Au_n\| + \frac{1}{\rho\mu}\|g\|\|AB_n\| + c_1c(M)\Big(\|Au_n\|^2 + \frac{1}{\rho\mu}\|AB_n\|^2\Big),\end{aligned} \tag{3.11}$$

where we have used Proposition 2.1 and Lemma 3.1.

By integrating (3.11) over $[0,\omega]$, we get

$$\begin{aligned}\int_0^\omega \Big(\nu\|Au_n\|^2 + \frac{\lambda}{\rho\mu}\|AB_n\|^2\Big)dt &\le 2M_0\beta^{-1}\Big(\int_0^\omega (\nu\|Au_n\|^2 + \frac{\lambda}{\rho\mu}\|AB_n\|^2)dt\Big)^{\frac{1}{2}}\\ &\quad + c_1c(M)\beta^{-1}\int_0^\omega \Big(\nu\|Au_n\|^2 + \frac{\lambda}{\rho\mu}\ \|AB_n\|^2\Big)dt.\end{aligned}$$

Choosing M sufficiently small such that $c_1c(M)\beta^{-1} < 1$, we obtain

$$\int_0^\omega \Big(\nu\|Au_n\|^2 + \frac{\lambda}{\rho\mu}\|AB_n\|^2\Big)dt \le c(M_0, M). \tag{3.12}$$

Due to (3.12), there exists $t_1^* \in [0,\omega]$ such that

$$\nu\|\nabla u_n(t_1^*)\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_n(t_1^*)\|^2 \le \lambda_1^{-1}\Big(\nu\|Au_n(t_1^*)\|^2 + \frac{\lambda}{\rho\mu}\|AB_n(t_1^*)\|^2\Big) \le \lambda_1^{-1}\omega^{-1}c(M_0, M).$$

Integrating (3.11) from t_1^* to $t + \omega (\forall t \in [0,\omega])$ easily yields

$$\sup_t \|\nabla u_n(t)\|, \quad \sup_t \|\nabla B_n(t)\| \le C(M_0, M). \tag{3.13}$$

Similarly, from (3.1) it follows that

$$(u_{nt} + \nu Au_n, u_{nt}) = (f - P(u_n \cdot \nabla)u_n, u_{nt}) + \frac{1}{\rho\mu}(P(B_n \cdot \nabla)B_n, u_{nt}),$$

$$\frac{1}{\rho\mu}(B_{nt} + \lambda AB_n, B_{nt}) = \frac{1}{\rho\mu}(g - P(u_n \cdot \nabla)B_n, B_{nt}) + \frac{1}{\rho\mu}(P(B_n \cdot \nabla)u_n, B_{nt}).$$

Summing them together and noticing Lemma 3.1 and (3.13) we have

$$\begin{aligned}&\|u_{nt}\|^2 + \frac{1}{\rho\mu}\|B_{nt}\|^2 + \frac{d}{dt}(\nu\|\nabla u_n\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_n\|^2)\\ &\le \|f\|^2 + \frac{1}{\rho\mu}\|g\|^2 + c(M_0, M)(\|\nabla u_{nt}\| + \|\nabla B_{nt}\|).\end{aligned} \tag{3.14}$$

Integrating this inequality over $[0,\omega]$ gives

$$\begin{aligned}&\int_0^\omega \Big(\|u_{nt}\|^2 + \frac{1}{\rho\mu}\|B_{nt}\|^2\Big)dt\\ &\le \Big(1 + \frac{1}{\rho\mu}\Big)M_0^2 + C(M_0, M)\int_0^\omega \Big(\sqrt{\nu}\|\nabla u_{nt}\| + \sqrt{\frac{\lambda}{\rho\mu}}\|\nabla B_{nt}\|\Big)dt.\end{aligned} \tag{3.15}$$

Differentiating the first two systems of equations in (3.1) with respect to t and multiplying them by $c'_{in}(t)$ and $\frac{1}{\rho\mu}d'_{in}(t)$ defined in (3.1) respectively, and summing up over i, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u_{nt}\|^2+\frac{1}{\rho\mu}\|B_{nt}\|^2\Big)+\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\\&=(f_t-P(u_{nt}\cdot\nabla)u_n,u_{nt})+\frac{1}{\rho\mu}\Big[(P(B_{nt}\cdot\nabla)B_n,u_{nt})\\&\quad+(P(B_{nt}\cdot\nabla)u_n,B_{nt})+(g_t-P(u_{nt}\cdot\nabla)B_n,B_{nt})\Big].\end{aligned}$$

Noticing Proposition 2.1 and the following interpolation inequality

$$\|A^\sigma v\|\le c_4\|A^{\gamma_1}v\|^{\sigma_1}\|A^{\gamma_2}v\|^{1-\sigma_1},\quad \forall v\in D(A^{\gamma_2})$$

with $\sigma=\gamma_1\sigma_1+\gamma_2(1-\sigma_1), 0<\gamma_1<\sigma<\gamma_2$ $(\sigma_1\ge 0)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u_{nt}\|^2+\frac{1}{\rho\mu}\|B_{nt}\|^2\Big)+\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|B_{nt}\|^2\\&\le\frac{1}{2\nu\lambda_1}\|f_t\|^2+\frac{\nu}{2}\|\nabla u_{nt}\|^2+\frac{1}{2\rho\mu\lambda\lambda_1}\|g_t\|^2\\&\quad+\frac{\lambda}{2\rho\mu}\|\nabla B_{nt}\|^2+c_2c_3c_4\|\nabla u_n\|\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{1+2\gamma}\\&\quad+\frac{c_2c_3c_4}{\rho\mu}(\|\nabla B_n\|\|B_{nt}\|^{1-2\gamma}\|\nabla B_{nt}\|^{2\gamma}\|\nabla u_{nt}\|\\&\quad+\|\nabla B_n\|\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{2\gamma}\|\nabla B_{nt}\|+\|\nabla u_n\|\|B_{nt}\|^{1-2\gamma}\|\nabla B_{nt}\|^{1+2\gamma}).\end{aligned}\tag{3.16}$$

In a similar way, we get

$$\begin{aligned}&\int_0^\omega\Big(\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt\\&\le\lambda_1^{-1}(\nu^{-1}+(\rho\mu\lambda)^{-1})M_1^2+2c_2c_3c_4c(M_0,M)\int_0^\omega\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{1+2\gamma}dt\\&\quad+\frac{2c_2c_3c_4}{\rho\mu}c(M_0,M)\Big(\int_0^\omega\|B_{nt}\|^{1-2\gamma}\|\nabla B_{nt}\|^{2\gamma}\|\nabla u_{nt}\|dt\\&\quad+\int_0^\omega\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{2\gamma}\|\nabla B_{nt}\|dt+\int_0^\omega\|B_{nt}\|^{1-2\gamma}\|\nabla B_{nt}\|^{1+2\gamma}dt\Big).\end{aligned}\tag{3.17}$$

Now since $\gamma=\frac{1}{4}$ for $N=3$, we see that (3.17) implies

$$\begin{aligned}&\int_0^\omega(\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2)dt\\&\le c(M_1)+c(M_0,M)\Big[\Big(\int_0^\omega\|u_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla u_{nt}\|^2dt\Big)^{\frac34}\\&\quad+\Big(\int_0^\omega\|B_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla u_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla B_{nt}\|^2dt\Big)^{\frac12}\\&\quad+\Big(\int_0^\omega\|B_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla B_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla u_{nt}\|^2dt\Big)^{\frac12}\\&\quad+\Big(\int_0^\omega\|B_{nt}\|^2dt\Big)^{\frac14}\Big(\int_0^\omega\|\nabla B_{nt}\|^2dt\Big)^{\frac34}\Big]\\&\le c(M_1)+c(M_0,M)\Big[\int_0^\omega(\|u_{nt}\|^2+\frac{1}{\rho\mu}\|B_{nt}\|^2\Big)dt\Big]^{\frac14}\Big[\int_0^\omega\Big(\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt\Big]^{\frac34}.\end{aligned}$$

Noticing (3.15), we know

$$\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt \le c(M_1) + c(M_0, M)\Big(\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2$$

$$+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt\Big)^{\frac{3}{4}} + c(M_0, M)\Big(\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt\Big)^{\frac{7}{8}},$$

which implies the boundedness

$$\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt \le c(M_0, M_1, M).$$

On the other hand, $\gamma = \frac{1}{2}$ for $N = 4$, from (3.17) it follows that

$$\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt \le c(M_1) + c(M_0, M)\int_0^\omega \Big(\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big)dt. \tag{3.18}$$

We can choose M_0, M sufficiently small such that $c(M_0, M) < 1$. Then from (3.18) we get

$$\int_0^\omega (\nu\|\nabla u_{nt}\|^2 + \frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2)dt \le c(M_0, M_1, M). \tag{3.19}$$

Hence, there exists $t_2^* \in [0, \omega]$ such that

$$\nu\|u_{nt}(t_2^*)\|^2 + \frac{\lambda}{\rho\mu}\|B_{nt}(t_2^*)\|^2 \le \lambda_1^{-1}\omega^{-1}c(M_0, M_1, M). \tag{3.20}$$

Consequently, integrating (3.16) from t_2^* to $t + \omega$ ($\forall t \in [0, \omega]$) yields

$$\sup_t \|u_{nt}\|, \quad \sup_t \|B_{nt}\| \le c(M_0, M_1, M). \tag{3.21}$$

Similarly, from (3.1) we have

$$(u_{nt} + \nu A u_n, A u_n) = (f - P(u_n\cdot\nabla)u_n, Au_n) + \frac{1}{\rho\mu}(P(B_n\cdot\nabla)B_n, Au_n),$$
$$(B_{nt} + \lambda AB_n, AB_n) = (g - P(u_n\cdot\nabla)B_n, AB_n) + (P(B_n\cdot\nabla)u_n, AB_n).$$

But

$$\|P(u_n\cdot\nabla)u_n\| \le c_1\|A^\gamma u_n\|\|Au_n\| \le c_1c(M)\|Au_n\|,$$
$$\|P(B_n\cdot\nabla)B_n\| \le c_1c(M)\|AB_n\|, \quad \|P(u_n\cdot\nabla)B_n\| \le c_1c(M)\|AB_n\|,$$
$$\|P(B_n\cdot\nabla)u_n\| \le c_1c(M)\|Au_n\|.$$

Due to the above inequalities and Holder inequality, it follows that

$$\nu\|Au_n\| + \frac{\lambda}{\rho\mu}\|AB_n\| \le c(M_0, M_1, M) + c_1c(M)\Big(\nu\|Au_n\| + \frac{\lambda}{\rho\mu}\|AB_n\|\Big).$$

We can choose M sufficiently small such that $c_1c(M) < 1$. Thus we obtain

$$\sup_t \|Au_n\|, \quad \sup_t \|AB_n\| \le c(M_0, M_1, M).$$

Differentiating the first two systems of equations in (3.1) with respect to t and taking scalar product in H with Au_{nt} and $\frac{1}{\rho\mu}AB_{nt}$ respectively, and then summing them together and noticing the interpolation inequality in Section 1, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla u_{nt}\|^2+\frac{1}{\rho\mu}\|\nabla B_{nt}\|^2\Big)+\nu\|Au_{nt}\|^2+\frac{\lambda}{\rho\mu}\|AB_{nt}\|^2\\
&\le c(M_1)+c(M_0,M_1,M)\Big[\|\nabla u_{nt}\|\|Au_{nt}\|+\frac{1}{\rho\mu}(\|\nabla B_{nt}\|\|Au_{nt}\|+\|\nabla u_{nt}\|\|AB_{nt}\|\\
&\quad+\|\nabla B_{nt}\|\|AB_{nt}\|)\Big]+c(M,M_0)\Big[\|A^{\gamma+\frac12}u_{nt}\|\|Au_{nt}\|+\frac{1}{\rho\mu}(\|A^{\gamma+\frac12}B_{nt}\|\|Au_{nt}\|\\
&\quad+\|A^{\gamma+\frac12}B_{nt}\|\|AB_{nt}\|+\|A^{\gamma+\frac12}u_{nt}\|\|AB_{nt}\|)\Big].
\end{aligned}\tag{3.22}
$$

As for the estimate (3.22), if $N=3$, we know

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla u_{nt}\|^2+\frac{1}{\rho\mu}\|\nabla B_{nt}\|^2\Big)+\nu\|Au_{nt}\|^2+\frac{\lambda}{\rho\mu}\|AB_{nt}\|^2\\
&\le c(M_1)+c(M_0,M_1,M)\Big[\|\nabla u_{nt}\|\|Au_{nt}\|+\frac{1}{\rho\mu}(\|\nabla B_{nt}\|\|Au_{nt}\|+\|\nabla u_{nt}\|\|AB_{nt}\|\\
&\quad+\|\nabla B_{nt}\|\|AB_{nt}\|)\Big]+c(M,M_0)\Big[\|\nabla u_{nt}\|^{\frac12}\|Au_{nt}\|^{\frac32}+\frac{1}{\rho\mu}(\|\nabla B_{nt}\|^{\frac12}\|AB_{nt}\|^{\frac12}\|Au_{nt}\|\\
&\quad+\|\nabla B_{nt}\|^{\frac12}\|AB_{nt}\|^{\frac32}+\|\nabla u_{nt}\|^{\frac12}\|Au_{nt}\|^{\frac12}\|AB_{nt}\|)\Big].
\end{aligned}\tag{3.23}
$$

If $N=4$, we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla u_{nt}\|^2+\frac{1}{\rho\mu}\|\nabla B_{nt}\|^2\Big)+\nu\|Au_{nt}\|^2+\frac{\lambda}{\rho\mu}\|AB_{nt}\|^2\\
&\le c(M_1)+c(M_0,M_1,M)\Big[\|\nabla u_{nt}\|\|Au_{nt}\|+\frac{1}{\rho\mu}(\|\nabla B_{nt}\|\|Au_{nt}\|+\|\nabla u_{nt}\|\|AB_{nt}\|\\
&\quad+\|\nabla B_{nt}\|\|AB_{nt}\|)\Big]+c(M,M_0)\Big[\|Au_{nt}\|^2+\frac{1}{\rho\mu}(2\|AB_{nt}\|\|Au_{nt}\|+\|AB_{nt}\|^2)\Big].
\end{aligned}\tag{3.24}
$$

Applying Young inequality in (3.23)–(3.24) yields

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla u_{nt}\|^2+\frac{1}{\rho\mu}\|\nabla B_{nt}\|^2\Big)+\frac{\nu}{2}\|Au_{nt}\|^2+\frac{\lambda}{2\rho\mu}\|AB_{nt}\|^2\\
&\le c(M_1)+c^2(M_0,M_1,M)\Big(\nu\|\nabla u_{nt}\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}\|^2\Big).
\end{aligned}\tag{3.25}
$$

Integrating (3.25) over $[0,\omega]$ and noticing (3.19),we know

$$
\sup_t\int_0^\omega\Big(\nu\|Au_{nt}\|^2+\frac{\lambda}{\rho\mu}\|AB_{nt}\|^2\Big)dt\le c(M_0,M_1,M).\tag{3.26}
$$

Hence, $\sup\limits_t\int_0^\omega\|Au_{nt}\|^2dt,\quad \sup\limits_t\int_0^\omega\|AB_{nt}\|^2dt\le c(M_0,M_1,M)$. From (3.26) it follows that there exists $t_3^*\in[0,\omega]$ such that $\nu\|Au_{nt}(t_3^*)\|^2+\frac{\lambda}{\rho\mu}\|AB_{nt}(t_3^*)\|^2\le\omega^{-1}c(M_0,M_1,M)$. Thus we know $\nu\|\nabla u_{nt}(t_3^*)\|^2+\frac{\lambda}{\rho\mu}\|\nabla B_{nt}(t_3^*)\|^2\le\lambda_1^{-1}\omega^{-1}c(M_0,M_1,M)$. By integrating (3.25) from t_3^* to $t+\omega$ $(\forall t\in[0,\omega])$ we get $\sup\limits_t\|\nabla u_{nt}\|,\quad \sup\limits_t\|\nabla B_{nt}\|\le c(M_0,M_1,M)$. Furthermore, from (3.1),we have

$$\begin{aligned}(u_{ntt} + \nu A u_{nt}, u_{ntt}) &= (f_t - P(u_{nt} \cdot \nabla)u_n - P(u_n \cdot \nabla)u_{nt}, u_{ntt}) \\ &\quad + \frac{1}{\rho\mu}(P(B_{nt} \cdot \nabla)B_n + P(B_n \cdot \nabla)B_{nt}, u_{ntt}), \\ (B_{ntt} + \lambda A B_{nt}, B_{ntt}) &= (g_t - P(u_{nt} \cdot \nabla)B_n - P(u_n \cdot \nabla)B_{nt}, B_{ntt}) \\ &\quad + (P(B_{nt} \cdot \nabla)u_n + P(B_n \cdot \nabla)u_{nt}, B_{ntt}).\end{aligned} \tag{3.27}$$

Due to (3.27) it follows that

$$\begin{aligned}\|u_{ntt}\|^2 \le 2\nu\|Au_{nt}\|^2 + 2\|f_t\|^2 + 4c_2^2c_3^2(\|\nabla u_{nt}\|^2\|Au_n\|^2 + \|Au_n\|^2\|Au_{nt}\|^2 \\ + \|\nabla B_{nt}\|^2\|AB_n\|^2 + \|AB_n\|^2\|Au_{nt}\|^2) + \frac{3}{4}\|u_{ntt}\|^2.\end{aligned}$$

Hence, from the results obtained as above, we get $\int_0^\omega \|u_{ntt}\|^2 dt \le c(M_0, M_1, M)$. Similarly, $\int_0^\omega \|B_{ntt}\|^2 dt \le c(M_0, M_1, M)$. Lemma 3.2 is proven.

§4. Proofs of Main Results

Proof of Theorem 2.1. Due to the estimates obtained in Lemma 3.1 and Lemma 3.2, standard compactness arguments show that there exists a subsequence $\{u_n(t), B_n(t)\}$ converging to functions $\{u(t), B(t)\}$ in the following ways:

$$\begin{aligned}&u_n \to u, \quad B_n \to B \quad \text{weakly-star in} \quad L^\infty(\omega; D(A)), \\ &u_n \to u, \quad B_n \to B \quad \text{strongly in} \quad L^\infty(\omega; V), \\ &u_{nt} \to u_t, \quad B_{nt} \to B_t \quad \text{weakly-star in} \quad L^\infty(\omega; V), \\ &u_{nt} \to u_t, \quad B_{nt} \to B_t \quad \text{strongly in} \quad L^\infty(\omega; H),\end{aligned} \tag{4.1}$$

and the function $\{u(t), B(t)\}$ satisfies

$$u(t), \quad B(t) \in H^2(\omega; H) \cap H^1(\omega; D(A)) \cap L^\infty(\omega; D(A)) \cap W^{1,\infty}(\omega; V).$$

Since the first three convergence results in (4.1) are evident, it is sufficient to show the last convergence result in (4.1). In fact, the sequences $|(u_{nt}(t), w_i)|$, $|(B_{nt}(t), w_i)|$ $(n = i, i{+}1, \cdots)$ are uniformly bounded and equicontinuous. Let $0 \le h \le \omega$. Then

$$\begin{aligned}|(u_{nt}(t+h) - u_{nt}(t), w_i)| \le c(M_0, M_1, M)|h|^{\frac{1}{2}}\|w_i\|, \\ |(B_{nt}(t+h) - B_{nt}(t), w_i)| \le c(M_0, M_1, M)|h|^{\frac{1}{2}}\|w_i\|,\end{aligned}$$

where the $w_i (i = 1, 2, 3, \cdots)$ is the complete orthonormal system in H consisting of the eigenvectors of A as mentioned above. Therefore, using the diagonal process we can finally select a subsequence $\{u_{nt}(t), B_{nt}(t)\}$ which converges weakly and uniformly with respect to $t \in [0, \omega]$ to two elements in H. Furthermore, considering the boundedness of $\|\nabla u_{nt}\|$ and $\|\nabla B_{nt}\|$ in Lemma 3.2, we obtain all of the convergence results in (4.1).

Next, considering the lemmas in Section 3, we show that $P(u \cdot \nabla)u$, $P(u \cdot \nabla)B$, $P(B \cdot \nabla)u$,

$P(B\cdot\nabla)B$ are well defined and

$$\|P(u_n\cdot\nabla)u_n - P(u\cdot\nabla)u\| \le \|P((u_n-u)\cdot\nabla)u_n\| + \|P(u\cdot\nabla)(u_n-u)\|$$
$$\le c_1\Big[\|A^{\gamma+\frac14}(u_n-u)\|\|A^{\frac34}u_n\| + \|A^{\gamma+\frac14}u\|\|A^{\frac34}(u_n-u)\|\Big]$$
$$\le c(M_0,M)\Big[\|u_n-u\|^{\frac34-\gamma} + \|u_n-u\|^{\frac14}\Big] \to 0 \text{ as } n\to\infty \text{ uniformly with respect to } t,$$
$$\|P(u_n\cdot\nabla)B_n - P(u\cdot\nabla)B\| \le \|P((u_n-u)\cdot\nabla)B_n\| + \|P(u\cdot\nabla)(B_n-B)\|$$
$$\le c_1\Big[\|A^{\gamma+\frac14}(u_n-u)\|\|A^{\frac34}u_n\| + \|A^{\gamma+\frac14}u\|\|A^{\frac34}(B_n-B)\|\Big]$$
$$\le c(M_0,M)\Big[\|u_n-u\|^{\frac34-\gamma} + \|B_n-B\|^{\frac14}\Big] \to 0 \text{ as } n\to\infty \text{ uniformly with respect to } t.$$

Similarly,

$$\|P(B_n\cdot\nabla)B_n - P(B\cdot\nabla)B\| \to 0 \quad \text{as} \quad n\to\infty \quad \text{uniformly with respect to} \quad t,$$
$$\|P(B_n\cdot\nabla)u_n - P(B\cdot\nabla)u\| \to 0 \quad \text{as} \quad n\to\infty \quad \text{uniformly with respect to} \quad t.$$

Consequently, we know

$$\Big(u_t + \nu Au + P(u\cdot\nabla)u - \frac{1}{\rho\mu}P(B\cdot\nabla)B, w_i\Big) = (f, w_i),$$
$$\Big(B_t + \lambda AB + P(u\cdot\nabla)B - P(B\cdot\nabla)u, w_i\Big) = (f, w_i),$$

where $i = 1,2,\cdots$ and $t\in(-\infty,\infty)$. Since $\{w_i\}(i=1,2,\cdots)$ are complete bases of H and $f, g\in H$, we know that the equations (2.1) hold. Then the proof of Theorem 2.1 is completed.

Proof of Theorem 2.2 Let $(u_i, B_i)(i=1,2)$ be two solutions of the problem (2.1) and $w = u_1-u_2$, $b = b_1-b_2$. Then we know

$$\begin{aligned} w_t + \nu Aw + P(w\cdot\nabla)u_1 + P(u_2\cdot\nabla)w - \frac{1}{\rho\mu}P(b\cdot\nabla)B_1 - \frac{1}{\rho\mu}P(B_2\cdot\nabla)b = 0,\\ b_t + \lambda Ab + P(w\cdot\nabla)B_1 + P(u_2\cdot\nabla)b - P(b\cdot\nabla)u_1 - P(B_2\cdot\nabla)w = 0. \end{aligned} \tag{4.2}$$

By taking scalar product in H of the first equality in (4.2) with w and of the second equality in (4.2) with $\frac{1}{\rho\mu}b$, we get

$$\frac12\frac{d}{dt}\Big(\|w\|^2 + \frac{1}{\rho\mu}\|b\|^2\Big) + \nu\|\nabla w\|^2 + \frac{\lambda}{\rho\mu}\|\nabla b\|^2$$
$$= (A^{-\gamma}P(w\cdot\nabla)w, A^\gamma u_1) - \frac{1}{\rho\mu}\Big[(A^{-\gamma}P(b\cdot\nabla)w, A^\gamma B_1)$$
$$- (A^{-\gamma}P(w\cdot\nabla)b, A^\gamma B_1 b) + (A^{-\gamma}P(b\cdot\nabla)b, A^\gamma u_1)\Big]$$
$$\le c_1\|\nabla w\|^2\|A^\gamma u_1\| + \frac{c_1}{\rho\mu}\Big[2\|\nabla b\|\|\nabla w\|\|A^\gamma B_1\| + \|\nabla b\|^2\|A^\gamma u_1\|\Big]$$
$$\le c_1 c(M)\Big(\nu\|\nabla w\|^2 + \frac{\lambda}{\rho\mu}\|\nabla b\|^2\Big).$$

We can choose M sufficiently small such that $c_1c(M) < 1$. Then it follows that

$$\frac{d}{dt}\Big(\|w\|^2 + \frac{1}{\rho\mu}\|b\|^2\Big) \le 2\beta\lambda_1(c_1c(M)-1)\Big(\|w\|^2 + \frac{1}{\rho\mu}\|b\|^2\Big) = -L\Big(\|w\|^2 + \frac{1}{\rho\mu}\|b\|^2\Big),$$

where $L = 2\beta(1-c_1c(M))\lambda_1 > 0$. Hence, it follows that

$$\|w(t)\|^2 + \frac{1}{\rho\mu}\|b(t)\|^2 \le (\|w(0)\|^2 + \frac{1}{\rho\mu}\|b(0)\|^2)\exp(-Lt), \forall t\in(0,\infty).$$

Since $w(t), b(t)$ are periodic in time t for any $t \in (-\infty, \infty)$, there exists a positive integer n_0 such that $t + n_0\omega > 0$ and

$$\|w(t)\|^2 + \frac{1}{\rho\mu}\|b(t)\|^2 = \|w(t+n_0\omega)\|^2 + \frac{1}{\rho\mu}\|b(t+n_0\omega)\|^2.$$

Hence, it follows that

$$\|w(t)\|^2 + \frac{1}{\rho\mu}\|b(t)\|^2 \le \Big(\|w(0)\|^2 + \frac{1}{\rho\mu}\|b(0)\|^2\Big)\exp(-Ln\omega) \quad (n > n_0),$$

which implies $\|w(t)\| = 0$ and $\|b(t)\| = 0$. The proof of Theorem 2.2 is completed.

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