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ON THE EXISTENCE OF FIXED POINTS FOR LIPSCHITZIAN SEMIGROUPS IN BANACH SPACES**

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Abstract

Let C be a nonempty bounded subset of a p-uniformly convex Banach space X, and $T = \{T(t) : t \in S\}$ be a Lipschitzian semigroup on C with $\lim_{n\to\infty} \inf_{t\in s} |||T(t)||| < \sqrt{N_p}$, where N_p is the normal structure coefficient of X. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties: $(\mathrm{P}_1)x \in E$ implies $\omega_w(x) \subset E$; $(\mathrm{P}_2)T$ is asymptotically regular on E. The authors prove that there exists a $z \in E$ such that $T(s)z = z$ for all $s \in S$. Further, under the similar condition, the existence of fixed points of Lipschitzian semigroups in a uniformly convex Banach space is discussed.

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§1. Introduction and Preliminaries

Let C be a nonempty subset of a Banach space X. Then a mapping $T : C \to C$ is said to be a Lipschitzian mapping if, for each integer $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$. A Lipschitzian mapping T is said to be uniformly k-Lipschitzian if $k_n = k$ for all $n \geq 1$, nonexpansive if $k_n = 1$ for all $n \geq 1$, respectively. Moreover, a mapping $T : C \to C$ is called asymptotically regular[1,19], if $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$ for all $x \in C$. Edelstein and O'Brien[3] proved that if T is nonexpansive, then the averaged mappings $T_a = aI + (1-a)T$, where $a \in (1, 0)$ and I is the identity operator of X, are asymptotically regular on C, i.e., $\lim_{n\to\infty} \|T_a^n x - T_a^{n+1} x\| = 0$ for all $x \in C$.

Recently, Gornicki proved several fixed point theorems[5,6] for asymptotically regular Lipschitzian mappings. And also Lim and Xu[14] gave the following fixed point theorem for uniformly k-Lipschitzian mappings in a Banach space with uniformly normal structure.

Theorem 1.1.[14] *Suppose X is a Banach space with uniformly normal structure, C is a nonempty bounded subset of X, and $T : C \to C$ is a uniformly k-Lipschitzian mapping*

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with $k < N(X)^{1/2}$, where $N(X)$ is the normal structure coefficient of X. Suppose also there exists a nonempty bounded closed convex subset E of C with the following property (P):

$$\text{(P)} \qquad x \in E \text{ inplies } \ \omega_w(x) \subset E,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\{y \in X : \ y = weak - \lim_{j\to\infty} T^{n_j} x \text{ for some } n_j \uparrow \infty\}.$$

Then T has a fixed point in E.

On the other hand, let C be a nonempty subset of a Banach space X, and S be an unbounded subset of $[0,\infty)$ such that $t+h \in S$ for all $t,\ h \in S$ and $t-h \in S$ for all $t,\ h \in S$ with $t > h$ (e.g., $S = [0,\infty)$ or $S = N$, the set of nonnegative integers). Then a one-parameter family $T = \{T(t) : \ t \in \ S\}$ of mapping of C into itself is said to be a Lipschitzian semigroup on C if T satisfies the following conditions:

(1) $T(0)x = x$ for all $c \in C$; (2) $T(t+s)x = T(t)T(s)x$ for all $t,\ s \in S$ and $x \in C$;

(3) for each $x \in C$, the mapping $s \to T(s)x$ from S into C is continuous when S has the relative topology of $[0,\infty)$; and

(4) for each $t \in S$, there exists a constant $k_t > 0$ such that

$$\|T(t)x - T(t)y\| \le k_t\|x-y\| \text{ for all } x,y \text{ in } C.$$

A Lipschitzian semigroup $T = \{T(t) : \ t \in S\}$ on C is called asymptotically regular on a subset E of C if there exists some $h > 0$ in S such that

$$\lim_{t\in S, t\to\infty} \|T(t+r) - T(t)x\| = 0 \text{ for all } x \in C, \text{ and } 0 \le r \le h,\ r \in S.$$

For each $t \in S$ we denote

$$|||T(t)||| := \sup\{\|T(t)x - T(t)y\|/\|x-y\| : x,y \in C, x \ne y\}.$$

We denote by $F(T)$ the set of common fixed points of $T(t), t \in S$, i.e.,

$$F(T) = \{x \in C : T(s)x = x \text{ for all } s \in S\}.$$

Let E be a nonempty bounded closed convex subset of a Banach space X and let $d(E) = \sup\{\|x-y\| : x,y \in E\}$ be the diameter of E. For each $x \in E$, let $r(x,E) = \sup\{\|x-y\| : y \in E\}$ and let $r(E) = \inf\{r(x,E) : x \in E\}$, the Chebyshev radius of E relative to itself. The normal structure coefficient of X is defined[17] as the number

$$N(X) = \inf\{d(E)/r(E) : E \text{ bounded closed convex subset of } X \text{ with } d(E) > 0\}.$$

A space X with $N(X) > 1$ is said to have uniformly normal structure. Recall that a Banach space with uniformly normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniformly normal structure (cf. e.g., [20]).

In 1993, Tan and Xu[11] showed a fixed point theorem for uniformly Lipschitzian semigroups in a p-uniformly convex Banach space. And also Zeng[9] established a fixed point theorem for Lipschitzian semigroups without convexity in a Hilbert space. Thus, their results generalized Mizoguchi and Takahashi's result [21,Theorem 1]. On the other hand, Tan and Xu[11] presented a new fixed point theorem for uniformly k-Lipschitzian semigroups in a uniformly convex Banach space. Further, Zeng[8] obtained one fixed point theorem for asymptotically regular Lipschitzian semigroups in a p-uniformly convex Banach space and the other fixed point theorem for asymptotically regular Lipschitzian semigroups in a uniformly convex Banach space. Zeng's results[8] extended the results of Gornicki[6], and Tan and Xu[11] to the asymptotically regular Lipschitzian semigroup setting. In addition, see also [10].

The purpose of the present paper is to prove the following result: Let C be a nonempty bounded subset of a p-uniformly convex Banach space X, and $T = \{T(t) : \ t \in S\}$ be a Lipschitzian semigroup on C with $\lim_{t\to\infty}\inf_{t\in S} |||T(t)||| < \sqrt{N_p}$, where N_P is the normal structure coefficient of X. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties: (P_1) $x \in E$ implies $\omega_w(x) \subset E$; (P_2) T is asymptotically regular on E. Then $F(T)$ is nonempty. Further, under the similar condition, we discuss the existence of fixed points of Lipschitzian semigroups in a uniformly convex Banach space. Our results extend the above theorem of Lim and Xu[14] to the case of Lipschitzian semigroups, and improve and generalize the theorems of Zeng[8] by removing the restriction, the asymptotic regularity of T on C.

We shall need the following lemmas in the sequel.

Lemma 1.1.[4] *Suppose X is a Banach space with uniformly normal structure. Then for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X, there exists y in $\overline{\text{co}}(\{x_n: n \geq 1\})$ such that*

$$\lim_{n\to\infty} \sup \|x_n - y\| \leq \tilde{N}(X)A(\{x_n\}),$$

where $\tilde{N}(X) = N(X)^{-1}$, $\overline{\text{co}}(D)$ is the closure of the convex hull of $D \subset X$, and

$$A(\{x_n\}) = \lim_{n\to\infty} (\sup\{\|x_i - x_j\|: i, j \geq n\})$$

is the asymptotic diameter of $\{x_n\}_{n=1}^{\infty}$.

Recall that the modulus of convexity of a Banach space X is the function δ_x defined on [0,2] by $\delta_x(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\|: x, y \in B_x \text{ with } \|x-y\| \geq \varepsilon\right\}$, where B_x is the closed unit ball of X. X is said to be uniformly convex if $\delta_x(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$, Also recall that X is said to have the modulus of convexity of power type $p \geq 2$ (and X is said to be p-uniformly convex) if there exists a constant $d > 0$ such that

$$\delta_x(\varepsilon) \geq d\varepsilon^p \quad \text{for} \quad \varepsilon \in (0,2].$$

The Hilbert space H is 2-uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - (1 - (\frac{1}{2}\varepsilon)^2)^{1/2} \geq \frac{1}{8}\varepsilon^2$) and an L^p space $(1 < p < \infty)$ is max $(2, p)$-uniformly convex.

Lemma 1.2.[15,16] *Let X be a p-uniformly convex Banach space. Then there exists a constant $d_p > 0$ such that*

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - d_p W_p(t)\|x-y\|^p$$

for all x, y in X and $0 \leq t \leq 1$, where $W_p(t) = t(1-t)^p + t^p(1-t)$.

When X is particularly an L^p space, we have the following lemma.

Lemma 1.3.[7,12,15,16] *Suppose that X is an L^p space, $1 < p < \infty$. Then*

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - d_p W_q(t)\|x-y\|^q$$

for all x, y in X and $0 \leq t \leq 1$, where $q = \max(2, p)$, $W_q(t) = t^q(1-t) + t(1-t)^q$ and

$$d_p = \begin{cases} (1 + t_p^{p-1}/)(1+t)^{p-1} & \text{if } 2 < p < \infty, \\ p - 1 & \text{if } 1 < p \leq 2, \end{cases}$$

with t_p being the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \ t \in (0,1).$$

Remark 1.1. Casini and Maluta[2] proved that the normal structure coefficient N_p of an L^p space $(1 < p \leq 2)$ satisfies $N_p \geq \sqrt{p}$.

Let C be a nonempty bounded subset of a Banach space X, and the Lipschitzian semigroup $T = \{T(t): t \in S\}$ on C be asymptotically regular at some $u \in C$ and satisfy

$$\lim_{t\to\infty} \inf_{t \in S} |||T(t)||| = k.$$

Let $\{t_n\} \subset S$ be a positive sequence that increases monotonously to $+\infty$ and satisfies

$$\lim_{t\to\infty} \inf_{t \in S} |||T(t)||| = \lim_{n\to\infty} |||T(t_n)||| = k.$$

Now we define a function $r(.): C \to [0, \infty]$ as follows:

$$r(x) = \lim_{n\to\infty} \sup \|x - T(t_n)u\| \quad \text{for each } x \in C.$$

Lemma 1.4. *If $r(x) = 0$, then $x \in F(T)$.*

Finally, we remind the readers of the following fact: the notation $\omega_w(x)$ stands for the weak ω-limit set of T at x, i.e., the set

$$\{y \in X : \ y = \text{weak} - \lim_{t_\alpha \to \infty} T(t_\alpha)x \text{ for some subset } \{t_\alpha\} \subset S\}.$$

§2. Fixed Point Theorem for Lipschitzian Semigroups in p-Uniformly Convex Banach Spaces

Theorem 2.1. *Let C be a nonempty bounded subset of a p-uniformly convex Banach space X, and $T = \{T(t) : \ t \in S\}$ be a Lipschitzian semigroup on C with $\lim_{t\to\infty} \inf_{t\in S} |||T(t)||| < \sqrt{N_p}$, where N_p is the normal structure coefficient of X. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:*

(P_1) *$x \in E$ implies $\omega_w(x) \subset E$;*

(P_2) *T is asymptotically regular on E.*

Then there exists a $z \in E$ such that $T(s)z = z$ for all $s \in S$.

Proof. Let $\{t_n\}$ be a positive sequence which increases monotonously to $+\infty$ and satisfies

$$\lim_{t\to\infty} \inf_{t\in S} |||T(t)||| = \lim_{n\to\infty} |||T(t_n)||| = k < \sqrt{N_p}. \tag{2.1}$$

Without loss of generality, let $k \geq 1$. Take any x_0 in E and consider, for each integer $n \geq 1$, the sequence $\{T(t_j)x_0\}_{j\geq n}$. According to Lemma 1.1, for every bounded sequence $\{T(t_j)x_0\}_{j\geq n}$ we have a $y_n \in \overline{\text{co}}\{T(t_j)x_0 : \ j \geq n\}$ (here $\overline{\text{co}}$ denotes the closed convex hull) such that

$$\lim_{j\to\infty} \sup \|T(t_j)x_0 - y\| \leq \frac{1}{N_p} \cdot A(\{T(t_j)x_0\}_{j\geq n}), \tag{2.2}$$

where $A(z_n)$ denotes the asymptotic diameter of the sequence $\{z_n\}$, i.e., the number

$$\lim_{n\to\infty} (\sup\{z_i - z_j : \ i, j \geq n\}).$$

Since X is reflexive, $\{y_n\}$ admits a subsequence $\{y_{n'}\}$ converging weakly to some $x_1 \in X$. Form (2.2) and the w-l.s.c. of the functional $\lim_{n\to\infty} \sup \|T(t_n)x_0 - y\|$, it follows that

$$\lim_{n\to\infty} \sup \|T(t_n)x_0 - x_1\| \leq \frac{1}{N_P} \cdot A(\{T(t_n)x_0\}). \tag{2.3}$$

It is also easily seen that x_1 belongs to the set $\bigcap_{n=1}^{\infty} \overline{\text{co}}\{T(t_j)x_0 : \ j \geq n\}$ and that

$$\|z - x_1\| \leq \lim_{n\to\infty} \sup \|z - T(t_n)x_0\| \quad \text{for all } z \in X. \tag{2.4}$$

Observing the property (P_1) and the fact that $\bigcap_{n=1}^{\infty} \overline{\text{co}}\{T(t_j)x_0 : \ j \geq n\} = \overline{\text{co}}\omega'_w(x_0)$, which is easy to be proven by using the Separation Theorem[22], we know that x_1 actually lies in E, where $\omega'_w(x_0)$ is the weak ω-limit set of the sequence $\{T(t_n)x_0\}$, i.e., the set

$$\{y \in X : \ y = \text{weak} - \lim_{j\to\infty} T(t_{n_j})x_0 \text{ for some } n_j \uparrow \infty\}.$$

So, we can repeat the above process and obtain a sequence $\{x_n\}_{n=1}^{\infty}$ in E with the properties: for all integer $m \geq 1$,

$$\lim_{n\to\infty} \sup \|T(t_n)x_{m-1} - x_m\| \leq \frac{1}{N_p} \cdot A(\{T(t_n)x_{m-1}\}), \tag{2.5}$$

$$\|z - x_m\| \leq \lim_{n\to\infty} \sup \|z - T(t_n)x_{m-1}\| \qquad \text{for all } z \in X. \tag{2.6}$$

For each integer $m \geq 0$, we write

$$D_m = \lim_{n\to\infty} \sup \|x_m - T(t_n)x_m\|, \quad r_m = \lim_{n\to\infty} \sup \|x_{m+1} - T(t_n)x_m\|.$$

By the property (P_2) it is easy to prove

$$\lim_{j\to\infty} \sup \|T(t_i + t_j)x_m - T(t_j)x_m\| = 0 \qquad \text{for all } \ i \geq 1.$$

By choosing two appropriate subsequences $\{p_i\}, \ \{q_j\}$ of $\{n\}_{n=1}^{\infty}$, we obtain from (2.1) and (2.5),

$$\begin{aligned} r_m &= \lim_{i\to\infty} \sup \|x_{m+1} - T(t_i)x_m\| \leq \frac{1}{N_p} \lim_{n\to\infty} (\sup(\|T(t_i)x_m - T(t_j)x_m\| : \ i, j \geq n)) \\ &\leq \frac{1}{N_p} \lim_{p_i\to\infty} \sup(\lim_{q_j\to\infty} \sup(|||T(t_{p_i})||| \ \|x_m - T(t_{q_j})x_m\| + \|T(t_{p_i} + t_{q_j})x_m - T(t_{q_j})x_m\|)) \\ &\leq \frac{k}{N_p} D_m \qquad \text{for } m = 0, \ 1, \ 2, \ \cdots . \end{aligned} \tag{2.7}$$

On the other hand, by Lemma 1.4 we have for each integer $i, \ j \geq 1$,

$$\begin{aligned} &\|(1-\lambda)x_1 + \lambda T(t_j)x_1 - T(t_i)x_0\|^p + d_p W_p(\lambda)\|x_1 - T(t_j)x_1\|^p \\ &\leq \lambda(\|T(t_j)x_1 - T(t_j + t_i)x_0\| + \|T(t_j + t_i)x_0 - T(t_i)x_0\|)^p \\ &\quad + (1-\lambda)\|x_1 - T(t_i)x_0\|^p. \end{aligned} \tag{2.8}$$

Using (2.4) and the asymptotic regularity of T on E, we derive

$$\|\lambda(x_1 - T(t_j)x_1)\|^p + d_p W_p(\lambda)\|x_1 - T(t_j)x_1\|^p \leq [\lambda|||T(t_j)|||^p + (1-\lambda)] \cdot r_0^p,$$

and hence $\lambda^p D_1^p + d_p W_p(\lambda) D_1^p \leq [\lambda k^p + (1-\lambda)] \cdot r_0^p$. Now letting $\lambda \to 1^-$, we get $D_1 \leq k r_0$. It follows from (2.7) that $D_1 \leq \frac{k^2}{N_p} \cdot D_0$. By induction, we obtain $D_{m+1} \leq A_p^{m+1} \cdot D_0$, where $A_p = \frac{k^2}{N_p} < 1$. By the triangle inequality we infer

$$\|x_{m+1} - x_m\| \leq D_m + r_m \leq \Big(1 + \frac{k}{N_p}\Big) \cdot A_p^m \cdot D_0 \to 0$$

as $m \to \infty$. Therefore, $\{x_m\}$ is a Cauchy sequence in E. Let $z = \lim_{m\to\infty} x_m$. Obviously, we deduce

$$\begin{aligned} \lim_{i\to\infty} \sup \|z - T(t_i)z\| &\leq \lim_{i\to\infty} \sup \Big[\|z - x_m\| + \|T(t_i)z - T(t_i)x_m\| + \|T(t_i)x_m - x_m\|\Big] \\ &\leq (1+k)\|z - x_m\| + A_p^m \cdot D_0 \to 0 \qquad \text{as } m \to \infty. \end{aligned}$$

Finally, by Lemma 1.4, we have $T(s)z = z$ for all $s \in S$.

Corollary 2.1. *Let C be a nonempty bounded subset of a p-uniformly convex Banach space X, and T be a Lipschitzian mapping of C into itself with $\lim_{n\to\infty} \inf |||T^n||| < \sqrt{N_p}$, where N_p is the normal stricture coefficient of X. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:*

(P_1) *$x \in E$ implies $\omega_w(x) \subset E$;*

(P_2) *$T: \ C \to C$ is asymptotically regular on E.*

Then the set of fixed points of T is nonempty.

Corollary 2.2. *Let C be a nonempty bounded subset of an L^p space $(1 < p < +\infty)$, and $T = \{(T(t) : \ t \in S\}$ be a Lipschitzian semigroup on C with $\lim_{t\to\infty} \inf_{t\in S} |||T(t)||| < \sqrt{N_p}$, where N_p is the normal structure coefficient of space L^p. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:*

(P_1) *$x \in E$ implies $\omega_w(x) \subset E$;*

(P_2) *T is asymptotically regular on E.*

Then there exists a $z \in E$ such that $T(s)z = z$ for all $s \in S$.

§3. Fixed Point Theorem for Lipschitzian Semigroups in Uniformly Convex Banach Spaces

Recall that a Banach space X is strictly convex if its unit sphere does not contain any line segment, that is, X is strictly convex if the following implication holds:

$$\Big[\|x\|=1,\ \|y\|=1,\ \text{ and }\ \Big\|\frac{1}{2}(x+y)\Big\|=1 \Rightarrow x=y\Big].$$

In order to measure the degree of strict convexity (rotundity) of X, we define its modulus of convexity $\delta_x:\ [0,2]\to[0,1]$ by

$$\delta_x(\varepsilon)=\inf\Big\{1-\frac{1}{2}\|x-y\|:\|x\|\le 1,\ \|y\|\le 1\ \text{ and }\ \|x-y\|\ge\varepsilon\Big\}.$$

The characteristic of convexity ε_0 of X is also defined by $\varepsilon_0=\varepsilon_0(x)=\sup\{\varepsilon:\ \delta_x(\varepsilon)=0\}$. It is well-known (see [13]) that the modulus of convexity δ_x satisfies the following properties:

(a) δ_x is increasing on [0,2], and moreover strictly increasing on $[\varepsilon_o,2]$;

(b) δ_x is continuous on [0,2) (but not necessarily at $\varepsilon=2$);

(c) $\delta_x(2)=1$ if and only if X is strictly convex;

(d) $\delta_x(0)=0$ and $\lim\limits_{\varepsilon\to 2^-}\delta_x(\varepsilon)=1-\frac{1}{2}\varepsilon_0$;

(e) $\|a-x\|\le r,\ \|a-y\|\le r$ and $\|x-y\|\ge\varepsilon\Rightarrow\|a-\frac{1}{2}(x+y)\|\le r(1-\delta_x(\varepsilon/r))$.

A Banach space X is said to be uniformly convex if $\delta_x(\varepsilon)>0$ for all positive ε; equivalently $\varepsilon_0=0$. Obviously, any uniformly convex Banach space is both strictly convex and reflexive. By properties above, we can see that if X is uniformly convex, then δ_x is strictly increasing and continuous on [0,2]. In addition, Bynum[17] and Maluta[18] have proven that if X is uniformly convex then $N(X)\ge\frac{1}{1-\delta_x(1)}$. Further, Xu[11] has also proven that if X is uniformly convex and $\gamma>1$ is the unique solution of the equation $\gamma[1-\delta_x(\frac{1}{\gamma})]=1$, then $N(X)>\gamma$. We note that for a Hilbert space H, we have $N(H)=\sqrt{2}$, and $\gamma=\frac{\sqrt{5}}{2}$.

Now we give the main result in this section.

Theorem 3.1. *Let C be a nonempty bounded subset of a uniformly convex Banach space X, and $T=\{T(t):\ t\in S\}$ be a Lipschitzian semigroup on C with*

$$\lim_{t\to\infty}\inf_{t\in S}|||T(t)|||<[\gamma_0 N(X)]^{\frac{1}{2}}, \tag{3.1}$$

where $\gamma_0=\inf\{\gamma:\gamma(1-\delta_x(\frac{1}{\gamma}))\ge\frac{1}{2}\}$. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:

(P_1) *$x\in E$ implies $\omega_w(x)\subset E$;*

(P_2) *T is asymptotically regular on E.*

Then the fixed point set $F(T)$ of T is nonempty.

Proof. Let $\{t_n\}$ be a positive sequence which increases monotonously to $+\infty$ and satisfies

$$\lim_{t\to\infty}\inf_{t\in S}|||T(t)|||=\lim_{n\to\infty}|||T(t_n)|||=k. \tag{3.2}$$

Take any x_0 in E. Recall the proof of Theorem 2.1. By exploiting exactly the same method as that in Theorem 2.1, we obtain the sequence $\{x_n\}_{n=1}^{\infty}$ in E with properties: for all integers $m\ge 1$,

$$\lim_{n\to\infty}\sup\|T(t_n)x_{m-1}-x_m\|\le\tilde{N}(X)A(\{T(t_n)x_{m-1}\}), \tag{3.3}$$

$$\|z-x_m\|\le\lim_{n\to\infty}\sup\|z-T(t_n)x_{m-1}\|\quad\text{for all}\ \ z\in X. \tag{3.4}$$

For each integer $m\ge 0$, we write

$$D_m=\lim_{n\to\infty}\sup\|x_m-T(t_n)x_m\|,\qquad r_m=\lim_{n\to\infty}\sup\|x_{m+1}-T(t_n)x_m\|.$$

By the property (P_2) it is easy to prove

$$\overline{\lim_{j}}\|T(t_i+t_j)x_m - T(t_j)x_m\| = 0 \quad \text{for} \quad i = 1,\ 2,\ \cdots .$$

By choosing two appropriate subsequences $\{p_i\},\ \{q_j\}$ of $\{n\}_{n=1}^{\infty}$, we obtain from (3.3),

$$\begin{aligned} r_m &= \lim_{n\to\infty}\sup\|T(t_n)x_m - x_{m+1}\| \\ &\le \tilde{N}(X)\lim_{n\to\infty}(\sup(\|T(t_i)x_m - T(t_j)x_m\| :\ i,j\ge n)) \\ &\le \tilde{N}(X)\overline{\lim_{p_i\to\infty}}\Big(\overline{\lim_{q_j\to\infty}}\Big(|||T(t_{p_i})|||\cdot\|x_m - T(t_{q_j})x_m\| + \|T(t_{p_i}+t_{q_j})x_m - T(t_{q_j})x_m\|\Big)\Big) \\ &\le \tilde{N}(X)\overline{\lim_{p_i\to\infty}}|||T(t_{p_i})|||\cdot\overline{\lim_{q_j\to\infty}}\|x_m - T(t_{q_j})x_m\| \\ &\le \tilde{N}(X)kD_m \qquad \text{for} \quad m = 0,\ 1,\ 2,\ \cdots . \end{aligned} \tag{3.5}$$

We may assume $D_m > 0$ for all integers $m \ge 0$. Let $m \ge 0$ be fixed and let $\varepsilon > 0$ be small enough. First choose an integer $j \ge 1$ such that

$$\|T(t_j)x_{m+1} - x_{m+1}\| > D_{m+1} - \varepsilon, \quad |||T(t_j)||| < k + \varepsilon,$$

and then choose an integer $n_0 \ge 1$ so large that

$$\|T(t_n)x_m - x_{m+1}\| < r_m + \varepsilon, \quad \|T(t_n)x_m - T(t_n+t_j)x_m\| < \varepsilon,$$

$$\begin{aligned} \|T(t_n)x_m - T(t_j)x_{m+1}\| &\le \|T(t_n+t_j)x_m - T(t_j)x_{m+1}\| + \|T(t_n)x_m - T(t_n+t_j)x_m\| \\ &\le |||T(t_j)|||\cdot\|T(t_n)x_m - x_{m+1}\| + \|T(t_n)x_m - T(t_n+t_j)x_m\| \\ &\le (k+\varepsilon)(r_m+\varepsilon) + \varepsilon \end{aligned}$$

for all integer $n \ge n_0$. It then follows that

$$\begin{aligned} &\Big\|T(t_n)x_m - \frac{1}{2}(x_{m+1} + T(t_j)x_{m+1})\Big\| \\ &\le [(k+\varepsilon)(r_m+\varepsilon)+\varepsilon]\cdot\Big(1 - \delta_x\Big(\frac{D_{m+1}-\varepsilon}{(k+\varepsilon)(r_m+\varepsilon)+\varepsilon}\Big)\Big) \end{aligned}$$

for $n \ge n_0$ and hence by using (3.4) we have

$$\begin{aligned} \frac{1}{2}(D_{m+1}-\varepsilon) &< \Big\|\frac{1}{2}(T(t_j)x_{m+1} - x_{m+1})\Big\| \\ &\le \lim_{n\to\infty}\sup\Big\|T(t_n)x_m - \frac{1}{2}(x_{m+1} + T(t_j)x_{m+1})\Big\| \\ &\le [(k+\varepsilon)(r_m+\varepsilon)+\varepsilon]\cdot\Big(1 - \delta_x\Big(\frac{D_{m+1}-\varepsilon}{(k+\varepsilon)(r_m+\varepsilon)+\varepsilon}\Big)\Big). \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ we obtain $\frac{1}{2}D_{m+1} \le kr_m\Big(1 - \delta_x\Big(\frac{D_{m+1}}{kr_m}\Big)\Big)$, which together with (3.5) leads to $D_{m+1} \le \frac{k}{r_0}r_m \le \frac{k^2}{\gamma_0 N(X)}D_m$, where $\gamma_0 = \inf\{\gamma :\ \gamma(1-\delta_x(\frac{1}{\gamma})) \ge \frac{1}{2}\}$. Hence $D_m \le AD_{m-1} \le A^n D_0$, where $A = k^2[\gamma_0 N(X)]^{-1} < 1$ by assumption. Noticing

$$\begin{aligned} \|x_{m+1} - x_m\| &\le \lim_{n\to\infty}\sup\|T(t_n)x_m - x_m\| + \lim_{n\to\infty}\sup\|T(t_n)x_m - x_{m+1}\| \\ &\le D_m + r_m \le (1 + k\tilde{N}(X))D_m, \end{aligned} \tag{3.6}$$

we see from (3.6) that $\{x_m\}$ is norm Cauchy and hence strongly convergent. Let $z = \lim_{m\to\infty} x_m$. Then we have

$$\lim_{i\to\infty}\sup\|z - T(t_i)z\| \le (1+k)\|z - x_m\| + A^m\cdot D_0 \to 0 \quad \text{as} \quad m \to \infty.$$

Finally, be Lemma 1.4, we deduce $T(s)z = z$ for all $s \in S$.

Corollary 3.1. *Let C be a nonempty bounded subset of a uniformly convex Banach space X, and T be a Lipschitzian mapping of C into itself with $\lim_{n\to\infty}\inf |||T^n||| < [\gamma_0 N(X)]^{\frac{1}{2}}$, where $\gamma_0 = \inf\{\gamma : \gamma(1-\delta_x(\frac{1}{r})) \geq \frac{1}{2}\}$. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:*

(P$_1$) *$x \in E$ implies $\omega_w(x) \subset E$;*

(P$_2$) *T is asymptotically regular on E.*

Then the fixed point set $F(T)$ of T is nonempty.

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