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NONLINEAR HYPERBOLIC CAUCHY PROBLEMS IN GEVREY CLASSES

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Abstract

The authors prove well posedness in Gevrey classes of Cauchy problem for nonlinear hyperbolic equations of constant multiplicity with Hölder dependence on the time variable.

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§1. Introduction

Any linear hyperbolic operator P with C^χ coefficients with respect to time, $0 < \chi < 1$, has a well posed Cauchy problem in suitable Gevrey classes G^σ whereas this is not true in C^∞. Focusing on operators with characteristics of constant multiplicity, we have the following bounds for the Gevrey index:

$$\sigma < \frac{1}{1-\chi}, \quad \text{if } P \text{ is strictly hyperbolic}^{[1,5,8,4]}, \tag{0.1}$$

$$\sigma < \min\Big\{1+\chi, \frac{r}{r-1}\Big\}, \quad r \geq 2 \quad \text{the largest multiplicity}^{[9]}. \tag{0.2}$$

From (0.1) and (0.2) we have well posedness for

$$\sigma < \frac{r}{r-\chi}, \quad r \geq 1$$

and this seems to be the natural condition on σ for any r (see Section 3).

Our purpose in this paper is to show that this still holds in nonlinear framework so improving, in the case of constant multiplicity, a result by K. Kajitani[6].

The result is obtained by the same method of [2] and [3] where we considered weakly hyperbolic equations with Levi conditions and smooth dependence on all variables. So we have found a uniform approach to the hyperbolic Cauchy problem with characteristics of constant multiplicity.

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§1. Main Results

For $\sigma > 1$ and $A > 0$ let us denote by $G_A^\sigma = G_A^\sigma(\mathbf{R}^n)$ the space of all functions f satisfying

$$\|f\|_{G_A^\sigma} := \sup_{\alpha\in\mathbf{Z}_+^n}\sup_{x\in\mathbf{R}^n}|D^\alpha f(x)|A^{-|\alpha|}\alpha!^{-\sigma} < \infty$$

and by $G_A^{\sigma,1} = G_A^{\sigma,1}(\mathbf{R}^n\times W), W$ an open set in $\mathbf{R}^\ell$, the space of all functions f such that

$$\|f\|_{G_A^{\sigma,1}} := \sup_{(\alpha,\beta)\in\mathbf{Z}_+^n\times\mathbf{Z}_+^\ell}\ \sup_{(x,w)\in\mathbf{R}^n\times W}|D_x^\alpha D_w^\beta f(x,w)|A^{-|\alpha|-|\beta|}\alpha!^{-\sigma}\beta!^{-1} < \infty;$$

so $G^\sigma := \bigcup\limits_{A>0} G_A^\sigma$ and $G^{\sigma,1} := \bigcup\limits_{A>0} G_A^{\sigma,1}$ are Gevrey spaces.

Then, let us consider the quasilinear Cauchy problem

$$\begin{cases} \sum\limits_{|\alpha|\le m} a_\alpha(t,x,D^{m'}u)D_{t,x}^\alpha u = f(t,x,D^{m'}u), \\ D_t^j u_{|t=0} = 0, \quad 0\le j<m, \end{cases} \tag{1.1}$$

where $D^{m'}u = D_{t,x}^{m'}u$ denotes the vector $(D_{t,x}^\beta u; |\beta|\le m')$, $m'\le m-1$, the functions $a_\alpha(t,x,w)$ and $f(t,x,w)$ are defined for $t\in[0,T], x\in\mathbf{R}^n, w\in W_0$ a neighborhood of the origin in $\mathbf{R}^\ell$, and the maps $x\to f(t,x,w)$ have supports contained in a compact set of $\mathbf{R}^n$. As it concerns the regularity of the equation with respect to the variable (t,x,w) we assume:

(R)
$$\begin{aligned} &a_\alpha\in C^\chi(0,T;G_A^{\sigma,1}),\ 0<\chi<1 && \text{for } |\alpha|=m; \\ &f, a_\alpha\in C^0(0,T;G_A^{\sigma,1}) && \text{for } |\alpha|<m. \end{aligned}$$

The linear differential operator

$$P(t,x,w,D_t,D_x) = \sum_{|\alpha|\le m} a_\alpha(t,x,w)D_{t,x}^\alpha$$

is assumed to be hyperbolic with characteristics roots of constant multiplicity, that is, for positive integers s and $m_j, j=1,\cdots,s$, independent of (t,x,w), the principal part P_m satisfies

(H)
$$P_m(t,x,w,\tau,\xi) = \prod_{j=1}^s\big(\tau-\lambda_j(t,x,w,\xi)\big)^{m_j}$$

with real $\lambda_j,\ j=1,\cdots,s$, and $|\lambda_h(t,x,w,\xi)-\lambda_k(t,x,w,\xi)|>c|\xi|$ if $h\ne k$.

The number $r:=\max\{m_j\ ;\ j=1,\cdots,s\}$ denotes the largest multiplicity.

Now we can state our main result:

Theorem 1.1. *Assume conditions* (R),(H) *and*

(G)
$$1<\sigma\le\frac{r}{r-\chi},$$

(N)
$$m'\le m-\min\{r,2\}.$$

Then, there are $T_0<T$ *and* $A_0>A$ *such that the problem* (1.1) *has a unique solution* $u\in C^m(0,T_0;G_{A_0}^\sigma)$.

As it concerns the degree of nonlinearity, we note that (N) is an empty condition for strictly hyperbolic equations ($r=1$) since we can allow $m'=m-1$. It reduces to $m'\le m-2$ in case of weak hyperbolicity.

§2. Notation and Preliminary Results

This section is devoted to introducing the space of functions and related pseudodifferential operators that we use in the proof of Theorem 1.1.

We denote by S^m_ℓ, $m \in \mathbf{R}$, $\ell \in \mathbf{Z}_+$, the class of all symbols $a(x,\xi) \in S^m_{1,0}(\mathbf{R}^n)$ with norm

$$\|a\|_{S^m_\ell} = \sup_{|\alpha|+|\beta|\le \ell} \sup_{\mathbf{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \langle\xi\rangle^{|\beta|-m}, \quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}.$$

In a corresponding way to the spaces of functions G^σ_A introduced in Section 1, we denote by $S^{m,\sigma}_{\ell,A}$ the space of all symbols $a(x,\xi)$ of order m such that

$$\|a\|_{S^{m,\sigma}_{\ell,A}} = \sup_{\alpha\in\mathbf{Z}^n_+, |\beta|\le\ell} \sup_{\mathbf{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \langle\xi\rangle^{|\beta|-m} A^{-|\alpha|} \alpha!^{-\sigma} < \infty$$

and define

$$S^{m,\sigma}_\ell := \varinjlim_{A\to+\infty} S^{m,\sigma}_{\ell,A}, \quad S^{m,\sigma} := \varprojlim_{\ell\to+\infty} S^{m,\sigma}_\ell.$$

For a symbol $a(x,\xi)$ in $S^{m,\sigma}_{\ell,A}$ and $\Lambda = \tau\langle D_x\rangle^{1/\sigma}$, $\tau \in \mathbf{R}$, let us denote by $a_\Lambda(x,D_x)$ the operator $a_\Lambda = e^\Lambda a e^{-\Lambda}$. Then, we have[6]

Proposition 2.1. *Let $a(t,x,\xi) \in C^k(0,T; S^{m,\sigma}_{\ell,A})$, $\Lambda = \Lambda(t) = \lambda(2T-t)\langle D_x\rangle^{1/\sigma}$, $\lambda \in \mathbf{R}$. Then there are $T_0 = T_0(A,\sigma,n) > 0$ and for every ℓ' a positive integer $\ell_0 = \ell_0(\ell',\sigma,n)$ such that*

$$\|a_\Lambda\|_{C^k(0,T;S^m_{\ell'})} \le C\|a\|_{C^k(0,T;S^{m,\sigma}_{\ell,A})} \quad , \quad |\lambda T| \le T_0 \ , \ \ell \ge \ell_0 \tag{2.1}$$

with a constant $C = C(A,\ell',\sigma,k,n) > 0$ independent of $a(t,x,\xi)$.

We use also symbols depending on a parameter $w \in W = W_0 \times \mathbf{R}^{d_M}$ where W_0 is a given open set in $\mathbf{R}^\ell$, $M \in \mathbf{Z}_+$ and d_M is the number of all $\alpha \in \mathbf{Z}^n_+$ such that $|\alpha| \le M$. In fact, we denote by $S^{m,\sigma,1}_{\ell,A}(\mathbf{R}^n \times W \times \mathbf{R}^n)$ the space of all symbols $a(x,w,\xi)$ of order m such that

$$\|a\|_{S^{m,\sigma,1}_{\ell,A}} = \sup_{\alpha,\beta,|\gamma|\le\ell} \sup_{x,w,\xi} |\partial_x^\alpha \partial_w^\beta \partial_\xi^\gamma a(x,w,\xi)| \langle\xi\rangle^{|\gamma|-m} A^{-|\alpha|-|\beta|} \alpha!^{-\sigma} \beta!^{-1} < \infty$$

and we set

$$S^{m,\sigma,1}_\ell := \varinjlim_{A\to+\infty} S^{m,\sigma,1}_{\ell,A}, \quad S^{m,\sigma,1} := \varprojlim_{\ell\to+\infty} S^{m,\sigma,1}_\ell.$$

Next we introduce Gevrey-Sobolev spaces. For $\tau,\mu \ge 0$, $\sigma > 1$ we denote by $H^{\tau,\sigma,\mu}(\mathbf{R}^n)$ the space of all functions u such that

$$\|u\|_{H^{\tau,\sigma,\mu}} := \|e^{\tau<D>^{1/\sigma}} u\|_{H^\mu} < \infty,$$

where $H^\mu = H^\mu(\mathbf{R}^n)$ the usual Sobolev space.

From Paley-Wiener theorem, it follows that

$$\|u\|_{H^{\tau,\sigma,\mu}} \le C\|u\|_{G^\sigma_A}, \quad u \in G^\sigma_A \cap C^\infty_0 \quad , \quad 0 \le \tau \le \tau_0 \tag{2.2}$$

with $\tau_0 = \tau_0(A,\sigma,n)$ and $C = C(A,\sigma,\mu,n)$ positive constants. Conversely, we have

$$H^{\tau,\sigma,\mu} \subset G^\sigma_A \quad , \quad \tau > 0 \ , \ A > c_\sigma \tau^{-\sigma}, \quad \mu > \frac{n}{2} \tag{2.3}$$

with continuous embedding.

Moreover for $\mu > \dfrac{n}{2}$, $H^{\tau,\sigma,\mu}$ is a Banach algebra:

$$\|uv\|_{H^{\tau,\sigma,\mu}} \le C_\mu \|u\|_{H^{\tau,\sigma,\mu}} \|v\|_{H^{\tau,\sigma,\mu}}. \tag{2.4}$$

Together with the space $H^{\tau,\sigma,\mu}$ we consider the following classes of functions and symbols depending on $t \in [0,T]$:

$$C_T^k(H^{\lambda,\sigma,\mu}) = \{u(t,x)\ ;\ t \to e^{\lambda(2T-t)\langle D_x\rangle^{1/\sigma}} D_t^j u(t,\cdot) \text{ is continuous from } [0,T] \text{ to } H^{\mu-j}(\mathbf{R}^n)\ ,\ j = 0,\cdots,k\} \tag{2.5}$$

with norm

$$\|u\|_{C_T^k(H^{\lambda,\sigma,\mu})} := \sup_{0\le j\le k}\sup_{0\le t\le T} \|D_t^j u(t,\cdot)\|_{H^{\lambda(2T-t),\sigma,\mu-j}};$$

$$C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu}) = \{a(t,x,\xi) \in C^k(0,T;S_{1,0}^m);\ \partial_x^\alpha\partial_\xi^\beta a(\cdot,\cdot,\xi) \in C_T^k(H^{\lambda,\sigma,\mu}),\ |\alpha|+|\beta| \le \ell\ ,\ \xi \in \mathbf{R}^n\} \tag{2.6}$$

with norm

$$\|a\|_{C_T^k(S_\ell^m;H^{\lambda,\sigma,\mu})} := \sup_{|\alpha|+|\beta|\le\ell}\sup_{\xi\in\mathbf{R}^n} \|\partial_x^\alpha\partial_\xi^\beta a(\cdot,\cdot,\xi)\|_{C_T^k(H^{\lambda,\sigma,\mu})}.$$

Both in definitions (2.5) and (2.6) we assume $k \in \mathbf{Z}_+,\ \lambda > 0,\ \mu > \frac{n}{2}$.

Note that, as a consequence of (2.3), we have $C_T^k(H^{\lambda,\sigma,\mu}) \subset C^k(0,T;G_A^\sigma),\ A > c_\sigma(\lambda T)^{-\sigma}$, $\mu > \frac{n}{2} + k$, since $2T - t \ge T$ for $t \in [0,T]$. This fact can not be used to obtain an estimate of type (2.1) for $a \in C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu})$ because the condition $A > c_\sigma T_0^{-\sigma}(A,\sigma,n)$ could not be satisfied. So we need the following result[6]:

Proposition 2.2. *Let $a \in C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu})$ and $\Lambda = \Lambda(t) = \lambda(2T-t)\langle D_x\rangle^{1/\sigma}$. Then for every $\ell' \in \mathbf{Z}_+$ there is a positive integer $\ell_0 = \ell_0(\ell',\sigma,n)$ such that*

$$\|a_{\pm\Lambda}\|_{C^k(0,T;S_{\ell'}^m)} \le C\|a\|_{C_T^k(S_\ell^m;H^{\lambda,\sigma,\mu})}\quad,\quad \ell \ge \ell_0 \tag{2.7}$$

with a constant $C = C(\mu,\ell',\sigma,k,n) > 0$ independent of $a(t,x,\xi)$.

Concerning the composition of $u \in C_T^k(H^{\lambda,\sigma,\mu})$ with smooth maps, we have[6]

Proposition 2.3. *Let $f(t,x,w) \in C^k(0,T;G_A^{\sigma,1})$ with $f(t,x,0) = 0$ and let $u \in C_T^k(H^{\lambda,\sigma,\mu})$ with $\|u\|_{C_T^k(H^{\lambda,\sigma,\mu})} \le r,\ r > 0$. Then, there are positive $\mu_0 = \mu_0(\sigma,n),\ T_0 = T_0(\sigma,\mu),\ r_0 = r_0(k,\mu,n), C = C(f,\sigma,\mu)$ such that*

$$\|f(t,x,u(t,x)\|_{C_T^k(H^{\lambda,\sigma,\mu})} \le C\|u\|_{C_T^k(H^{\lambda,\sigma,\mu})},\quad \mu \ge \mu_0\ ,\ \lambda T A^{1/\sigma} < T_0\ ,\ rA \le r_0. \tag{2.8}$$

In particular, for $a(t,x,w,\xi) \in C^k(0,T;S^{m,\sigma,1})$ and $u \in C_T^k(H^{\lambda,\sigma,\mu_0+\ell})$ we have

$$a(t,x,u(t,x),D_x) = a_0(t,x,D_x) + a_1(t,x,u(t,x),D_x)$$

with

$$a_0 = a(t,x,0,\xi) \in C^k(0,T;S^{m,\sigma}),\quad a_1 \in C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu_0})$$

provided that λT and $\|u\|_{C_T^k(H^{\lambda,\sigma,\mu_0+\ell})}$ are sufficiently small. We shall write

$$a = a_0 + a_1 \in C^k(0,T;S^{m,\sigma}) + C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu_0})$$

in such a situation. Then, one can use Proposition 2.1 and Proposition 2.2 to have a precise estimate of the norm of $a(t,x,u(t,x),D_x)$ as a continuous operator from $C_T^k(H^{\lambda,\sigma,\mu})$ to $C_T^k(H^{\lambda,\sigma,\mu-m})$ in view of the well-known continuity of $a \in S_{\ell'}^m$ in usual Sobolev spaces.

It is reasonable to conjecture that Proposition 2.3 holds assuming f of class G^σ in (x,w), but, as far as we know, this more general result has not been obtained yet. So in Theorem 1.1 we assume the analytic regularity of a_α and f with respect to the variable w.

§3. The Linear Problem

Given $u \in C_T^{m'+1}(H^{\lambda,\sigma,\mu})$, $\|u\|_{C_T^{m'}(H^{\lambda,\sigma,\mu})} \le r_0$, $\|u\|_{C_T^{m'+1}(H^{\lambda,\sigma,\mu})} \le r_1$, $\lambda T \le T_0$, let us consider the linear Cauchy problem for the unknown v,

$$\begin{cases} P(t,x,D^{m'}u,D_t,D_x)v = f(t,x,D^{m'}u), \\ D_t^j v_{|t=0} = 0\ ,\ \ 0 \le j < m, \end{cases} \tag{3.1}$$

with $P(t,x,w,D_t,D_x) = \sum\limits_{|\alpha|\le m} a_\alpha(t,x,w)D_{t,x}^\alpha, f(t,x,w),\sigma,m'$, satisfying all conditions (H), (R),(G),(N) in Theorem 1.1.

We want to show that it is possible to take large λ,μ and small r_0,T_0 in order to have a unique solution $v \in C_T^m(H^{\lambda,\sigma,\mu})$.

The first step is a factorization of the operator P. From assumptions (H) and (R), the characteritic roots $\lambda_j(t,x,w,\xi)$, $j=1,\cdots,s$, are symbols of pseudodifferential operators of order 1 in $\mathbf{R}^n$, depending on the parameter (t,w), after a modification in a neighborhood of $\xi=0$. Precisely, we have

$$\lambda_j \in C^\chi(0,T;S^{1,\sigma,1}). \tag{3.2}$$

We extend each $\lambda_j(t,x,D^{m'}u(t,x),\xi)$ to a Hölder continuous function of $t \in]-\infty,T]$, $\lambda_j(t,x,D^{m'}u(t,x),\xi) = \lambda_j(0,x,D^{m'}u(0,x),\xi)$ for $t<0$, then we regularize it by means of a Friedrichs' mollifier $J_\varepsilon, \varepsilon = |\xi|^{-1}$, defining

$$\begin{aligned} &\tilde\lambda_j(t,x,D^{m'}u,\xi) = \int \lambda_j(s,x,D^{m'}u(s,x),\xi)\varphi((t-s)|\xi|)|\xi|ds, \\ &\varphi \in C_0^\infty(\mathbb{R}^n),\ \text{supp}\ \varphi \subset \mathbf{R}_+, 0 \le \varphi \le 1, \int \varphi = 1. \end{aligned} \tag{3.3}$$

Obviously $\partial_t^h\tilde\lambda_j(t,x,w,\xi) \in C^0(0,T;S^{1+h,\sigma,1}), h \ge 0$, but from (3.2) we have a gain of order equal to χ for $\Delta_j := \lambda_j - \tilde\lambda_j$ and $\partial_t^h\tilde\lambda_j$, $h \ge 1$. In fact, for sufficiently small T_0 and r_0, Proposition (2.3), the property (3.2) and the estimate

$$\|D^{m'}u(t,x) - D^{m'}u(s,x)\|_{H^{\lambda(2T-t),\sigma,\mu-m'-1}} \le (t-s)\|u\|_{C_T^{m'+1}(H^{\lambda,\sigma,\mu})},\quad s \le t,$$

give

$$\begin{aligned} &\Delta_j(t,x,D^{m'}u(t,x),\xi) = \Delta_{j,0}(t,x,\xi) + \Delta_{j,1}(t,x,D^{m'}u(t,x),\xi), \\ &\Delta_{j,0} + \Delta_{j,1} \in C^0(0,T;S^{1-\chi,\sigma}) + C_T^0(S_\ell^{1-\chi};H^{\lambda,\sigma,\mu-m'-\ell-1}); \end{aligned} \tag{3.4}$$

$$\|\Delta_{j,1}\|_{C_T^0(S_\ell^{1-\chi};H^{\lambda,\sigma,\mu-m'-\ell-1})} \le C_\ell(1+r_1); \tag{3.5}$$

$$\begin{aligned} &\partial_t^h\tilde\lambda_j(t,x,D^{m'}u(t,x),\xi) = \partial_t^h\tilde\lambda_{j,0}(t,x,\xi) + \partial_t^h\tilde\lambda_{j,1}(t,x,D^{m'}u(t,x),\xi), \\ &\partial_t^h\tilde\lambda_{j,0} + \partial_t^h\tilde\lambda_{j,1} \in C^0(0,T;S^{1+h-\chi,\sigma}) + C_T^0(S^{1+h-\chi};H^{\lambda,\sigma,\mu-m'-\ell-1}),\quad h \ge 1; \end{aligned} \tag{3.6}$$

$$\|\partial_t^h\tilde\lambda_{j,1}\|_{C_T^0(S_\ell^{1+h-\chi};H^{\lambda,\sigma,\mu-m'-\ell-1})} \le C_\ell(1+r_1). \tag{3.7}$$

So we get the following factorization of the operator $P(t,x,D^{m'}u,D_t,D_x):$

$$\begin{aligned} &P = \prod_{j=1}^s (D_t - \tilde\lambda_j(t,x,D^{m'}u,D_x))^{r_j} + R(t,x,D_*^{M_0+m'}u,D_t,D_x), \\ &D_*^{M_0+m'}u := D_x^{M_0}D_{t,x}^{m'}u, \end{aligned} \tag{3.8}$$

where the remainder R is of order $m-\chi$ and has the structure

$$R=\sum_{0}^{m-1} R_j(t,x,D_*^{M_0+m'}u,D_x)D_t^j$$

with

$$R_j=R_j^0+R_j^1\in C^0(0,T;S^{m-j-\chi,\sigma})+C_T^0(S_\ell^{m-j-\chi};H^{\lambda,\sigma,\mu-M_0-m'-\ell-1}),$$
$$\|R_j^1\|_{C_T^0(S_\ell^{m-j-\chi};H^{\lambda,\sigma,\mu-M_0-m'-\ell-1})}\le C_\ell(1+r_1);$$

the positive integer M_0 depends only on $m_j,\ j=1,\cdots,s$, and the dimension n.

Our next aim is to reduce the problem (3.1) to a first order system by means of the factorization (3.8). In doing so, only to have a simpler notation, let us treat the case of an operator P with two characteristics roots. We can factorize P in two different ways

$$P=P_1P_2+R,\quad P=P_2P_1+\tilde R,\quad P_j=(D_t-\tilde\lambda_j)^{m_j},\ j=1,2$$

with remainders R and $\tilde R$ of order $m-\chi$.

Then we define the vector $(w_0,\cdots,w_{2m-1})$ as follows:

$$\begin{array}{ll}
w_0=v, & w_m=v,\\
w_1=(D_t-\tilde\lambda_1)v, & w_{m+1}=(D_t-\tilde\lambda_2)v,\\
\quad\vdots & \quad\vdots\\
w_{m_1-1}=(D_t-\tilde\lambda_1)^{m_1-1}v, & w_{m+m_2-1}=(D_t-\tilde\lambda_2)^{m_2-1}v,\\
w_{m_1}=P_1v, & w_{m_2}=P_2v,\\
w_{m_1+1}=(D_t-\tilde\lambda_2)P_1v, & w_{m+m_2+1}=(D_t-\tilde\lambda_1)P_2v,\\
\quad\vdots & \quad\vdots\\
w_{m-1}=(D_t-\tilde\lambda_2)^{m_2-1}P_1v, & w_{2m-1}=(D_t-\tilde\lambda_1)^{m_1-1}P_2v,
\end{array}\tag{3.9}$$

so obtaining an obvious equivalence between the equations $P_1P_2v=f,P_2P_1v=f$ and a diagonal hyperbolic system of order 1. To reduce the remainders R and $\tilde R$ to operators of order $\varrho<1$, we put a weight $\langle D_x\rangle^{\varrho(m-1-j)}$ on w_j and w_{m+j} defining

$$v_j=\langle D_x\rangle^{\varrho(m-1-j)}w_j,\quad v_{m+j}=\langle D_x\rangle^{\varrho(m-1-j)}w_{m+j},\ j=0,\cdots,m-1.$$

We have

Lemma 3.1. *For every $k<m$ there is an operator $Q=Q(t,x,D_*^{M_0+m'}u,D_x)$ of order $d=k-m+r(1-\varrho)+\varrho$ such that*

$$D^kv=QV,\quad V=(v_0,\cdots,v_{2m-1}).\tag{3.11}$$

Furthermore, for sufficiently small T_0 and r_0 we have

$$\begin{aligned}&Q(t,x,D_*^{M_0+m'}u,\xi)=Q^0(t,x,\xi)+Q^1(t,x,D_*^{M_0+m'}u,\xi),\\&Q^0+Q^1\in C^0(0,T;S^{d,\sigma})+C_T^0(S_\ell^d;H^{\lambda,\sigma,\mu-M_0-m'-\ell-1});\end{aligned}\tag{3.12}$$

$$\|Q^1\|_{C_T^0(S_\ell^d;H^{\lambda,\sigma,\mu-M_0-m'-\ell-1})}\le C_\ell(1+r_1).\tag{3.13}$$

Proof. From (3.9) and (3.10) we get immediatly

$$\begin{aligned}&D_t^kv=\sum_{j=0}^{k}a_j^{(k)}(t,x,D_*^{M_0+m'}u,D_x)v_j,\\&a_j^{(k)}(t,x,w,\xi)\in C^0(S^{k-j-\varrho(m-1-j),\sigma,1}),\ 0\le k\le m-1,\end{aligned}\tag{3.14}$$

which proves the lemma in the case $r = m$. Note that we do not need (3.6) but only $\partial_t^h \tilde{\lambda}_j(t,x,w,\xi) \in C^0(0,T;S^{1+h,\sigma,1})$ to get (3.14).

In the general case, we use (3.14) and Lagrange's interpolation formula. So, for $0 \le k \le m-1, m = m_1 + m_2$, let us write the identity

$$\tau^k = \Big(\frac{\tau - \tilde{\lambda}_1}{\tilde{\lambda}_2 - \tilde{\lambda}_1} - \frac{\tau - \tilde{\lambda}_2}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\Big)^{m-1-k}\Big(\frac{\tau - \tilde{\lambda}_1}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\tilde{\lambda}_2 - \frac{\tau - \tilde{\lambda}_2}{\tilde{\lambda}_2 - \tilde{\lambda}_1}\tilde{\lambda}_1\Big)^k.$$

Then, applying Newton's formula we have

$$\tau^k = \sum_{k_1=0}^{m-1-k}\sum_{k_2=0}^{k} a^{(k)}_{k_1,k_2}(\tau - \tilde{\lambda}_1)^{k_1+k_2}(\tau - \tilde{\lambda}_2)^{m-1-k_1-k_2}, \quad \operatorname{ord} a^{(k)}_{k_1,k_2} \le k+1-m.$$

We write the terms in the sum with $k_1 + k_2 < m_1$ in the form

$$a^{(k)}_{k_1,k_2}(\tau - \tilde{\lambda}_1)^{k_1+k_2}(\tau - \tilde{\lambda}_2)^{m_2}[(\tau - \tilde{\lambda}_1) + (\tilde{\lambda}_1 - \tilde{\lambda}_2)]^{m_1-1-k_1-k_2}$$

and the terms with $k_1 + k_2 \ge m_1$ in the form

$$a^{(k)}_{k_1,k_2}(\tau - \tilde{\lambda}_1)^{m_1}(\tau - \tilde{\lambda}_2)^{m-1-k_1-k_2}[(\tau - \tilde{\lambda}_2) + (\tilde{\lambda}_2 - \tilde{\lambda}_1)]^{k_1+k_2-m_1}.$$

Applying again Newton's formula to the powers in square brackets, we obtain

$$r^k = \sum_{j=0}^{m_1-1} b^{(k)}_{1,j}(\tau - \tilde{\lambda}_1)^j(\tau - \tilde{\lambda}_2)^{m_2} + \sum_{j=0}^{m_2-1} b^{(k)}_{2,j}(\tau - \tilde{\lambda}_2)^j(\tau - \tilde{\lambda}_1)^{m_1},$$

$$\operatorname{ord} b^{(k)}_{1,j} \le k - j - m_2, \quad \operatorname{ord} b^{(k)}_{2,j} \le k - j - m_1.$$

Carrying the above equality to the operators level, from (3.9),(3.10) and (3.6) we have

$$\begin{aligned} D_t^k v &= \sum_{j=0}^{m_1-1} q^{(k)}_{1,j} v_{m+m_2+j} + \sum_{j=0}^{m_2-1} q^{(k)}_{2,j} v_{m_1+j} + \sum_{j=0}^{m_1-1} r^{(k)}_j D_t^j v, \\ q^{(k)}_{i,j} &= q^{(k)}_{i,j}(t,x,D_*^{M_0+m'}u,D_x), \quad r^{(k)}_j = r^{(k)}_j(t,x,D_*^{M_0+m'}u,D_x), \\ \operatorname{ord} q^{(k)}_{i,j} &\le k - m + m_i(1-\varrho) + \varrho, \quad \operatorname{ord} r^{(k)}_j \le k - j - \chi. \end{aligned} \tag{3.15}$$

In the last sum in (3.15) one can replace $D_t^j v$ with the expression given by the same equality with j instead of k, so obtaining a similar representation of $D_t^k v$ but with new operators $r_j^{(k)}$ of order $k - j - 2\chi$.

One repeats this process till it is possible to get (3.11) replacing in the third sum $D_t^j v$ with the expression given by (3.14).

The properties (3.12) and (3.13) follow from (3.7).

Applying Lemma 3.1 to Rv and $\tilde{R}v$, we have

$$Rv = Q_1 V, \quad \tilde{R}v = \tilde{Q}_1 V$$

with Q_1 and $\tilde{Q}_1$ of order $\varrho < 1$ if we choose $\varrho = \dfrac{r-\chi}{r}$. Then (3.1) is equivalent to the problem

$$\begin{cases} D_t V - K_0(t,x,D^{m'}u,D_x)V + A_0(t,x,D_*^{M_0+m'}u,D_x)V = F_0(t,x,D^{m'}u), \\ V_{|t=0} = 0, \end{cases} \tag{3.16}$$

where K_0 is diagonal of order 1 with $\tilde{\lambda}_j$ as entries, A_0 is of order $\varrho = \dfrac{r-\chi}{r}$,

$$\begin{aligned}
&A_0(t,x,D_*^{M_0+m'}u,\xi) = A_{0,0}(t,x,\xi) + A_{0,1}(t,x,D_*^{M_0+m'}u,\xi),\\
&A_{0,0}+A_{0,1} \in C^0(0,T;S^{\varrho,\sigma}) + C_T^0(S_\ell^\varrho; H^{\lambda,\sigma,\mu-M_0-m'-\ell-1}),\\
&\|A_{0,1}\|_{C_T^0(S_\ell^\varrho;H^{\lambda,\sigma,\mu-M_0-m'-\ell-1})} \le C_\ell(1+r_1)
\end{aligned} \tag{3.17}$$

for sufficiently small T_0 and r_0; $F_0 = (0,\cdots,f,0,\cdots,f)$.

It is well known (e.g. see [7]) that one has to assume $\sigma \le \frac{1}{\varrho}$ to have existence and uniqueness of a local in time solution $V(t,\cdot) \in G^\sigma$ ($\sigma < \frac{1}{\varrho}$ for a global solution) of the problem (3.16). So the condition (G) in Theorem 1.1 $\sigma \le \frac{r}{r-\chi}$ appears in a natural way.

In a fixed point scheme, it is convenient to take derivatives ∂_x^β in (3.1) with $|\beta| \le M$ and $M > M_0$ in order to control the argument $D_*^{M_0+m'}u$ of A_0 in (3.16). So, for $v^{(\beta)} := \partial_x^\beta v$, we write

$$\begin{cases} Pv^{(\beta)} + [\partial_x^\beta, P]v = \partial_x^\beta f,\\ D_t^j v^{(\beta)}_{|t=0} = 0, \quad 0 \le j < m. \end{cases} \tag{3.18}$$

Then we define $V^{(\beta)} = (v_0^{(\beta)},\cdots,v_{2m-1}^{(\beta)})$ in the same way as V in (3.9),(3.10) with $v^{(\beta)}$ in place of v and set

$$W = (V^{(\beta)}; |\beta| \le M). \tag{3.19}$$

From Lemma 3.1 we have

$$[\partial_x^\beta, P]v = G(t,x,D_*^{M+m'}u)B(t,x,D_*^{M_0+m'}u,D_x)W$$

with $G \in C^0(0,T;G^{\sigma,1})$ and B of order $\frac{r\chi-\chi}{r} < \frac{r-\chi}{r} = \varrho$,

$$\begin{aligned}
&B(t,x,D_*^{M_0+m'}u,\xi) = B_0(t,x,\xi) + B_1(t,x,D_*^{M_0+m'}u,\xi),\\
&B_0+B_1 \in C^0(0,T;S^{\varrho,\sigma}) + C_T^0(S_\ell^\varrho; H^{\lambda,\sigma,\mu-M_0-m'-\ell-1}),\\
&\|B_1\|_{C_T^0(S_\ell^\varrho;H^{\lambda,\sigma,\mu-M_0-m'-\ell-1})} \le C_\ell(1+r_1)
\end{aligned} \tag{3.20}$$

for small T_0 and r_0.

Hence (3.18) is equivalent to the problem

$$\begin{cases} (D_t - K + A + GB)W = F,\\ W_{|t=0} = 0, \end{cases} \tag{3.21}$$

where $-K+A$ is block diagonal, $-K+A = (-K_0+A_0)\otimes\cdots\otimes(-K_0+A_0)$ with N factors, N the number of $\beta \in \mathbf{Z}_+^n$ such that $|\beta| \le M$. The entries of F are equal to $\partial_x^\beta f$ or to 0.

Our next aim is to prove an a priori estimate for the system

$$L = D_t - K + A + GB.$$

Let μ_0 be as in Proposition 2.3; we have

Proposition 3.1. *It is possible to choose $M > M_0$ sufficiently large, T_0 and r_0 sufficiently small and to find a positive h_0 such that if $\mu = M + m' + \mu_0$, $\lambda \ge h_0(1+r_1)$, then*

$$\|W(t)\|_{H^{\lambda(2T-t),\sigma,\mu_0}} \le \|W(0)\|_{H^{\lambda 2T,\sigma,\mu_0}} + 2\int_0^t \|LW(s)\|_{H^{\lambda(2T-s),\sigma,\mu_0}}ds \tag{3.22}$$

for every $W \in C_T^1(H^{\lambda,\sigma,\mu_0+1})$.

Proof. Let $K^*(t)$ denote the adjoint operator of $K(t)$ in H^{μ_0}. Then, the operator E defined by $E_\Lambda = K_\Lambda - K^*_{-\Lambda}$ is of order $\varrho \le \frac{1}{\sigma}$ as A and B.

By Propositions 2.1, 2.2, 2.3, the properties (3.17) and (3.20), we can fix $\ell \in \mathbf{Z}_+, M = M_0 + \ell + 1, \mu = M + m' + \mu_0$ and small r_0, T_0 in order to have continuous operators $E(t), A(t), B(t)$ from $H^{\lambda(2T-t),\sigma,\mu_0+1/\sigma}$ to $H^{\lambda(2T-t),\sigma,\mu_0}$ with norm controlled by a constant $C(1+r_1)$.

For the factor $G = G(t, x, D_*^{M+m'}u)$, we use the Banach algebra property (2.4),

$$\begin{aligned}\|GBW(t)\|_{H^{\lambda(2T-t),\sigma,\mu_0}} &\le \|G\|_{H^{\lambda(2T-t),\sigma,\mu_0}}\|BW(t)\|_{H^{\lambda(2T-t),\sigma,\mu_0}}\\ &\le C_0(1+r_0)(1+r_1)\|W(t)\|_{H^{\lambda(2T-t),\sigma,\mu_0+1/\sigma}}.\end{aligned}$$

Thus, denoting by $(U,V)_{H^{\lambda(2T-t),\sigma,\mu_0}}$ the scalar product in $H^{\lambda(2T-t),\sigma,\mu_0}$ we obtain for $S := i(L - D_t)$,

$$\mathrm{Re}(SW(t), W(t))_{H^{\lambda(2T-t),\sigma,\mu_0}} \ge -h_0(1+r_1)(\langle D_x\rangle^{1/\sigma}W(t), W(t))_{H^{\lambda(2T-t),\sigma,\mu_0}}.$$

Taking $\lambda \ge h_0(1+r_1)$, this gives

$$2\mathrm{Re}(iLW(t), W(t))_{H^{\lambda(2T-t),\sigma,\mu_0}} \ge \frac{\partial}{\partial t}\|W(t)\|^2_{H^{\lambda(2T-t),\sigma,\mu_0}},$$

which implies (3.22).

By the usual energy method, from the estimate (3.22) we get the following result of well posedness for the problem (3.21):

Theorem 3.1. *Let all conditions in Proposition* 3.1 *be fulfilled. Then the Cauchy problem* (3.21) *has a unique solution* $W \in C_T^1(H^{\lambda,\sigma,\mu_0})$. *Furthermore, the solution satisfies*

$$\|W(t)\|_{H^{\lambda(2T-t),\sigma,\mu_0}} \le 2\int_0^t \|F(s,\cdot,D_*^{M+m'}u(s,\cdot))\|_{H^{\lambda(2T-s),\sigma,\mu_0}}ds. \tag{3.23}$$

From Lemma 3.1 there is an operator $Q = Q(t, x, D_*^{M_0+m'}u, D_x)$ of order

$$-\delta = m' + 1 - m + \frac{\chi(r-1)}{r}$$

such that $D_*^{M+m'}v = QW$. So we obtain the main result of this section:

Theorem 3.2. *Under the hypotheses of Theorem* 3.1 *the Cauchy problem* (3.1) *has a unique solution* $v \in C_T^m(H^{\lambda,\sigma,\mu})$. *Furthermore, the solution satisfies*

$$\|v\|_{C_T^{m'}(H^{\lambda,\sigma,\mu+\delta})} \le C(1+r_1)\int_0^t \|f(s,\cdot,D^{m'}u(s,\cdot))\|_{H^{\lambda(2T-s),\sigma,\mu-m'}}ds,$$

$$\delta = m - m' - 1 - \frac{\chi(r-1)}{r}. \tag{3.24}$$

Condition (N) in Theorem 1.1 means $\delta \ge 0$. Precisely, we have $\delta = 0$ in the strictly hyperbolic case $r = 1$ allowing $m' = m - 1$; it is $\delta > 0$ if $m' \le m - 2$.

§4. Fixed Point

Let λ, μ, r_0, T_0 be as in Theorem 3.2 and take $T < \frac{T_0}{\lambda}$.

When $m' \le m - 2$, from Lemma 3.1 there is also an operator Q of order $1 - \delta$ such that $D_*^{M+m'+1}v = QW$. So we can replace $\|v\|_{C_T^{m'}(H^{\lambda,\sigma,\mu+\delta})}$ by $\|v\|_{C_T^{m'+1}(H^{\lambda,\sigma,\mu+\delta})}$ in (3.24) obtaining

$$\|v\|_{C_T^{m'+1}(H^{\lambda,\sigma,\mu+\delta})} \le CT(1+r_1)(1+r_0).$$

Since in this case it is $\delta > 0$, we fix $r_1 = r_0$, then we take T so small to have in Theorem 3.2 a compact map $v = S(u)$ from the ball

$$\{u \in C_T^{m'+1}(H^{\lambda,\sigma,\mu}); \|u\|_{C_T^{m'+1}(H^{\lambda,\sigma,\mu})} \le r_1\}$$

into itself. The fixed point of S is the unique solution of the nonlinear problem (1.1) in $C_T^{m'+1}(H^{\lambda,\sigma,\mu})$.

When $m' = m-1, r = 1$, it is $\delta = 0$ in (3.24). In this case Theorem 3.2 gives

$$\|v\|_{C_T^{m-1}(H^{\lambda,\sigma,\mu})} \le CT(1+r_1)(1+r_0)$$

and also

$$\|v\|_{C_T^{m}(H^{\lambda,\sigma,\mu})} \le C_0(1+r_0) + CT(1+r_1)(1+r_0)$$

estimating $\partial_t^m v$ from the equation $Pv = f$.

This time, we fix $r_1 = 2C_0(1+r_0)$ and take T so small to have a well defined map $v = S(u)$ from the set

$$\{u \in C_T^{m}(H^{\lambda,\sigma,\mu}); \|u\|_{C_T^{m}(H^{\lambda,\sigma,\mu})} \le r_1,\ \|u\|_{C_T^{m-1}(H^{\lambda,\sigma,\mu})} \le r_0\}$$

into itself. So, there is a subsequence of $u_0 = 0,\ \ u_{k+1} = S(u_k)$ converging in $C_T^m(H^{\lambda,\sigma,\mu-1})$ to a solution u of the problem (1.1). As usual, for small T, the estimate (3.24) gives also the uniqueness in any bounded subset of $C_T^m(H^{\lambda,\sigma,\mu-1})$ because the difference $u_1 - u_2$ of two solutions has to satisfy

$$\|u_1 - u_2\|_{C_T^{m-1}(H^{\lambda,\sigma,\mu-2})} \le CT\|u_1 - u_2\|_{C_T^{m-1}(H^{\lambda,\sigma,\mu-2})}.$$

Since $x \to f(t,x,w)$ has compact support, the same thing holds for the support of any solution $u \in C^m(0,T;G_A^\sigma)$ in view of the hyperbolicity of $P(t,x,w,D_t,D_x)$. Thus we have existence and uniqueness in $C^m(0,T;G_A^\sigma)$ from the inclusions (2.2) and (2.3) taking a smaller T if necessary.

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