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ASYMPTOTIC ANALYSIS OF LINEARLY ELASTIC SHALLOW SHELLS WITH VARIABLE THICKNESS

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Abstract

The author considers a linearly elastic shallow shell with variable thickness and shows that, as the thickness of the shell goes to zero, the solution of the three-dimensional equations converges to the solution of the two-dimensional shallow shell equations with variable thickness.

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§1. Introduction

In this paper, we identify the two-dimensional model of a shallow shell with variable thickness. More precisely, we consider a family of linearly elastic shallow shells with variable thickness. We show that, if the applied forces are of specific order of magnitude, the covariant components of the scaled displacement field converge, as the thickness of the shell goes to zero, to a two dimensional problem that constitutes the model of a shallow shell with variable thickness. The key to the convergent analysis lies in establishing a generalized Korn's inequality (cf. Lemma 4.2).

In the case of shallow shells with constant thickness, Ciarlet and Miara[6] have justified the two dimensional equations of shallow shells and S. Kesavan and N. Sabu[12] have studied the corresponding eigenvalue problem, both in Cartesian coordinates. Busse, Ciarlet and Miara[2] have justified the two dimensional shallow shell model in curvilinear coordinates. Busse[1] has considered the case of linear membrane and flexural shells with variable thickness and Roquefort[14] has studied the nonlinear membrane and flexural shells with variable thickness.

This paper is organized as follows. Section 2 begins with some preliminaries from differential geometry needed for defining the geometry of the shell and then we describe the three-dimensional problem for a linearly elastic shell. In Section 3, we transform the problem over a fixed domain and in Section 4, we obtain the a priori estimates that will be used in the convergence analysis, which is studied in Section 5. In Section 6, we describe the two-dimensional model.

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§2. The Three-Dimensional Problem

Latin indices takes their values in the set $\{1, 2, 3\}$ and Greek indices takes their values in the set $\{1, 2\}$; the repeated index convention for summation is systematically used in conjunction with the above rules.

Let ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ such that ω is locally on one side of γ. Let γ_0 be a portion of γ with length $\gamma_0 > 0$. For all $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon), \quad \Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, \varepsilon], \quad \Gamma^\varepsilon = \gamma \times [-\varepsilon, \varepsilon], \quad \Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}.$$

Let $x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, x_3^\varepsilon)$ denote a generic point in $\overline{\Omega}^\varepsilon$ and let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$. Let ∂_ν denote the outer normal derivative along γ_0. We assume that for each $\varepsilon > 0$, we are given a function $\theta^\varepsilon : \overline{\omega} \to \mathbb{R}$ of class $\mathcal{C}^3$. We then define the map $\varphi^\varepsilon : \overline{\omega} \to \mathbb{R}^3$ by

$$\varphi^\varepsilon(x_1, x_2) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) \text{ for all } (x_1, x_2) \in \overline{\omega}. \tag{2.1}$$

At each point of the surface $S^\varepsilon = \varphi^\varepsilon(\overline{\omega})$, we define the normal vector

$$\boldsymbol{a}^\varepsilon = (|\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1)^{-1/2}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1).$$

The variable thickness of the shell is governed by a function $e \in W^{2,\infty}(\omega)$ such that there exists a constant e_0 such that $0 < e_0 \le e(x_1, x_2)$ for all $(x_1, x_2) \in \overline{\omega}$. For each $\varepsilon > 0$, we define the mapping $\boldsymbol{\Phi}^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ by

$$\boldsymbol{\Phi}^\varepsilon(x^\varepsilon) = \varphi^\varepsilon(x_1, x_2) + x_3^\varepsilon e(x_1, x_2)\boldsymbol{a}^\varepsilon(x_1, x_2), \quad \forall x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon.$$

Hence at the point $\boldsymbol{\Phi}^\varepsilon(x^\varepsilon)$, the thickness is $2\varepsilon e(x_1, x_2)$.

One can show (cf. [10, Propostion 3.2]) that there exits $\varepsilon_0 > 0$ such that the vectors

$$\boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \partial_i^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)$$

are linearly independent at all points $x^\varepsilon \in \overline{\Omega}^\varepsilon$ and the mapping $\boldsymbol{\Phi}^\varepsilon$ is injective for all $0 < \varepsilon \le \varepsilon_0$. The vectors $\boldsymbol{g}_i^\varepsilon(x^\varepsilon)$ form the covariant basis of the tangent space of $\boldsymbol{\Phi}^\varepsilon(\Omega^\varepsilon)$ at $\boldsymbol{\Phi}^\varepsilon(x^\varepsilon)$. For $0 < \varepsilon \le \varepsilon_0$, the vectors $\boldsymbol{g}^{i,\varepsilon}$ defined by the relation $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}_i^\varepsilon = \delta_i^j$ form the contravariant basis of the tangent space of $\boldsymbol{\Phi}^\varepsilon(\Omega^\varepsilon)$ at $\boldsymbol{\Phi}^\varepsilon(x^\varepsilon)$.

The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\varepsilon = \boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}_j^\varepsilon, \; g^{ij,\varepsilon} = \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}.$$

The volume element is given by $\sqrt{g^\varepsilon}dx$ where $g^\varepsilon = \det(g_{ij}^\varepsilon)$. The Christoffel symbols are defined by $\Gamma_{ij}^{p,\varepsilon} = \boldsymbol{g}^{p,\varepsilon} \cdot \partial_j^\varepsilon \boldsymbol{g}_i^\varepsilon$.

For each $0 < \varepsilon \le \varepsilon_0$, the set $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is the reference configuration of an elastic shell, with middle surface $S = \varphi(\overline{\omega})$ and thickness 2ε. We assume that the material constituting the shell is homogeneous and isotropic and that $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$ is a natural state, so that the material is characterized by its two Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. Finally, we assume that the shell is subjected to a homogeneous boundary condition of place (i.e., of vanishing displacements) along the portion $\boldsymbol{\Phi}(\Gamma_0^\varepsilon)$ of the lateral face $\boldsymbol{\Phi}(\Gamma^\varepsilon)$. The unknown of the problem is the vector field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$, where the three functions $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}$ are the covariant components of the displacement field $\boldsymbol{u}_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the points of the shell.

We define the space of admissible displacements by

$$\boldsymbol{V}(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in H^1(\Omega^\varepsilon) | \boldsymbol{v}^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\}. \tag{2.2}$$

Then it is classical (cf. [4]) that the variational formulation of the corresponding three-dimensional problem expressed in terms of the curvilinear coordinates (x^ε) of the reference configuration $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$ of the shell consists in finding $\boldsymbol{u}^{\varepsilon} \in \boldsymbol{V}(\Omega^{\varepsilon})$ such that

$$\int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k||l}(\boldsymbol{u}^{\varepsilon}) e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon}) \sqrt{g^{\varepsilon}} dx^{\varepsilon} = \int_{\Omega^{\varepsilon}} f^{i,\varepsilon} v^{\varepsilon}_i \sqrt{g^{\varepsilon}} dx^{\varepsilon} \text{ for all } \boldsymbol{v}^{\varepsilon} \in \boldsymbol{V}(\Omega^{\varepsilon}), \tag{2.3}$$

where

$$A^{ijkl,\varepsilon} := \lambda^{\varepsilon} g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^{\varepsilon}(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{2.4}$$

designate the contravariant components of the three-dimensional elasticity tensor,

$$e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon}) := \frac{1}{2}(\partial^{\varepsilon}_i v^{\varepsilon}_j + \partial^{\varepsilon}_j v^{\varepsilon}_i) - \Gamma^{p,\varepsilon}_{ij} v^{\varepsilon}_p \tag{2.5}$$

designate the covariant components of the linearized strain tensor associated with an arbitrary displacement field $\boldsymbol{v}^{\varepsilon}_i \boldsymbol{g}^{i,\varepsilon}$ of the surface S, and $f^{i,\varepsilon} \in L^2(\Omega^{\varepsilon})$ are the contravariant components of the applied body force density.

§3. The Scaled Problem

We follow here the same method as in [2, 6, 10].

Let $\Omega = \omega \times (-1,1), \Gamma_0 = \gamma_0 \times [-1,1], \Gamma = \gamma \times [-1,1], \Gamma_{\pm} = \omega \times \{\pm 1\}$.

Let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let $\partial_i = \partial/\partial x_i$. With $x^{\varepsilon} = (x^{\varepsilon}_i) \in \overline{\Omega}^{\varepsilon}$, we associate the point $x = (x_i) \in \overline{\Omega}$ defined by $x_{\alpha} = x^{\varepsilon}_{\alpha}$ and $x_3 = (1/\varepsilon)x^{\varepsilon}_3$; we thus have $\partial^{\varepsilon}_{\alpha} = \partial_{\alpha}$ and $\partial^{\varepsilon}_3 = (1/\varepsilon)\partial_3$.

With the unknown $\boldsymbol{u}^{\varepsilon} = (u^{\varepsilon}_i) : \overline{\Omega}^{\varepsilon} \to \mathbb{R}^3$ and the vector fields $\boldsymbol{v}^{\varepsilon} = (v^{\varepsilon}_i) : \overline{\Omega}^{\varepsilon} \to \mathbb{R}^3$ appearing in the three-dimensional problem (2.3)–(2.5), we associate the scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ and the scaled vector fields $\boldsymbol{v} = (v_i)$ defined by

$$u^{\varepsilon}_{\alpha}(x^{\varepsilon}) = \varepsilon^2 u^{\varepsilon}_{\alpha}(x) \text{ and } u^{\varepsilon}_3(x^{\varepsilon}) = \varepsilon u^{\varepsilon}_3(x) \text{ for all } x^{\varepsilon} \in \Omega, \tag{3.1}$$

$$v^{\varepsilon}_{\alpha}(x^{\varepsilon}) = \varepsilon^2 v^{\varepsilon}_{\alpha}(x) \text{ and } v^{\varepsilon}_3(x^{\varepsilon}) = \varepsilon v^{\varepsilon}_3(x) \text{ for all } x^{\varepsilon} \in \Omega. \tag{3.2}$$

We next assume that there exist constants $\lambda > 0$ and $\mu > 0$ independent of ε and functions $f^i \in L^2(\Omega)$ and $\theta \in \mathcal{C}^3(\omega)$ independent of ε such that

$$\lambda^{\varepsilon} = \lambda, \quad \mu^{\varepsilon} = \mu, \tag{3.3}$$

$$f^{\alpha,\varepsilon}(x^{\varepsilon}) = \varepsilon^2 f^{\alpha}(x) \text{ and } f^{3,\varepsilon}(x^{\varepsilon}) = \varepsilon^3 f^3(x) \text{ for all } x \in \Omega \tag{3.4}$$

$$\theta^{\varepsilon}(y) = \varepsilon\theta(y) \text{ for all } y \in \omega. \tag{3.5}$$

Then the function $\boldsymbol{u}(\varepsilon)$ satisfies

$$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) = \{\boldsymbol{v} = (v_i) \in H^1(\Omega) | \boldsymbol{v} = 0 \text{ on } \Gamma_0\}, \tag{3.6}$$

$$\int_{\Omega} A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon; \boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon; \boldsymbol{v}(\varepsilon)) \sqrt{g(\varepsilon)} dx = \int_{\Omega} f^i v_i \sqrt{g(\varepsilon)} dx \tag{3.7}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, where the functions $A^{ijkl}(\varepsilon) : \overline{\Omega} \to \mathbb{R}, g(\varepsilon) : \overline{\Omega} \to \mathbb{R}$ and $e_{i||j}(\varepsilon; \boldsymbol{v}) \in L^2(\Omega)$ are defined by the relations

$$A^{ijkl,\varepsilon}(x^{\varepsilon}) = A^{ijkl}(\varepsilon)(x), \; g^{\varepsilon}(x^{\varepsilon}) = g(\varepsilon)(x) \text{ for all } x \in \Omega, \tag{3.8}$$

$$e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon})(x^{\varepsilon}) = \varepsilon^2 e_{i||j}(\varepsilon; \boldsymbol{v})(x) \text{ for all } x \in \Omega. \tag{3.9}$$

Remark 3.1. The assumption (3.5) is the shallowness assumption, originally introduced by Ciarlet and Paumier.

§4. Preliminaries

We gather in the following two lemmas, which are generalization of Lemma 4.1 and Lemma 4.2 of [2], various results that will be needed in the proof of convergence. In what follows, $|\cdot|_{0,\Omega}$ and $||\cdot||_{1,\Omega}$ denote the $L^2(\Omega)$ norm, and the $H^1(\Omega)$ norm for both scalar-valued and vector-valued functions.

We denote by $C_1, C_2, \cdots$ positive constants that are independent of ε but they may depend on the function θ.

Lemma 4.1. *The functions $e_{i||j}(\varepsilon, \boldsymbol{v})$ defined in* (3.9) *are of the form*

$$e_{\alpha||\beta}(\varepsilon; \boldsymbol{v}) = \widetilde{e}_{\alpha\beta}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{\alpha||\beta}(\varepsilon; \boldsymbol{v}),$$
$$e_{\alpha||3}(\varepsilon; \boldsymbol{v}) = \frac{1}{\varepsilon}\{\widetilde{e}_{\alpha 3}(\boldsymbol{v}) + \varepsilon^2 e^{\#}_{\alpha||3}(\varepsilon; \boldsymbol{v})\},$$
$$e_{3||3}(\varepsilon; \boldsymbol{v}) = \frac{1}{\varepsilon^2}\widetilde{e}_{33}(\boldsymbol{v}),$$

where

$$\widetilde{e}_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{v_3}{e}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e), \tag{4.4}$$
$$\widetilde{e}_{\alpha 3}(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \tag{4.5}$$
$$\widetilde{e}_{33}(\boldsymbol{v}) = \partial_3 v_3, \tag{4.6}$$

and there exists a constant C_1 such that

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{\alpha,j} \|e^{\#}_{\alpha j}(\varepsilon; \boldsymbol{v})\| \le C_1||\boldsymbol{v}||_{1,\Omega} \textit{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{4.7}$$

The functions $g(\varepsilon)$ defined in (3.8) *are of the form*

$$g(\varepsilon) = e^2 + \varepsilon^2 g^{\#}(\varepsilon) \tag{4.8}$$

and there exists a constant C_2 such that

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{x\in\Omega}|g^{\#}(\varepsilon)(x)| \le C_2. \tag{4.9}$$

The functions $A^{ijkl}(\varepsilon)$ are of the form

$$A^{ijkl}(\varepsilon) = A^{ijkl}(0) + \varepsilon^2 A^{ijkl}_{\#}(\varepsilon), \tag{4.10}$$

where

$$A^{\alpha\beta\gamma\delta}(0) = \lambda\delta^{\alpha\beta}\delta^{\gamma\tau} + \mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}), \tag{4.11}$$
$$A^{\alpha\beta\sigma 3}(0) = 0, \quad A^{\alpha\beta 33}(0) = \frac{1}{e^2}\lambda\delta^{\alpha\beta}, \quad A^{\alpha 3\sigma 3}(0) = \frac{1}{e^2}\mu\delta^{\alpha\sigma}, \tag{4.12}$$
$$A^{\alpha 333}(0) = 0, \quad A^{3333}(0) = \frac{1}{e^4}(\lambda + 2\mu), \tag{4.13}$$

and there exists a constant C_3 such that

$$\sup_{0<\varepsilon\le\varepsilon_0)} \max_{x\in\Omega} |A^{ijkl}_{\#}(\varepsilon)(x)| \le C_3. \tag{4.14}$$

Also there exists a constant C_4 such that

$$A^{ijkl}(\varepsilon)(x)t_{kl}t_{ij} \ge C_4 t_{ij}t_{ij} \tag{4.15}$$

for all $0 < \varepsilon \le \varepsilon_0$, for all $x \in \overline{\Omega}$, and for all symmetic matrices (t_{ij}).

Proof. With the vectors $g_i^\varepsilon : \overline{\Omega}^\varepsilon \to I\!R^3$ and the functions $g_{ij}^\varepsilon, g^{ij,\varepsilon}, \Gamma_{ij}^{p,\varepsilon} : \overline{\Omega}^\varepsilon \to I\!R$, we associate the vectors $g_i(\varepsilon) : \overline{\Omega} \to I\!R^3$ and the functions $g_{ij}(\varepsilon)$, $g^{ij}(\varepsilon), \Gamma_{ij}^p(\varepsilon) : \overline{\Omega} \to I\!R$, by the relations

$$g_i^\varepsilon(x^\varepsilon) = g_i(\varepsilon)(x), \quad g_{ij}^\varepsilon(x^\varepsilon) = g_{ij}(\varepsilon)(x), \tag{4.16}$$

$$g^{ij,\varepsilon}(x^\varepsilon) = g^{ij}(\varepsilon)(x), \quad \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon) = \Gamma_{ij}^p(\varepsilon)(x). \tag{4.17}$$

Then a simple calculation using the assumption (3.5) shows that

$$g_\alpha(\varepsilon) = \begin{pmatrix} \delta_{\alpha 1} - \varepsilon^2 x_3[e\partial_{\alpha 1}\theta + \partial_1\theta\partial_\alpha e] + O(\varepsilon^4) \\ \delta_{\alpha 2} - \varepsilon^2 x_3[e\partial_{\alpha 2}\theta + \partial_2\theta\partial_\alpha e] + O(\varepsilon^4) \\ \varepsilon[\partial_\alpha\theta + x_3\partial_\alpha e] + O(\varepsilon^3) \end{pmatrix}, \tag{4.18}$$

$$g_3(\varepsilon) = \begin{pmatrix} \varepsilon e\partial_1\theta + O(\varepsilon^3) \\ \varepsilon e\partial_2\theta + O(\varepsilon^3) \\ e + O(\varepsilon^2) \end{pmatrix}, \tag{4.19}$$

$$g_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} + \varepsilon^2[\partial_\alpha\theta\partial_\beta\theta - 2x_3(e\partial_{\alpha\beta}\theta + \partial_\alpha\theta\partial_\beta e)] + O(\varepsilon^4), \tag{4.20}$$

$$g_{\alpha 3}(\varepsilon) = O(\varepsilon), g_{33}(\varepsilon) = e^2 + O(\varepsilon^2), \Gamma_{\alpha\beta}^\sigma(\varepsilon) = O(\varepsilon^2), \tag{4.21}$$

$$\Gamma_{\alpha\beta}^3(\varepsilon) = \frac{\varepsilon}{e}[\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e] + O(\varepsilon^3), \quad \Gamma_{\alpha 3}^\sigma = O(\varepsilon). \tag{4.22}$$

The announced results then follow from the above relations.

In the next lemma, we establish a generalized Korn's inequality, which involves the functions $\widetilde{e}_{ij}(\boldsymbol{v})$ defined in (4.4)–(4.6), which generalize the traditional functions

$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \tag{4.23}$$

as well as the functions $\widetilde{e}_{ij}(\boldsymbol{v})$ used in [2]. This inequality will yield the fundamental a priori estimates that the scaled unknowns $(\boldsymbol{u}(\varepsilon))$ satisfy.

Lemma 4.2. *Let $\theta \in \mathcal{C}^3(\overline{\omega})$ be a given function and let the functions $\widetilde{e}_{ij}(\boldsymbol{v})$ be defined as in* (4.4)–(4.6)*. Then there exists a constant C_5 such that*

$$||\boldsymbol{v}||_{1,\Omega} \le C_5\Big\{\sum_{i,j}||\widetilde{e}_{ij}(\boldsymbol{v})||_{0,\Omega}\Big\}^{1/2} \tag{4.24}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, where $\boldsymbol{V}(\Omega)$ is the space defined in (3.6).

Proof. For clarity the proof is divided into four steps.

(i) Let the space $\boldsymbol{E}^\theta(\Omega)$ be defined by $\boldsymbol{E}^\theta(\Omega) = \{\boldsymbol{v} = (v_i) \in L^2(\Omega); \widetilde{e}_{ij}(\boldsymbol{v}) \in L^2(\Omega)\}$. Then

$$\boldsymbol{E}^\theta(\Omega) = \boldsymbol{H}^1(\Omega). \tag{4.25}$$

Let $\boldsymbol{v} = (v_i)$ be an element in $\boldsymbol{E}^\theta(\Omega)$. Then

$$e_{\alpha\beta}(\boldsymbol{v}) = \widetilde{e}_{\alpha\beta}(\boldsymbol{v}) + \frac{v_3}{e}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e) \in L^2(\Omega), \quad e_{i3}(\boldsymbol{v}) = \widetilde{e}_{i3}(\boldsymbol{v}) \in L^2(\Omega),$$

where the functions $e_{ij}(\boldsymbol{v})$ are defined in (4.23). The classical identity

$$\partial_{jk}v_i = \partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v})$$

shows that $\partial_{jk}v_i \in H^{-1}(\Omega)$. Also $\boldsymbol{v} \in \boldsymbol{E}^\theta(\Omega) \Rightarrow \partial_j v_i \in H^{-1}(\Omega)$. Hence by a lemma of J. L. Lions (cf. [4, Theorem 1.7.1]), we have $\partial_j v_i \in L^2(\Omega)$ and hence $\boldsymbol{E}^\theta(\Omega) \subset \boldsymbol{H}^1(\Omega)$. The opposite inclusion is obvious and hence the equality (4.25) follows.

(ii) The mapping $\|\cdot\|$ defined by

$$||\boldsymbol{v}|| = \Big\{||\boldsymbol{v}||_{0,\Omega} + \sum_{i,j}||\widetilde{e}_{ij}(\boldsymbol{v})||_{0,\Omega}^2\Big\}^{1/2} \tag{4.26}$$

is a norm over the space $\boldsymbol{H}^1(\Omega)$, and there exists a constant C_6 such that

$$||\boldsymbol{v}||_{1,\Omega} \leq C_6||\boldsymbol{v}|| \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{4.27}$$

Clearly there exists a constant C_7 such that

$$||\boldsymbol{v}|| \leq C_7||\boldsymbol{v}||_{1,\Omega} \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

Hence the identity mapping from the space $\boldsymbol{H}^1(\Omega)$ equipped with the norm $\|\cdot\|_{1,\Omega}$ into the space $\boldsymbol{E}^\theta(\Omega)$ equipped with the norm $\|\cdot\|$ is continuous, and it is also surjective since $\boldsymbol{E}^\theta(\Omega) = \boldsymbol{H}^1(\Omega)$ by the step (i). Since the space $\boldsymbol{E}^\theta(\Omega)$ is a Hilbert space when it is equipped with the norm $||.||$, the open mapping theorem implies the existence of a constant C_6 satisfying (4.27).

(iii) The semi-norm $|\cdot|^\theta$ defined by

$$|\boldsymbol{v}|^\theta = \Big\{\sum_{ij}||\widetilde{e}_{ij}(\boldsymbol{v})||^2_{0,\Omega}\Big\}^{1/2} \tag{4.28}$$

is a norm over the space $\boldsymbol{V}(\Omega)$.

The only property that remains to be checked is that

$$\boldsymbol{v} \in \boldsymbol{V}(\Omega) \text{ and } |\boldsymbol{v}|^\theta = 0 \Rightarrow \boldsymbol{v} = 0.$$

We next note a theorem due to Hörmander (cf. [11, Theorem 2.4]). If $P(x,\xi) = \sum\limits_{ij} a_{ij}\xi_i\xi_j$ where $a_{ij}(x)$ are Lipschitz-continuous in a neighbourhood of zero, $P(x,\xi)$ is elliptic and if $u \in H^1(\omega)$ satisfies $|P(x,D)u| \leq C \sum\limits_{|\alpha|\leq 1} |D^\alpha u|$ then $u = 0$ in ω if u vanishes in a neighbourhood of a point in ω.

Let $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ be such that $\widetilde{e}_{ij}(\boldsymbol{v}) = 0$. Since $e_{i3}(\boldsymbol{v}) = \widetilde{e}_{i3}(\boldsymbol{v}) = 0$, a standard argument (cf. [3, Chapter 1]) implies that there exist functions $\eta_\alpha \in H^1(\omega), \eta_3 \in H^2(\omega)$, $\eta_i = \partial_\nu\eta_3 = 0$ on γ_0 such that $v_\alpha = \eta_\alpha - x_3\partial\eta_3, v_3 = \eta_3$. The relation $\widetilde{e}_{\alpha\beta}(\boldsymbol{v}) = 0$ then implies that

$$\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - \frac{\eta_3}{e}\partial_{\alpha\beta}\theta = x_3\Big(\partial_{\alpha\beta}\eta_3 + \frac{\eta_3}{e}\partial_{\alpha\beta}e\Big)$$

and whence $\partial_{\alpha\beta}\eta_3 + \frac{\eta_3}{e}\partial_{\alpha\beta}e = 0$ in ω since the left-hand side of the above equality is only a function of (x_1, x_2).

In particular, $\eta_3 \in H^2(\omega)$ satisfies

$$\begin{aligned} &\Delta\eta_3 + \frac{\eta_3}{e}\Delta e = 0 \text{ in } \omega, \\ &\eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0. \end{aligned} \tag{4.29}$$

Let $\omega^{'}$ be a domain which contains γ_0 in its interior. Then the function $\eta_3^{'}$ defined by

$$\eta_3^{'} = \begin{cases} \eta_3 & \text{in } \omega, \\ 0 & \text{in } \omega^{'} - \omega \end{cases} \tag{4.30}$$

satisfies $\eta_3^{'} \in H^2(\omega^{'})$,

$$\begin{aligned} &\Delta\eta_3^{'} + \frac{\eta_3^{'}}{e}\Delta e = 0 \text{ in } \omega^{'}, \\ &\eta_3^{'} = 0 \text{ in } \omega^{'} - \omega, \end{aligned} \tag{4.31}$$

and whence $||\Delta\eta_3^{'}||_{0,\omega'} \leq C||\eta_3^{'}||_{0,\omega'}$ and $\eta_3^{'} = 0$ in $\omega^{'} - \omega$. Hence by Hörmander's theorem, we have $\eta_3^{'} = 0$ in $\omega^{'}$ and hence $\eta_3 = 0$ in ω.

The functions η_α then satisfies $\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha = 0$ in ω, $\eta_\alpha = 0$ on γ_0 and hence $\eta_\alpha = 0$ on ω.

(iv) There exists a constant C_8 such that

$$||\boldsymbol{v}||_{1,\Omega} \le C_8|\boldsymbol{v}|^\theta \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{4.32}$$

Suppose the property is false. Then there exists functions $\boldsymbol{v}^k \in \boldsymbol{V}(\Omega), k = 1, 2, \cdots$ such that

$$||\boldsymbol{v}^k||_{1,\Omega} = 1 \text{ for all } k \ge 1, \quad |\boldsymbol{v}^k|^\theta \to 0 \text{ as } k \to \infty.$$

Since the sequence is bounded in $\boldsymbol{H}^1(\Omega)$, there exists a subsequence $(\boldsymbol{v})_{l=0}^\infty$ that converges strongly in the space $\boldsymbol{L}^2(\Omega)$ by the Rellich-Kondrasov theorem. Since $|\boldsymbol{v}^l|^\theta \to \mathbf{0}$ as $l \to \infty$, this subsequence is a Cauchy sequence with respect to the norm $\|\cdot\|$. Since this norm is equivalent to the norm $\|\cdot\|_{1,\Omega}$ by the step (ii), and since the space $\boldsymbol{H}^1(\Omega)$ is complete, the subsequence $(\boldsymbol{v}^l)_{l=1}^\infty$ converges in the space $\boldsymbol{H}^1(\Omega)$. On the one hand,

$$||\boldsymbol{v}||_{1,\Omega} = \lim_{l\to\infty} ||\boldsymbol{v}^l||_{1,\Omega} = 1. \tag{4.33}$$

On the other hand,

$$|\boldsymbol{v}|^\theta = \lim_{l\to\infty} |\boldsymbol{v}^l|^\theta = 0 \tag{4.34}$$

and hence $\boldsymbol{v} = 0$ by the step (iii), which is impossible by (4.33).

§5. The Limit Problem

In the following theorem, we establish that, as $\varepsilon \to 0$, the family $(\boldsymbol{u}(\varepsilon))$ converges strongly in $\boldsymbol{H}^1(\Omega)$ and we also identify the variational problem that the limit of the family satisfies.

Theorem 5.1. (a) *There exists* $\boldsymbol{u} = (u_i) \in \boldsymbol{V}(\Omega)$ *such that* $\boldsymbol{u}(\varepsilon) \to \boldsymbol{u}$ *in* $\boldsymbol{H}^1(\Omega)$.

(b) *Define the space*

$$\boldsymbol{V}(\omega) = \{\boldsymbol{\eta} = \eta_i \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}. \tag{5.1}$$

Then there exists $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{V}(\omega)$ *such that*

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3 = \zeta_3. \tag{5.2}$$

(c) *The function* $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{V}(\omega)$ *solves the following variational equation*:

$$\begin{aligned} &-\int_\omega m^{\alpha\beta}\partial_{\alpha\beta}\eta_3 e d\omega - \int_\omega (n^{\alpha\beta}\partial_{\alpha\beta}\theta + m^{\alpha\beta}\partial_{\alpha\beta}e)\eta_3 e d\omega + \int_\omega n^{\alpha\beta}\partial_\beta\eta_\alpha e d\omega \\ &= \int_\omega p^i\eta_i e d\omega - \int_\omega q^\alpha\partial_\alpha\eta_3 e d\omega \end{aligned} \tag{5.3}$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, *where*

$$m^{\alpha\beta} = -\frac{4\lambda\mu}{3(\lambda+2\mu)}\Big(\Delta\zeta_3 + \zeta_3\frac{\Delta e}{e}\Big)\delta_{\alpha\beta} + \frac{4\mu}{3}\Big(\partial_{\alpha\beta}\zeta_3 + \zeta_3\frac{\partial_{\alpha\beta}e}{e}\Big), \tag{5.4}$$

$$n^{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu}\hat{e}_{\sigma\sigma}(\boldsymbol{\zeta})\delta_{\alpha\beta} + 4\mu\hat{e}_{\alpha\beta}(\boldsymbol{\zeta}), \tag{5.5}$$

$$\hat{e}_{\alpha\beta}(\boldsymbol{\zeta}) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\frac{\partial_{\alpha\beta}\theta}{e} = \frac{1}{2}\int_{-1}^1 \widetilde{e}_{\alpha\beta}(\zeta)dx_3, \tag{5.6}$$

$$p^i = \int_{-1}^1 f^i dx_3, \tag{5.7}$$

$$q^\alpha = \int_{-1}^1 x_3 f^\alpha dx_3. \tag{5.8}$$

Proof. For clarity, the proof is divided into six steps.

(i) There exists constants $\varepsilon_1, 0 < \varepsilon_1 \leq 1$ and $C_9 > 0$ such that

$$||\boldsymbol{u}(\varepsilon)||_{1,\Omega} \leq C_9 \text{ for all } 0 < \varepsilon \leq \varepsilon_1. \tag{5.9}$$

Letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equation (3.7), using the relations (4.8), (4.9), and (4.15), we have

$$\begin{aligned}\frac{1}{\sqrt{2}}C_4 \sum_{i,j} ||e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} &\leq \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon;\boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))\sqrt{g(\varepsilon)}dx \\ &= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx \\ &\leq \sqrt{e^2+\varepsilon^2 C_2}\Big\{\sum_i ||f^i||^2_{0,\Omega}\Big\}^{1/2} ||\boldsymbol{u}(\varepsilon)||_{0,\Omega}\end{aligned} \tag{5.10}$$

for $\varepsilon \leq \min\{\varepsilon_0, (2C_2)^{-1/2}\}$. Hence for ε sufficiently small, there exists a constant C_{10} such that

$$\sum_{i,j} \|e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))\|^2_{0,\Omega} \leq C_{10}||\boldsymbol{u}(\varepsilon)||_{0,\Omega}. \tag{5.11}$$

The relations (4.1)–(4.7), the inequality $(A-B)^2 \geq A^2/2 - B^2$ and the generalized Korn's inequality (4.24) together show that for $\varepsilon \leq \min\{\varepsilon_0, 1\}$,

$$\begin{aligned}\sum_{i,j} ||e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} &\geq \sum_{\alpha,\beta} ||\widetilde{e}_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + \varepsilon^2 e^{\#}_{\alpha\beta}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} \\ &\quad + 2\sum_\alpha ||\widetilde{e}_{\alpha 3}(\boldsymbol{u}(\varepsilon)) + \varepsilon^2 e^{\#}_{\alpha 3}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} + ||\widetilde{e}_{33}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} \\ &\geq \frac{1}{2}\sum_{i,j} ||\widetilde{e}_{ij}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} - 8\varepsilon^4 C_1^2 ||\boldsymbol{u}(\varepsilon)||^2_{1,\Omega} \\ &\geq \Big\{\frac{1}{2}C_5^{-2} - 8\varepsilon^4 C_1^2\Big\} ||\boldsymbol{u}(\varepsilon)||^2_{1,\Omega}.\end{aligned} \tag{5.12}$$

Hence for ε sufficiently small, there exists a constant C_{11} such that

$$||\boldsymbol{u}(\varepsilon)||^2_{1,\Omega} \leq C_{11} \sum_{i,j} ||e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))||^2_{0,\Omega}. \tag{5.13}$$

The relation (5.9) is then a consequence of relations (5.11) and (5.13).

(ii) Define the tensors $\widetilde{\boldsymbol{K}}(\varepsilon) = (\widetilde{K}_{ij}(\varepsilon)) \in L^2(\Omega)$ by

$$\widetilde{K}_{\alpha\beta}(\varepsilon) := \widetilde{e}_{\alpha\beta}(\boldsymbol{u}(\varepsilon)), \widetilde{K}_{\alpha 3}(\varepsilon) := \frac{1}{\varepsilon}\widetilde{e}_{\alpha 3}(\boldsymbol{u}(\varepsilon)), \widetilde{K}_{33}(\varepsilon) := \frac{1}{\varepsilon^2}\widetilde{e}_{33}(\boldsymbol{u}(\varepsilon)). \tag{5.14}$$

Then there exists a constant C_{12} such that

$$||\widetilde{\boldsymbol{K}}(\varepsilon)||_{0,\Omega} \leq C_{12} \text{ for all } 0 < \varepsilon \leq \varepsilon_1. \tag{5.15}$$

Using the definitions (5.14) and the relations (4.1)–(4.3), we have

$$\begin{aligned}||\widetilde{\boldsymbol{K}}(\varepsilon)||_{0,\Omega} &= \sum_{\alpha,\beta} ||e_{\alpha||\beta}(\varepsilon;\boldsymbol{u}(\varepsilon)) - \varepsilon^2 e^{\#}_{\alpha\beta}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} \\ &\quad + 2\sum_{\alpha} ||e_{\alpha||3}(\varepsilon;\boldsymbol{u}(\varepsilon)) - \varepsilon e^{\#}_{\alpha3}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} + ||e_{3||3}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} \\ &\le \sum_{i,j} ||e_{i||j}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} + 2\varepsilon^4 \sum_{\alpha,\beta} ||e^{\#}_{\alpha\beta}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega} \\ &\quad + 4\varepsilon^2 \sum_{\alpha} ||e^{\#}_{\alpha3}(\varepsilon,\boldsymbol{u}(\varepsilon))||^2_{0,\Omega}. \end{aligned} \tag{5.16}$$

Hence the relation (5.15) follows by using the inequalities (4.7) and the boundedness of the families $(\boldsymbol{u}(\varepsilon)) \in \boldsymbol{H}^1(\Omega)$.

(iii) By the step (i), there exists a subsequence, indexed by ε for notational convenience, and there exists a function $\boldsymbol{u} = u_i \in \boldsymbol{V}(\Omega)$ such that

$$\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{H}^1(\Omega) \text{ as } \varepsilon \to 0.$$

Then there exists functions $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$ satisfying $\zeta_i = \partial_\nu \zeta_3 = 0$ on γ_0 such that

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3 = \zeta_3. \tag{5.17}$$

From the definitions (4.23) and (5.14) and the boundedness of $(\widetilde{\boldsymbol{K}}(\varepsilon))$, we deduce that

$$||e_{\alpha 3}(\boldsymbol{u}(\varepsilon))||_{0,\Omega} \le \varepsilon C_{13} \text{ and } ||e_{\alpha 3}(\boldsymbol{u}(\varepsilon))||_{0,\Omega} \le \varepsilon^2 C_{13}$$

for some constant $C_{13} > 0$. Since a norm is a weakly lower semi-continuous function,

$$||e_{i3}(\boldsymbol{u})||_{0,\Omega} \le \liminf_{\varepsilon\to 0} ||e_{i3}(\boldsymbol{u}(\varepsilon))||_{0,\Omega} = 0$$

and whence $e_{i3}(\boldsymbol{u}) = 0$. Then a standard arguement (cf. [3, Chapter 1]) implies that the components u_i of the limit $\boldsymbol{u}$ are of the form (5.17).

(iv) By the step (iii), there exists a subsequence, still indexed by ε for convenience, and a function $\widetilde{\boldsymbol{K}} = (\widetilde{K}_{ij}) \in L^2(\Omega)$ such that $\widetilde{\boldsymbol{K}}(\varepsilon) \rightharpoonup \widetilde{\boldsymbol{K}}$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Then

$$\widetilde{K}_{\alpha\beta} = \widetilde{e}_{\alpha\beta}(\boldsymbol{u}), \quad \widetilde{K}_{\alpha3} = 0, \quad \widetilde{K}_{33} = -\frac{\lambda e^2}{\lambda+2\mu}\widetilde{e}_{\sigma\sigma}(\boldsymbol{u}). \tag{5.18}$$

Since $\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ (by the step (iii)), the definitions (4.4) of the functions $\widetilde{e}_{\alpha\beta}$ shows that $\widetilde{K}_{\alpha\beta}(\varepsilon) = \widetilde{e}_{\alpha\beta}(\boldsymbol{u}(\varepsilon))$ converges weakly in $L^2(\Omega)$ to the function $\widetilde{e}_{\alpha\beta}(\boldsymbol{u})$. We next note the following result (cf. [3, Chapter 1]): let $w \in L^2(\Omega)$ be given; then

$$\int_\Omega w\partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0 \Rightarrow w = 0. \tag{5.19}$$

With the relations (4.1)–(4.3), (4.8), (4.10) and the definitions (5.14) of the functions $\widetilde{K}_{ij}(\varepsilon)$, the variational equations (3.7) can be written as

$$\begin{aligned}&\int_\Omega \Big(\{[A^{\alpha\beta\sigma\tau}(0) + \varepsilon^2 A^{\alpha\beta\sigma\tau}_{\#}(\varepsilon)][\widetilde{K}_{\sigma\tau}(\varepsilon) + \varepsilon^2 e^{\#}_{\sigma\tau}(\varepsilon;\boldsymbol{u}(\varepsilon))] \\ &+ [A^{\alpha\beta33}(0) + \varepsilon^2 A^{\alpha\beta33}_{\#}(\varepsilon)]\widetilde{K}_{33}(\varepsilon)\} \\ &\Big\{\frac{1}{2}\partial_\alpha v_\beta + \frac{1}{2}\partial_\beta v_\alpha - \frac{v_3}{e}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e) + \varepsilon^2 e^{\#}_{\alpha\beta}(\varepsilon;\boldsymbol{v})\Big\}\end{aligned}$$

$$
\begin{aligned}
&+\{4[A^{\alpha 3\sigma 3}(0)+\varepsilon^2 A_{\#}^{\alpha 3\sigma 3}(\varepsilon)][\widetilde{K}(\varepsilon)_{\sigma 3}+\varepsilon e^{\#}_{\sigma 3}(\varepsilon;\boldsymbol{u}(\varepsilon))]\}\\
&\Big\{\frac{1}{2\varepsilon}\partial_\alpha v_3+\frac{1}{2\varepsilon}\partial_3 v_\alpha+\varepsilon e^{\#}_{\alpha 3}(\varepsilon;\boldsymbol{v})\Big\}\\
&+\{[A^{33\sigma\tau}(0)+\varepsilon^2 A_{\#}^{33\sigma\tau}(\varepsilon)][\widetilde{K}_{\sigma\tau}(\varepsilon)+\varepsilon^2 e^{\#}_{\sigma\tau}(\varepsilon;\boldsymbol{u}(\varepsilon))]\\
&+[A^{3333}(0)+\varepsilon^2 A_{\#}^{3333}(\varepsilon)]\widetilde{K}_{33}(\varepsilon)\}\Big\{\frac{1}{\varepsilon^2}\partial_3 v_3\Big\}\Big)\sqrt{e^2+\varepsilon^2 g^{\#}(\varepsilon)}dx\\
=&\int_\Omega f^i v_i\sqrt{e^2+\varepsilon^2 g^{\#}(\varepsilon)}dx \text{ for all } \boldsymbol{v}\in V(\Omega). \qquad (5.20)
\end{aligned}
$$

Multiplying these equations by ε, letting $v_3=0$, and using (4.11)–(4.13), we find that

$$
2\int_\Omega A^{\alpha 3\sigma 3}(0)K_{\alpha 3}(\varepsilon)\partial_3 v_\alpha e dx=2\mu\int_\Omega\frac{\widetilde{K}_{\alpha 3}(\varepsilon)}{e}\partial_3 v_\alpha dx=\varepsilon R(\varepsilon;\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v}) \qquad (5.21)
$$

with

$$
\sup_{0<\varepsilon\le\varepsilon_1}|R(\varepsilon;\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v})|\le C_{14}||\boldsymbol{v}||_{1,\Omega} \qquad (5.22)
$$

for all $(v_\alpha)\in\boldsymbol{H}^1(\Omega)$ such that $(v_\alpha)=0$ on Γ_0. For each such (v_α), the left-hand side of (5.21) goes to $2\mu\int_\Omega\frac{1}{e}\widetilde{K}_{\alpha 3}\partial_3 v_\alpha dx$ as $\varepsilon\to 0$ and the right-hand side goes to zero. Hence $\int_\Omega\frac{1}{e}\widetilde{K}_{\alpha 3}\partial_3 v_\alpha dx=0$ and thus $\frac{1}{e}\widetilde{K}_{\alpha 3}=0$ by (5.19). Since $e(x_1,x_2)>0$, it follows that $\widetilde{K}_{\alpha 3}=0$.

Multiplying the equations (5.20) by ε^2, letting $(v_\alpha)=0$ and using (4.11)–(4.13), we find that

$$
\begin{aligned}
&\int_\Omega\{A^{33\sigma\tau}(0)\widetilde{K}_{\sigma\tau}(\varepsilon)+A^{3333}(0)\widetilde{K}_{33}(\varepsilon)\}\partial_3 v_3 e dx\\
=&\int_\Omega\Big\{\frac{\lambda}{e^2}\widetilde{K}_{\sigma\sigma}(\varepsilon)+\frac{(\lambda+2\mu)}{e^4}\widetilde{K}_{33}(\varepsilon)\Big\}\partial_3 v_3 e dx\\
=&\,\varepsilon S(\varepsilon,\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v}) \qquad (5.23)
\end{aligned}
$$

with

$$
\sup_{0<\varepsilon\le\varepsilon_1}|S(\varepsilon,\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v})|\le C_{15}||\boldsymbol{v}||_{1,\Omega} \qquad (5.24)
$$

for all $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$. Passing to the limit as $\varepsilon\to 0$, we get

$$
\int_\Omega\frac{1}{e^3}\{e^2\lambda\widetilde{K}_{\sigma\sigma}+(\lambda+2\mu)\widetilde{K}_{33}\}\partial_3 v_3 dx=0, \qquad (5.25)
$$

and thus the last relation follows by another application of (5.19).

(v) The function $\boldsymbol{\zeta}=(\zeta_i)$ solves the variational equation (5.3).

Restricting the function $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ to the space

$$
\boldsymbol{V}_{KL}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega):\boldsymbol{v}=0\text{ on }\Gamma_0, e_{i3}(\boldsymbol{v})=0\text{ in }\Omega\} \qquad (5.26)
$$

we see that (5.20) reduce to

$$
\int_\Omega\{\lambda\widetilde{K}_{pp}(\varepsilon)\delta_{\alpha\beta}+2\mu\widetilde{K}_{\alpha\beta}(\varepsilon)\}\widetilde{e}_{\alpha\beta}(\boldsymbol{v})e dx=\int_\Omega f^i v_i e dx+\varepsilon T(\varepsilon,\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v}) \qquad (5.27)
$$

with

$$
\sup_{0<\varepsilon\le\varepsilon_1}|T(\varepsilon,\widetilde{\boldsymbol{K}}(\varepsilon),\boldsymbol{u}(\varepsilon),\boldsymbol{v})|\le C_{16}||\boldsymbol{v}||_{1,\Omega} \qquad (5.28)
$$

for all $\boldsymbol{v}\in\boldsymbol{V}_{KL}(\Omega)$.

Passing to the limit as $\varepsilon \to 0$ and taking into account the relation (5.18), we are left with

$$\int_\Omega \{\lambda \widetilde{K}_{pp}\delta_{\alpha\beta} + 2\mu \widetilde{K}_{\alpha\beta}\} e_{\alpha\beta}(\boldsymbol{v}) e dx = \int_\Omega \Big\{\frac{2\lambda\mu}{\lambda+2\mu}\widetilde{e}_{\sigma\sigma}\delta_{\alpha\beta} + 2\mu\widetilde{e}_{\alpha\beta}(\varepsilon)\Big\}\widetilde{e}_{\alpha\beta}(\boldsymbol{v}) e dx$$

$$= \int_\Omega f^i v_i e dx \tag{5.29}$$

for all $\boldsymbol{v} \in \boldsymbol{V}_{KL}(\Omega)$. Once the components u_i of $\boldsymbol{u}$ have been replaced by their expression (5.17) and the components v_i of $\boldsymbol{v}$ have been replaced by

$$v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3 \text{ and } v_3 = \eta_3 \text{ with } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega),$$

it is verified that the equation (5.29) coincides with the equations (5.3).

(vi) The variational equations (5.3) have a unique solution $\boldsymbol{\zeta} \in \boldsymbol{V}(\omega)$.

It is easy to see that the mapping $\boldsymbol{\eta} \in \boldsymbol{V}(\omega) \to ((\eta_\alpha - x_3\partial_\alpha\eta_3), \eta_3) \in \boldsymbol{V}_{KL}(\Omega)$ is an isomorphism. Hence it is sufficient to show that the variational equations (5.29) have a unique solution. But that follows from the fact that the bilinear form $B(..)$ defined by the left-hand side of that equation satisfies

$$B(\boldsymbol{v},\boldsymbol{v}) \geq 2\mu\sum_{\alpha\beta}||\widetilde{e}_{\alpha\beta}(\boldsymbol{v})||^2_{0,\Omega} = 2\mu\sum_{i,j}||\widetilde{e}_{ij}(\boldsymbol{v})||^2_{0,\Omega} \text{ for all } \boldsymbol{v} \in V_{KL}(\Omega) \tag{5.30}$$

and from the fact that $\Big\{\sum_{i,j}||\widetilde{e}_{ij}(\boldsymbol{v})||^2_{0,\Omega}\Big\}^{1/2}$ is a norm in the space $V(\Omega)$ (note that $\boldsymbol{V}_{KL}(\Omega)$ is a closed subspace of $\boldsymbol{V}(\Omega)$), equivalent to $||.||_{1,\Omega}$.

(vii) The strong convergence of $(\boldsymbol{u}(\varepsilon))$ to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ follows as in [2].

§6. Two-Dimensional Equations

We now "descale" the functions ζ_i and u_i found in Theorem 5.1 to obtain the two-dimensional model approximating the three dimensional problem. In view of the scaling (3.1)–(3.2), we define functions $\zeta_i^\varepsilon : \overline{\omega} \to I\!R$ and $u_i^\varepsilon(0) : \{\Omega^\varepsilon\}^- \to I\!R$ by

$$\zeta_\alpha^\varepsilon(y) = \varepsilon^2\zeta_\alpha(y) \text{ and } \zeta_3^\varepsilon(y) = \varepsilon\zeta_3(y) \text{ for all } y \in \omega, \tag{6.1}$$

$$u_\alpha^\varepsilon(0)(x^e) = \varepsilon^2 u_\alpha(x) \text{ and } u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(x) \text{ for all } x \in \Omega, \tag{6.2}$$

where the points $x^\varepsilon \in \Omega^\varepsilon$ and $x \in \Omega$ are related as in Section 3.

Theorem 6.1. *The functions ζ_i^ε defined in* (6.1) *satisfies* $(\zeta_i^\varepsilon) \in \boldsymbol{V}(\omega)$,

$$-\int_\omega m^{\alpha\beta,\varepsilon}\partial_{\alpha\beta}\eta_3 e d\omega - \int_\omega (n^{\alpha\beta,\varepsilon}\partial_{\alpha\beta}\theta^\varepsilon + m^{\alpha\beta,\varepsilon}\partial_{\alpha\beta}e)\eta_3 e d\omega + \int_\omega n^{\alpha\beta,\varepsilon}\partial_\beta\eta_\alpha e d\omega$$

$$= \int_\omega p^{i,\varepsilon}\eta_i e d\omega - \int_\omega q^{\alpha,\varepsilon}\partial_\alpha\eta_3 e d\omega \tag{6.3}$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, where

$$m^{\alpha\beta,\varepsilon} = -\varepsilon^3\Big\{\frac{4\lambda\mu}{3(\lambda+2\mu)}\Big(\Delta\zeta_3^\varepsilon + \zeta_3^\varepsilon\frac{\Delta e}{e}\Big)\delta_{\alpha\beta} + \frac{4\mu}{3}\Big(\partial_{\alpha\beta}\zeta_3^\varepsilon + \zeta_3^\varepsilon\frac{\partial_{\alpha\beta}e}{e}\Big)\Big\}, \tag{6.4}$$

$$n^{\alpha\beta,\varepsilon} = \varepsilon\Big\{\frac{4\lambda\mu}{\lambda+2\mu}\hat{e}^\varepsilon_{\sigma\sigma}(\boldsymbol{\zeta}^\varepsilon)\delta_{\alpha\beta} + 4\mu\hat{e}^\varepsilon_{\alpha\beta}(\boldsymbol{\zeta})\Big\}, \tag{6.5}$$

$$\hat{e}^\varepsilon_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) = \frac{1}{2}(\partial_\alpha\zeta_\beta^\varepsilon + \partial_\beta\zeta_\alpha^\varepsilon) - \zeta_3^\varepsilon\frac{\partial_{\alpha\beta}\theta^\varepsilon}{e}, \tag{6.6}$$

$$p^{i,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^\varepsilon, \tag{6.7}$$

$$q^\alpha = \int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon f^{\alpha,\varepsilon} dx_3^\varepsilon. \tag{6.8}$$

Remark 6.1. (1) It is also possible to make other assumptions on the data that will yield the same shallow shell equations. More precisely, if s is any real number, then replacing the relations (3.3)–(3.4) by the more general relation

$$f^{\alpha,\varepsilon}(x^e) = \varepsilon^{s+2} f^\alpha(x) \text{ and } f^{3,\varepsilon}(x^e) = \varepsilon^{s+3} f^3(x) \text{ for all } x \in \Omega,$$
$$\lambda^\varepsilon = \varepsilon^s \lambda \text{ and } \mu^\varepsilon = \varepsilon^3 \mu,$$

where the functions $f^i \in L^2(\Omega)$ and the constants $\lambda > 0$ and $\mu > 0$ are independent of ε, will yield the same conclusions as those found here.

(2) It can be verified that when $e = 1$, the equations (6.3) coinside with the two dimensional equations of shallow shell with constant thickness obtained in [2].

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