Chin. Ann. of Math.
22B: 4(2001),471-474.

A NOTE ON THE CONSISTENCY OF LS ESTIMATES IN LINEAR MODELS**

CHEN XIRU*

Abstract

It is well known that when the random errors are iid. with finite variance, the week and the strong consistency of LS estimate of multiple regression coefficients are equivalent. This note, by constructing a counter-example, shows that this equivalence no longer holds true in case that the random errors possess only the r-th moment with $1 \le r < 2$.

Keywords Linear regression model, Consistency, LS estimate
2000 MR Subject Classification 60E05
Chinese Library Classification O211.3 **Document Code** B
Article ID 0252-9599(2000)01-0471-04

Consider the linear regression model

$$y_i = x_i'\beta + e_i, \qquad 1 \le i \le n, \quad n \ge 1, \tag{1}$$

where $x_1, x_2, \cdots$ are known p-vectors, β is the unknown p-dimensional vector of regression coefficients, $e_1, e_2, \cdots$ is a sequence of iid. random errors, and $y_1, y_2, \cdots$ are known observations of the dependent variable. Denote by F the common distribution of $e_1, e_2, \cdots$, and write

$$\Im_r = \Big\{F : \int_{\infty}^{\infty} x dF = 0, \qquad 0 < \int_{\infty}^{\infty} |x|^r dF < \infty\Big\}, \qquad 1 \le r \le 2.$$

The Least Squares (LS) estimate of β is

$$\hat{\beta}_n = \sum_{i=1}^{n} S_n^{-1} x_i y_i, \quad S_n = \sum_{i=1}^{n} x_i x_i'.$$

Here we tacitly assume that S_n^{-1} exists for large n. For simplicity of wording we write

$$\hat{\beta}_n \sim WC(F)$$

if $\hat{\beta}$ is a weakly consistent estimate of β for some specified F. If $\hat{\beta}_n \sim WC(F)$ for all $F \in \Im_r$, we write

$$\hat{\beta}_n \sim WC_r.$$

The meaning of the symbols $SC(F)$ and SC_r are defined similarly, only that weak consistency is replaced by strong consistency.

Manuscript received March 13, 2000.
*The Graduate School at Beijing, University of Science and Technology of China, Beijing 100039, China.
**Project supported by the National Natural Science Foundation of China (No.19631040).

A peculiar and somewhat unexpected property of $\hat{\beta}_n$ in the case $r = 2$ is that for any $F \in \Im_2$, $WC(F)$ and $SC(F)$ are equivalent (see [1, 2]). A question naturally arises whether this still holds true for $r < 2$. This note, by constructing a counter-example, gives the answer 'no' to this question.

We construct the counter-example for the case of $p = 1$. Let a be the largest integer not exceeding a . Define a function

$$G(T) = [T^{2-r}] + \sum_{i=1}^{[T^{2-r}]} [T^r/i], \quad T = 1, 2, \cdots, \quad 1 \le r < 2.$$

Find a subsequence $\mathfrak{m}_1 < \mathfrak{m}_2 < \cdots$ of positive integers such that

$$[\mathfrak{m}_j^r] \ge G(\mathfrak{m}_{j-1}), \quad j \ge 2, \tag{2}$$

$$\sum_{i=1}^{j-1} \mathfrak{m}_i^r < \mathfrak{m}_j^\zeta \text{ for some } \ \zeta \in (0,\, 2-r), \quad j \ge 2. \tag{3}$$

Since $0 < 2 - r \le 1$, from (3) we have

$$\sum_{i=1}^{j-1} \mathfrak{m}_i^{2-r} \le \sum_{i=1}^{j-1} \mathfrak{m}_i \le \sum_{i=1}^{j-1} \mathfrak{m}_i^r \le \mathfrak{m}_j^\zeta < \mathfrak{m}_j^{2-r} \le \mathfrak{m}_j. \tag{4}$$

For any given positive integer j, define a $G(\mathfrak{m}_j)$-dimensional vector λ_j as follows

$$\begin{aligned} \lambda_j &= (\lambda_j(1), \cdots, \lambda_j(G(\mathfrak{m}_j))) \\ &= (\mathfrak{m}_j^{-1}, \cdots, \mathfrak{m}_j^{-1},\, 1, \cdots, k\mathfrak{m}_j^{-1}, \cdots, k\mathfrak{m}_j^{-1}, \\ &\quad 1, \cdots, [\mathfrak{m}_j^{2-r}]\mathfrak{m}_j^{-1}, \cdots, [\mathfrak{m}_j^{2-r}]\mathfrak{m}_j^{-1},\, 1). \end{aligned}$$

There are $[\mathfrak{m}_j^r/k]$ components equal to $k\mathfrak{m}_j^{-1}$, followed by number 1, so there are

$$\sum_{k=1}^{[\mathfrak{m}_j^{2-r}]} [\mathfrak{m}_{j/k}^r] + \mathfrak{m}_j^{2-r}] = G(\mathfrak{m}_j)$$

components in the vector λ_j. Define

$$x_i = \lambda_j(i), \quad G(\mathfrak{m}_{j-1}) < i \le G(\mathfrak{m}_j), \quad j \ge 1 \quad (G(\mathfrak{m}_0) = 0) \tag{5}$$

and consider the model (1) in which x_i is defined by (5). We proceed to show that for this model, $\hat{\beta}_n$ is WC_r but not SC_r. This means that there exists $F \in \Im_r$ such that $\hat{\beta}_n \sim WC(F)$ but $\hat{\beta}_n \sim SC(F)$ is not true.

For convenience of reference we denote by $\{j, k\}$ the set of the subscripts of the consecutive components $k\mathfrak{m}_j^{-1}, \cdots, k\mathfrak{m}_j^{-1}, 1$ in λ_j. It is easy to verify that

(i)

$$\sum_{i \in \{j,k\}} \lambda_j^r(i) \approx k^{r-1};$$

$$\sum_{t=1}^{k} \sum_{i \in \{j,t\}} \lambda_j^r(i) \approx k^r.$$

Here and in the following $a_k \approx b_k$ means $a_k = O(b_k), \ \ b_k = O(a_k)$ as $k \to \infty$.

(ii)

$$1 \leq \sum_{i \in \{j,k\}} \lambda_j^r(i) \leq 2.$$

(iii) The first section $\{\mathfrak{m}_j^{-1}, \cdots, \mathfrak{m}_j^{-1}, 1\}$ of λ_j has a length greater than the length of the whole λ_{j-1}.

Take a positive integer n. Assume that $n \in \{j,k\}$. From (4) and (i),(iii), noticing that

$$(2-r)r \leq r, \quad (2-r)r \geq 2-r > \zeta,$$

we have

$$\sum_{i=1}^{n} x_i^r \approx \mathfrak{m}_1^{(2-r)r} + \cdots + \mathfrak{m}_{j-1}^{(2-r)r} + k^r \approx \mathfrak{m}_{j-1}^{(2-r)r} + k^r,$$

$$S_n = \sum_{i=1}^{n} x_i^2 \geq \mathfrak{m}_1^{2-r} + \cdots + \mathfrak{m}_{j-1}^{(2-r)r} + k \approx \mathfrak{m}_{j-1}^{(2-r)r} + k,$$

and

$$\sum_{i=1}^{n} x_i^r / S_n^r + O(1).$$

According to Theorem 1 of [3], we have $\hat{\beta}_n \sim WC_r$.

On the other hand, from (4) and (ii), we have

$$S_n \leq 2(\mathfrak{m}_1^{2-r} + \cdots + \mathfrak{m}_{j-1}^{2-r}) + 2k < 2\mathfrak{m}_j^{\zeta} + 2k.$$

Therefore

$$\begin{aligned}
&\#\{n : G(\mathfrak{m}_{j-1}) < n \leq G(\mathfrak{m}_j) : x_n/S_n > (4\mathfrak{m}_j)_1\} \\
&\geq \sum_{k=1}^{[\mathfrak{m}_j^{2-r}]} [\mathfrak{m}_j^r/k] I(k\mathfrak{m}_j^{-1}/(2\mathfrak{m}_j^{\zeta} + 2k) > (4\mathfrak{m}_j)^{-1}) \\
&\geq \mathfrak{m}_j^r \sum_{k=[\mathfrak{m}_j^{\zeta}]+1}^{[\mathfrak{m}_j^{2-r}]} k^{-1} \\
&\approx (2-r-\zeta)\mathfrak{m}_j^r \log \mathfrak{m}_j.
\end{aligned}$$

Since $2-r-\zeta > 0$ and $\mathfrak{m}_j \to \infty$, we have

$$\begin{aligned}
&\#\{n : x_n/S_n > (4\mathfrak{m}_j)^{-1}\}/(4\mathfrak{m}_j)^r \\
&\geq \#\{n : G(\mathfrak{m}_{j-1}) < n < G(\mathfrak{m}_j), \quad x_n/S_n > (4\mathfrak{m}_j)^{-1}\} \\
&\geq (2-r-\zeta)\mathfrak{m}_j^r \log \mathfrak{m}_j/(4\mathfrak{m}_j)^r \to \infty.
\end{aligned}$$

For general p a counter-example can be constructed on the basis of the above result. We only need to put

$$\begin{aligned}
&x_{ip+j} = (0, \cdots, 0, \ a_{i+1}, \ 0, \cdots, 0)', \\
&1 \leq j \leq p, \quad i = 0, 1, \cdots,
\end{aligned}$$

where a_i equals x_i defined by (5).

The above counter-example shows that there exists $F \in \Im_r (1 \le r < 2)$ such that for some specially chosen $\{x_i\}$,

$$\hat{\beta}_n \sim WC(F)$$

but $\hat{\beta}_n \sim SC(F)$ is false. Of course, this does not mean that for any $F \in \Im_r$, '$\hat{\beta}_n \sim WC(F)$' and '$\hat{\beta}_n \sim SC(F)$' must not be equivalent. Indeed, since $\Im_2 \subset \Im_r$, such an F (for which $\hat{\beta} \sim WC(F)$ and $\hat{\beta}_n \sim SC(F)$ are equivalent) certainly exists. An unsolved question is whether we can find such a distribution F that $F \in \Im_r$ but $F \notin \Im_r$ for any $r' > r$.

References

[1] Drygas, H., Weak and strong consistency of the least squares estimates in regression model [J], *Wahrsch. Verw. Gebiete,* **34**(1976), 119–127.

[2] Lai, T. L. Robbins, H. & Wei, C. Z., Strong consistency of the least squares-estimates in multiple regression II [J], *J. Multivariate anal.,* **9**(1979), 343–362..

[3] Chen, X., Consistency of LS estimates multiple regression under a lower order moment condition [J], *Science in China, Series A,* **38:**12(1995), 1420–1431.

[4] Jin, M. & Chen, X., Strong consistency of least squares estimate in multiple regression when the error variance is infinite [J], *Statistica Sinica,* **9:**1(1999), 289–296.