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BOUNDARY FEEDBACK STABILIZATION OF NONUNIFORM TIMOSHENKO BEAM WITH A TIPLOAD***

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Abstract

The boundary stabilization problem of a Timoshenko beam attached with a mass at one end is studied. First, with linear boundary force feedback and moment control simultaneously at the end attached with the load, the energy corresponding to the closed loop system is proven to be exponentially convergent to zero as time $t \to \infty$. Then, some counterexamples are given to show that, in other cases, the corresponding closed loop system is, in general, not stable asymtotically, let alone exponentially.

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§1. Introduction

The purpose of this paper is to study the stabilization problem of Timoshenko beam attached with a load of mass M at one end and forced by linear boundary feedback controls. The system to be investigated in this paper is described as follows (see [1] for example):

$$\begin{cases} \rho\ddot{w} + (K(\varphi - w'))' = 0, \quad 0 \le x \le \ell,\ t > 0, \\ I_\rho\ddot{\varphi} - (EI\varphi')' + K(\varphi - w') = 0, \quad 0 \le x \le \ell,\ t > 0, \\ w(0,t) = \varphi(0,t) = 0, \\ M\ddot{w}(\ell,t) - K(\ell)(\varphi(\ell,t) - w'(\ell,t)) = u_1(t), \\ EI(\ell)\varphi'(\ell,t) = u_2(t). \end{cases} \tag{1.1}$$

Here $u_1(t)$ and $u_2(t)$ are the boundary feedback controls of force and moment respectively, the meanings of all the other variables, functions and coefficients are the same as those described in related papers, say paper [1] for example. In this paper, we always assume that there exists a positive constant c_1 satisfying

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Condition S. $\rho, I_\rho, K, EI \in C^1[0,\ell],\ \ \rho, I_\rho, K, EI \geq c_1$.

Here and afterwards, the prime and the dot always denote derivatives with respect to space and time variables, respectively.

Up to now, a lot of results on various boundary feedback stabilization problems of Timoshenko beam equation have been turned out (see [1,3,4]). In paper [1], the authors proved that with both force and moment feedback controls applied to just one end of a Timoshenko beam, the energy corresponding to the closed loop system decays uniformly to zero as time $t \to \infty$. In this paper, we consider the asymptotic behavior of a tiploaded Timoshenko beam with linear boundary controls. As will be seen below, this type of controls, under some conditions, can stabilize the Timoshenko Beam exponentially, while under others, the corresponding closed loop system is even not asymptotically stable.

This paper is arranged as follow. In Section 2, by virtue of semigroup theory of linear operators (see [5, 6]), we prove the well-posedness of the corresponding closed loop system. In Section 3, by applying the frequency domain multiplier method used in [9], it is shown that the closed loop system (1.1) and (2.1) is exponentially stable if none of the feedback constants $\alpha,\ \beta,\ \mu$ in (2.1) are zero. Finally in Section 4, we derive some counterexamples to show that, in general, the closed loop system (1.1) and (2.1) can not even be stabilized asymptotically in other cases.

§2. The Well-Posedness of the Closed Loop System

For the system (1.1), we apply the following linear boundary feedbacks

$$\begin{cases} u_1(t) = -\alpha\dot{w}(\ell,t) + \beta(\dot{\varphi}(\ell,t) - \dot{w}'(\ell,t)), \\ u_2(t) = -\mu\dot{\varphi}(\ell,t), \end{cases} \tag{2.1}$$

with $\alpha,\beta,\mu \geq 0$. Set

$$\xi(t) = \begin{cases} \frac{M}{\beta}\dot{w}(\ell,t) - (\varphi(\ell,t) - w'(\ell,t)), & \text{if } \beta \neq 0, \\ \dot{w}(\ell,t), & \text{if } \beta = 0. \end{cases} \tag{2.2}$$

Then we have

$$\dot{\xi}(t) = g(t) \triangleq \begin{cases} -\frac{K(\ell)}{\beta}\xi + \Big(\frac{MK(\ell)}{\beta^2} - \frac{\alpha}{\beta}\Big)\dot{w}(\ell,t), & \text{if } \beta \neq 0, \\ -\frac{\alpha}{M}\dot{w}(\ell,t) + \frac{K(\ell)}{M}\big(\varphi(\ell,t) - w'(\ell,t)\big), & \text{if } \beta = 0. \end{cases} \tag{2.3}$$

Now the closed loop system (1.1) and (2.1) becomes

$$\begin{cases} \rho\ddot{w} + (K(\varphi - w'))' = 0, \quad 0 \leq x \leq \ell,\ t > 0, \\ I_\rho\ddot{\varphi} - (EI\varphi')' + K(\varphi - w') = 0, \quad 0 \leq x \leq \ell,\ t > 0, \\ \dot{\xi}(t) - g(t) = 0, \\ w(0,t) = \varphi(0,t) = EI(\ell)\varphi'(\ell,t) + \mu\dot{\varphi}(\ell,t) = 0. \end{cases} \tag{2.4}$$

To incorporate the above closed loop system into a certain function space, we define a product Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = V_0^1 \times L^2_\rho(0,\ell) \times V_0^1 \times L^2_{I_\rho}(0,\ell) \times \mathbb{R},$$

where

$$V_0^k = \{\varphi \in H^k(0,\ell) \mid \varphi(0) = 0\}, \quad k = 1,2,$$

and $H^k(0,\ell)$ is the usual Sobolev space of order k. The inner product in $\mathcal{H}$ is defined as follows: for $Y_k = [w_k, z_k, \varphi_k, \psi_k, \xi_k]^\tau \in \mathcal{H},\ k = 1, 2,$

$$\Big(Y_1, Y_2\Big)_{\mathcal{H}} = \int_0^\ell K(\varphi_1 - w_1')(\overline{\varphi}_2 - \overline{w}_2')\,dx + \int_0^\ell EI\varphi_1'\overline{\varphi}_2'\,dx + \int_0^\ell \rho z_1\overline{z}_2\,dx + \int_0^\ell I_\rho\psi_1\overline{\psi}_2\,dx + \tau\xi_1\overline{\xi}_2,$$

where $\tau = \gamma$ if $\beta \neq 0$ and $\tau = M$ if $\beta = 0$, and $\gamma = \frac{K(\ell)\beta^2}{K(\ell)M+\alpha\beta}$. We define a linear operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} w \\ z \\ \varphi \\ \psi \\ \xi \end{bmatrix} = \begin{bmatrix} z \\ -\frac{(K(\varphi-w'))'}{\rho} \\ \psi \\ \frac{(EI\varphi')'}{I_\rho} - \frac{K}{I_\rho}(\varphi - w') \\ g \end{bmatrix}, \quad \begin{bmatrix} w \\ z \\ \varphi \\ \psi \\ \xi \end{bmatrix} \in \mathcal{D}(\mathcal{A}),$$

$$\mathcal{D}(\mathcal{A}) = \Big\{[w, z, \varphi, \psi, \xi]^\tau \in \mathcal{H} \;\Big|\; w, \varphi \in V_0^2,\ z, \psi \in V_0^1,\ EI(\ell)\varphi'(\ell) + \mu\psi(\ell) = 0\Big\},$$

where

$$\xi = \begin{cases} \frac{M}{\beta}z(\ell) - \varphi(\ell) + w'(\ell), & \text{if } \beta \neq 0, \\ z(\ell), & \text{if } \beta = 0, \end{cases}$$

$$g = \begin{cases} -\frac{K(\ell)}{\beta}\xi + \Big(\frac{MK(\ell)}{\beta^2} - \frac{\alpha}{\beta}\Big)z(\ell), & \text{if } \beta \neq 0, \\ -\frac{\alpha}{M}z(\ell) + \frac{K(\ell)}{M}\big(\varphi(\ell) - w'(\ell)\big), & \text{if } \beta = 0. \end{cases}$$

Then we can write the closed loop system (2.4) as the following linear evolution equation in $\mathcal{H}$:

$$\frac{dY(t)}{dt} = \mathcal{A}Y(t), \tag{2.5}$$

where $Y(t) = [w(\cdot, t), \dot{w}(\cdot, t), \varphi(\cdot, t), \dot{\varphi}(\cdot, t), \xi(t)]^\tau$.

Theorem 2.1. *Let $\mathcal{A}$ be defined as above, then $\mathcal{A}$ generates a C_0 semigroup $T(t)$ of contraction in $\mathcal{H}$. Moreover, $\mathcal{A}$ has compact resolvent and $0 \in \rho(\mathcal{A})$.*

Proof. For any $Y = [w, z, \varphi, \psi, \xi]^\tau \in D(\mathcal{A})$, integrating by parts and referring to the boundary condition of $Y \in D(\mathcal{A})$, we have

$$\mathrm{Re}\,(\mathcal{A}Y, Y)_{\mathcal{H}} = \begin{cases} -K(\ell)\gamma\beta^{-1}|\varphi(\ell) - w'(\ell)|^2 - \alpha\gamma M\beta^{-2}|z(\ell)|^2 - \mu|\psi(\ell)|^2, & \text{if } \beta \neq 0, \\ -\mu|\psi(\ell)|^2 - \alpha|z(\ell)|^2, & \text{if } \beta = 0, \end{cases}$$

which implies the dissipativity of $\mathcal{A}$.

For the maximal dissipativity of $\mathcal{A}$, it is sufficient to show that $\forall \widetilde{Y} = [\widetilde{w}, \widetilde{z}, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{\xi}]^\tau \in \mathcal{H}$, there exists $Y = [w, z, \varphi, \psi, \xi]^\tau \in D(\mathcal{A})$, such that $\mathcal{A}Y = \widetilde{Y}$. This assertion, however, can be easily obtained via the direct calculation.

It is easy to show the compactness of the resolvent of $\mathcal{A}$ by using the Sobolev embedding theorem. Finally a direct calculation shows that $0 \in \rho(\mathcal{A})$. The proof is finished.

Thus according to the semigroup theory, we obtain:

Theorem 2.2. *For any $Y_0 \in \mathcal{H}$,* (2.5)*, and hence the closed loop system* (1.1) *and* (2.1)*, has a unique weak solution $Y(t) = T(t)Y_0$, where $T(t)$ is the linear semigroup of contraction generated by $\mathcal{A}$. Moreover, if $Y_0 \in \mathcal{D}(\mathcal{A})$, $Y(t) = T(t)Y_0$ becomes the strong solution to* (2.5).

§3. Exponential Decay of the Closed Loop System

We now discuss the exponential stability of the closed loop system (2.5) in the case of $\alpha\beta\mu \neq 0$. The energy corresponding to the solution to the closed loop system (2.5) is

$$E(t) = \frac{1}{2}\Big[\int_0^\ell EI|\varphi'|^2dx + \int_0^\ell K|\varphi - w'|^2dx + \int_0^\ell \rho|\dot{w}|^2dx + \int_0^\ell I_\rho|\dot{\varphi}|^2dx + \tau|\xi|^2\Big],$$

where $Y(t) = [w(\cdot,t), \dot{w}(\cdot,t), \varphi(\cdot,t), \dot{\varphi}(\cdot,t), \xi(t)]^\tau$ is the solution to (2.5). It is easy to check that in the case of $Y_0 \in \mathcal{D}(\mathcal{A})$ and $\alpha\beta\mu \neq 0$,

$$\dot{E}(t) = -K(\ell)\gamma\beta^{-1}|\varphi(\ell) - w'(\ell)|^2 - \alpha\gamma M\beta^{-2}|z(\ell)|^2 - \mu|\psi(\ell)|^2. \tag{3.1}$$

Let $\boldsymbol{i}\mathbb{R}$ denote the imaginary axis.

Lemma 3.1. $\boldsymbol{i}\mathbb{R} \subset \rho(\mathcal{A})$, *the resolvent set of* $\mathcal{A}$.

Proof. Since $\mathcal{A}$ has compact resolvent, it is sufficient to prove $\boldsymbol{i}\mathbb{R} \cap \sigma_p(A) = \emptyset$. Assuming the contrary, then there exists an eigenvalue $\boldsymbol{i}\lambda \in \boldsymbol{i}\mathbb{R}$ of $\mathcal{A}$. Obviously $\lambda \neq 0$. Let $\Psi = [w, z, \varphi, \psi, \xi]^\tau \in D(\mathcal{A})$ be an eigenfunction of $\mathcal{A}$ corresponding to $\boldsymbol{i}\lambda$. From the assumption that $\alpha,\ \beta,\ \mu > 0$ and the fact that

$$\mathrm{Re}\,(\mathcal{A}\Psi, \Psi)_{\mathcal{H}} = -K(\ell)\gamma\beta^{-1}|\varphi(\ell) - w'(\ell)|^2 - \alpha\gamma M\beta^{-2}|z(\ell)|^2 - \mu|\psi(\ell)|^2 = 0,$$

it follows that w and φ satisfy

$$\begin{cases} (K(w' - \varphi))' + \lambda^2\rho w = 0, \\ (EI\varphi')' - K(\varphi - w') + \lambda^2 I_\rho\varphi = 0, \\ w(\ell) = \varphi(\ell) = w'(\ell) = \varphi'(\ell) = 0, \\ \xi - \frac{M}{\beta}z(\ell) + \varphi(\ell) - w'(\ell) = 0. \end{cases} \tag{3.2}$$

Thus, according to the general theory of ordinary differential equations, we get $w = \varphi = \xi = 0$, and hence $\Psi = 0$, a contradiction.

Theorem 3.1. *Suppose that Condition S holds. Then in the case of* $\alpha\beta\mu > 0$*, the energy corresponding to the closed loop system* (2.5) *decays exponentially, that is, for every* $Y_0 \in \mathcal{H}$*, there exist positive constants* M, ω*, independent of* Y_0*, such that* $E(t) \leq Me^{-\omega t}\|Y_0\|^2$.

Proof. From Lemmas 2.1 and 3.1, and according to [7], we need only to prove that

$$\lim_{\lambda\in\mathbb{R},\ |\lambda|\to+\infty} \|(\boldsymbol{i}\lambda - \mathcal{A})^{-1}\| < +\infty.$$

Assuming the contrary, then there must be $\lambda_n \in \boldsymbol{i}\mathbb{R}$ and $Z_n = [w_n, z_n, \varphi_n, \psi_n, \xi_n]^\tau \in D(\mathcal{A}),\ n = 1, 2, \cdots$ such that

$$\|Z_n\|_{\mathcal{H}} = 1, \quad |\lambda_n| \to +\infty \text{ (as } n \to \infty),$$

$$(\lambda_n - \mathcal{A})Z_n = \widetilde{Z}_n \overset{\triangle}{=} (\widetilde{w}_n, \widetilde{z}_n, \widetilde{\varphi}_n, \widetilde{\psi}_n, \widetilde{\xi}_n)^\tau \to 0 \text{ (as } n \to \infty),$$

which means that as $n \to \infty$,

$$\int_0^\ell K|(\lambda_n(\varphi_n - w_n') - (\psi_n - z_n'))|^2dx + \int_0^\ell EI|(\lambda_n\varphi_n' - \psi_n')|^2dx = o(1), \tag{3.3}$$

$$\int_0^\ell |\lambda_n\rho z_n + (K(\varphi_n - w_n'))'|^2dx + \int_0^\ell |\lambda_n I_\rho\psi_n - (EI\varphi_n')' + K(\varphi_n - w_n')|^2dx = o(1), \tag{3.4}$$

$$\mathrm{Re}\,((\lambda_n - \mathcal{A})Z_n, Z_n)_{\mathcal{H}} = \mu|\psi_n(\ell)|^2 + \frac{\alpha\gamma M}{\beta^2}|z_n(\ell)|^2 + \frac{K\gamma}{\beta}|\varphi_n(\ell) - w_n'(\ell))|^2dx = o(1), \tag{3.5}$$

$$\Big(\lambda_n+\frac{K(\ell)}{\beta}\Big)\xi_n-\Big(\frac{MK(\ell)}{\beta^2}-\frac{\alpha}{\beta}\Big)z_n(\ell)=o(1). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$|\lambda_n\xi_n|,\ |\xi_n|,\ |z_n(\ell)|,\ |\psi_n(\ell)|,\ |\varphi_n(\ell)-w_n'(\ell)|=o(1). \tag{3.7}$$

From the definitions of $\mathcal{A}$ and $\widetilde{Z}_n$, we have

$$\begin{aligned}&\lambda_n\Big(\int_0^\ell K|\varphi_n-w_n'|^2dx+\int_0^\ell EI|\varphi_n'|^2dx\Big)\\&-\int_0^\ell K(\psi_n-z_n')(\overline{\varphi}_n-\overline{w}_n')dx-\int_0^\ell EI\psi_n'\overline{\varphi}_n'dx\\=&\int_0^\ell K(\widetilde{\varphi}_n-\widetilde{w}_n')(\overline{\varphi}_n-\overline{w}_n')dx+\int_0^\ell EI\widetilde{\varphi}_n'\overline{\varphi}_n'dx.\end{aligned}$$

Using Hölder inequality and the fact of $\|Z_n\|_{\mathcal{H}}=1$ and $\|\widetilde{Z}_n\|_{\mathcal{H}}\to 0$ as $n\to\infty$, we obtain

$$\begin{aligned}&\lambda_n\Big(\int_0^\ell K|\varphi_n-w_n'|^2dx+\int_0^\ell EI|\varphi_n'|^2dx\Big)\\&-\int_0^\ell K(\psi_n-z_n')(\overline{\varphi}_n-\overline{w}_n')dx-\int_0^\ell EI\psi_n'\overline{\varphi}_n'dx=o(1).\end{aligned} \tag{3.8}$$

Referring to the definition of $\mathcal{A}$ and $\widetilde{Z}_n$, and integrating by parts, we get

$$\begin{aligned}&\lambda_n\Big(\int_0^\ell(\rho|z_n|^2+I_\rho|\psi_n|^2)dx\Big)+\int_0^\ell K(\overline{\psi}_n-\overline{z}_n')(\varphi_n-w_n')dx+\int_0^\ell EI\overline{\psi}_n'\varphi_n'dx\\=&\int_0^\ell\Big(\rho\lambda_nz_n+(K(\varphi_n-w_n'))'\Big)\overline{z}_ndx+\int_0^\ell\Big(I_\rho\lambda_n\psi_n-(EI\varphi_n')'+K(\varphi_n-w_n')\Big)\overline{\psi}_ndx\\&+EI(\ell)\varphi_n'(\ell)\overline{\psi}_n(\ell)-K(\ell)(\varphi_n(\ell)-w_n'(\ell))\overline{z}_n(\ell)\\=&\int_0^\ell\rho\widetilde{z}_n\overline{z}_ndx+\int_0^\ell I_\rho\widetilde{\psi}_n\overline{\psi}_ndx-\mu|\psi_n(\ell)|^2+K(\ell)\Big(\xi_n-\frac{M}{\beta}z_n(\ell)\Big)\overline{z}_n(\ell).\end{aligned}$$

By the same argument as in proving (3.8) and by using (3.7), it follows that

$$\lambda_n\Big(\int_0^\ell(\rho|z_n|^2+I_\rho|\psi_n|^2)dx\Big)+\int_0^\ell K(\overline{\psi}_n-\overline{z}_n')(\varphi_n-w_n')dx+\int_0^\ell EI\overline{\psi}_n'\varphi_n'dx=o(1). \tag{3.9}$$

Hence from (3.8) and (3.9) we get

$$\lambda_n\Big(\int_0^\ell K|\varphi_n-w_n'|^2dx+\int_0^\ell EI|\varphi_n'|^2dx-\int_0^\ell(\rho|z_n|^2+I_\rho|\psi_n|^2)dx\Big)=o(1), \tag{3.10}$$

$$\int_0^\ell K|\varphi_n-w_n')|^2dx+\int_0^\ell EI|\varphi_n'|^2dx-\int_0^\ell(\rho|z_n|^2+I_\rho|\psi_n|^2)dx=o(1). \tag{3.11}$$

Consequently, using (3.3) and the fact that

$$\|Z_n\|=1,\quad \|\widetilde{Z}_n\|=o(1),\quad \lambda_nw_n=z_n+\widetilde{w}_n,\quad \lambda_n\varphi_n=\psi_n+\widetilde{\varphi}_n$$

and $\xi_n=o(1)$, we obtain

$$\begin{cases}\int_0^\ell K|\varphi_n-w_n'|^2dx+\int_0^\ell EI|\varphi_n'|^2dx-\frac12=o(1),\\ \int_0^\ell(\rho|z_n|^2+I_\rho|\psi_n|^2)dx-\frac12=o(1),\\ \int_0^\ell|\lambda_n|^2(\rho|w_n|^2+I_\rho|\varphi_n|^2)dx-\frac12=o(1).\end{cases} \tag{3.12}$$

We now prove that

$$\lambda_n w_n(\ell),\ \lambda_n \varphi_n(\ell) = o(1), \tag{3.13}$$

$$w_n'(\ell),\ \varphi_n'(\ell) = o(1), \tag{3.14}$$

$$w_n(\cdot),\ \varphi_n(\cdot) = o(1) \ \text{ in } L^2(0,\ell). \tag{3.15}$$

First we show (3.13). From (3.3), (3.7) and Condition S, we have

$$\int_0^\ell EI|\lambda_n \varphi_n' - \psi_n'|^2 dx = o(1), \quad \psi_n(\ell) = o(1),$$

and then by using Hölder inequality it follows that

$$\lambda_n \varphi_n(\ell) = \int_0^\ell (\lambda_n \varphi_n' - \psi_n') dx + \psi_n(\ell) = o(1).$$

Similarly, using (3.3), (3.7) and Hölder inequality, we get

$$\begin{aligned}\lambda_n w_n(\ell) &= \int_0^\ell \Big(\lambda_n(w_n' - z_n') - (\lambda_n \varphi_n - \psi_n)\Big) dx + \int_0^\ell (\lambda_n \varphi_n - \psi_n) dx + z_n(\ell) \\ &= \int_0^\ell \int_0^x (\lambda_n \varphi_n' - \psi_n') ds dx + o(1) = o(1).\end{aligned}$$

Next we prove (4.14). From (3.7) and (3.13), we obtain

$$w_n'(\ell) = w_n'(\ell) - \varphi_n(\ell) + \varphi_n(\ell) = o(1), \quad \varphi_n'(\ell) = -\mu EI(\ell)^{-1} \psi_n(\ell) = o(1).$$

Finally (3.15) follows directly from the third assertion of (3.12).

Based on the above estimates, we now show that

$$\lim_{n\to\infty} \int_0^\ell (|\lambda_n w_n|^2 + |\lambda_n \varphi_n|^2) = 0, \tag{3.16}$$

which contradicts (3.12), and hence the proof of the theorem will be finished.

We have

$$\lambda_n^2 w_n - \frac{1}{\rho}(K(w_n' - \varphi_n))' = \widetilde{z}_n + \lambda_n \widetilde{w}_n, \tag{3.17}$$

$$\lambda_n^2 \varphi_n - \frac{1}{I_\rho}[(EI\varphi_n')' - K(\varphi_n - w_n')] = \widetilde{\psi}_n + \lambda_n \widetilde{\varphi}_n. \tag{3.18}$$

Multiplying both sides of (3.17) and (3.18) by $\rho(e^{\eta x} - 1)\overline{w}_n'$ and $I_\rho(e^{\eta x} - 1)\overline{\varphi}_n'$ respectively, then integrating from 0 to ℓ and adding, we get

$$\begin{aligned}&\mathrm{Re}\Big(\int_0^\ell \lambda_n^2 \rho(e^{\eta x} - 1) w_n \overline{w}_n' dx + \int_0^\ell \lambda_n^2 I_\rho (e^{\eta x} - 1)\varphi_n \overline{\varphi}_n' dx \\ &- \int_0^\ell (e^{\eta x} - 1)(K(w_n' - \varphi_n))' \overline{w}_n' dx - \int_0^\ell (e^{\eta x} - 1)\Big((EI\varphi_n')' - K(\varphi_n - w_n')\Big)\overline{\varphi}_n' dx\Big) \\ &= \mathrm{Re}\Big((\widetilde{z}_n + \lambda_n \widetilde{w}_n, \widetilde{\psi}_n + \lambda_n \widetilde{\varphi}_n), (e^{\eta x} - 1)(w_n', \varphi_n')\Big)_{L^2_\rho \times L^2_{I_\rho}} \triangleq \Delta_n,\end{aligned} \tag{3.19}$$

where η is a positive constant to be determined. It is not difficult to check that

$$\Delta_n = o(1) \ (\text{as } n \to \infty).$$

Referring to (3.13) and integrating by parts, we obtain

$$\mathrm{Re}\int_0^\ell \lambda_n^2\rho(e^{\eta x}-1)w_n\overline{w}_n' dx = \frac{1}{2}\int_0^\ell \Big(\eta e^{\eta x}\rho + (e^{\eta x}-1)\rho'\Big)|\lambda_n w_n|^2 dx + o(1), \tag{3.20}$$

$$\mathrm{Re}\int_0^\ell \lambda_n^2 I_\rho(e^{\eta x}-1)\varphi_n\overline{\varphi}_n' dx = \frac{1}{2}\int_0^\ell \Big(\eta e^{\eta x}I_\rho + (e^{\eta x}-1)I_\rho'\Big)|\lambda_n \varphi_n|^2 dx + o(1). \tag{3.21}$$

From (3.3), (3.14) and (3.15), it follows that

$$\begin{aligned}
&\mathrm{Re}\int_0^\ell (e^{\eta x}-1)(K(w_n'-\varphi_n))'\overline{w}_n' dx\\
&= -\mathrm{Re}\int_0^\ell \eta e^{\eta x}K(w_n'-\varphi_n)\overline{w}_n' dx - \mathrm{Re}\int_0^\ell (e^{\eta x}-1)K(w_n'-\varphi_n)\overline{w}_n'' dx + o(1)\\
&= -\int_0^\ell \eta e^{\eta x}K|w_n'|^2 dx - \mathrm{Re}\int_0^\ell (e^{\eta x}-1)Kw_n'\overline{w}_n'' dx\\
&\quad + \mathrm{Re}\int_0^\ell (e^{\eta x}-1)K\varphi_n\overline{w}_n'' dx + o(1),
\end{aligned} \tag{3.22}$$

$$\mathrm{Re}\int_0^\ell (e^{\eta x}-1)Kw_n'\overline{w}_n'' dx = -\frac{1}{2}\int_0^\ell [\eta e^{\eta x}K + (e^{\eta x}-1)K']|w_n'|^2 dx + o(1), \tag{3.23}$$

$$\mathrm{Re}\int_0^\ell (e^{\eta x}-1)K\varphi_n\overline{w}_n'' dx = -\mathrm{Re}\int_0^\ell (e^{\eta x}-1)\varphi_n'\overline{w}_n' dx + o(1). \tag{3.24}$$

Combining (3.22), (3.23) and (3.24), we deduce

$$\begin{aligned}
&\mathrm{Re}\int_0^\ell (e^{\eta x}-1)(K(w_n'-\varphi))'\overline{w}_n' dx\\
&= -\frac{1}{2}\int_0^\ell \Big(\eta e^{\eta x}K - (e^{\eta x}-1)K'\Big)|w_n'|^2 dx\\
&\quad - \mathrm{Re}\int_0^\ell (e^{\eta x}-1)K\varphi_n'\overline{w}_n' dx + o(1).
\end{aligned} \tag{3.25}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Re}\int_0^\ell (e^{\eta x}-1)[(EI\varphi_n')' - K(\varphi - w_n')]\overline{\varphi}_n' dx\\
&= -\frac{1}{2}\int_0^\ell [\eta e^{\eta x}EI - (e^{\eta x}-1)EI']|\varphi_n'|^2 dx\\
&\quad + \mathrm{Re}\int_0^\ell (e^{\eta x}-1)Kw_n'\overline{\varphi}_n' dx + o(1).
\end{aligned} \tag{3.26}$$

From (3.19), (3.20), (3.25) and (3.26), we obtain

$$\begin{aligned}
&\int_0^\ell \Big(\eta\rho e^{\eta x} + (e^{\eta x}-1)\rho'\Big)|\lambda_n w_n|^2 dx + \int_0^\ell \Big(\eta e^{\eta x}I_\rho + (e^{\eta x}-1)I_\rho'\Big)|\lambda_n\varphi_n|^2 dx\\
&+ \int_0^\ell \Big(\eta e^{\eta x}K - (e^{\eta x}-1)K'\Big)|w_n'|^2 dx + \int_0^\ell \Big(\eta e^{\eta x}EI - (e^{\eta x}-1)EI'\Big)|\varphi_n'|^2 dx\\
&= o(1) \quad (\text{as } n\to\infty).
\end{aligned} \tag{3.27}$$

Therefore, the assertion (3.16) follows from (3.27). The proof is then complete.

§4. Asymptotic Behavior of the Closed Loop System

Throughout this section, we assume that all ρ, I_ρ, EI, K are positive constants. We now discuss the case of $\alpha\beta\mu = 0$, and prove that the closed loop system (2.5), in general, is not asymptotically stable. The main result in this section is the following

Theorem 4.1. *Assume that $\alpha\mu = 0$. Then there exist some constants ρ_0, K_0, I_{ρ_0} and EI_0 such that the corresponding closed loop system* (2.5) *with the coefficients ρ_0, K_0, I_{ρ_0} and EI_0 is not asymptotically stable.*

Proof. Since the resolvent of $\mathcal{A}$ is compact, according to [8], for the closed loop system (2.5) to be not asymptotically stable it is necessary and sufficient that there exists $\omega \in \mathbb{R}$ such that $\boldsymbol{i}\omega \in \sigma_p(\mathcal{A})$, the point spectrum of $\mathcal{A}$.

(1) $\alpha\beta \neq 0,\ \mu = 0$. Let $\mathcal{A}\Psi = \boldsymbol{i}\omega\Psi$ with $\Psi \neq 0$ and

$$\Psi = [w, z, \varphi, \psi, \xi]^\tau \in \mathcal{D}(\mathcal{A}).$$

It is obvious that $\omega \neq 0$. From $\mathrm{Re}\,(\mathcal{A}\Psi, \Psi)_{\mathcal{H}} = 0$, we get

$$z(\ell) = \varphi(\ell) - w'(\ell) = 0.$$

Then w and φ satisfy

$$\begin{cases} \frac{K}{\rho}(w'' - \varphi') + \omega^2 z = 0, \\ \frac{EI}{I_\rho}\varphi'' - \frac{K}{I_\rho}(\varphi - w') + \omega^2\psi = 0, \\ w(0) = \varphi(0) = w(\ell) = \varphi'(\ell) = \varphi(\ell) - w'(\ell) = 0, \end{cases} \tag{4.1}$$

$$\frac{K}{\beta}\xi + \boldsymbol{i}\omega\xi = \Big(\frac{MK}{\beta^2} - \frac{\alpha}{\beta}\Big)z(\ell).$$

(4.1) can be rewritten as

$$\begin{cases} Z' = \widetilde{A}Z, \\ B_1 Z(0) = 0, \\ B_2 Z(\ell) = 0, \end{cases} \tag{4.2}$$

where

$$Z(x) = [w(x), w'(x), \varphi(x), \varphi'(x)]^\tau, \quad \widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\rho_1^2\omega^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -c & c - \rho_2^2\omega^2 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix},$$

$$\rho_1 = \sqrt{\rho/K}, \quad \rho_2 = \sqrt{I_\rho/EI}, \quad a = \rho_1^2\omega^2, \quad b = \rho_2^2\omega^2, \quad c = K/EI.$$

Let $Z = PZ_1$, with

$$P = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 \\ \alpha_1^2 & -\alpha_1^2 & \alpha_2^2 & -\alpha_2^2 \\ a\beta_1 & -a\beta_1 & a\beta_2 & -a\beta_2 \\ a\alpha_1\beta_1 & a\alpha_1\beta_1 & a\alpha_2\beta_2 & a\alpha_2\beta_2 \end{bmatrix},$$

$$\alpha_1 = \boldsymbol{i}\sqrt{\frac{a+b+\sqrt{(a-b)^2+4ac}}{2}}, \tag{4.3}$$

$$\alpha_2 = \begin{cases} \boldsymbol{i}\sqrt{\frac{a+b-\sqrt{(a-b)^2+4ac}}{2}}, & \text{if } b > c, \\ \sqrt{\frac{\sqrt{(a-b)^2+4ac}-(a+b)}{2}}, & \text{if } b < c, \end{cases} \tag{4.4}$$

$$\beta_1 = 1 + \alpha_1^2/a = \frac{(a-b) - \sqrt{(a-b)^2+4ac}}{2a} < 0, \tag{4.5}$$

$$\beta_2 = 1 + \alpha_2^2/a = \frac{(a-b) + \sqrt{(a-b)^2+4ac}}{2a} > 0. \tag{4.6}$$

Then the first equation of (4.2) becomes $Z_1' = \widehat{A}Z_1$, where

$$\widehat{A} \triangleq P^{-1}\widetilde{A}P = \operatorname{diag}\{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2\}.$$

The general solution to this equations can be written as

$$Z_1(x) = \operatorname{diag}\{e^{\alpha_1 x}, e^{-\alpha_1 x}, e^{\alpha_2 x}, e^{-\alpha_2 x}\}\Theta$$

where Θ is a 4×1 constant vector. Denote

$$Q_1 \triangleq \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 \\ a\beta_1 & -a\beta_1 & a\beta_2 & -a\beta_2 \\ \alpha_1 e^{\alpha_1 \ell} & \alpha_1 e^{-\alpha_1 \ell} & \alpha_2 e^{\alpha_2 \ell} & \alpha_2 e^{-\alpha_2 \ell} \\ a\alpha_1\beta_1 e^{\alpha_1 \ell} & a\alpha_1\beta_1 e^{-\alpha_1 \ell} & a\alpha_2\beta_2 e^{\alpha_2 \ell} & a\alpha_2\beta_2 e^{-\alpha_2 \ell} \\ (a\beta_1 - \alpha_1^2)e^{\alpha_1 \ell} & (\alpha_1^2 - a\beta_1)e^{-\alpha_1 \ell} & (a\beta_2 - \alpha_2^2)e^{\alpha_2 \ell} & (\alpha_2^2 - a\beta_2)e^{-\alpha_2 \ell} \end{bmatrix}. \tag{4.7}$$

Then (4.2) has nontrivial solution if and only if

$$\operatorname{rank}(Q_1) = 2 + \operatorname{rank}\left(\begin{bmatrix} 1 & 1 \\ \alpha_1 \sinh \ell\alpha_1 & \alpha_2 \sinh \ell\alpha_2 \\ \beta_1 \cosh \alpha_1 & \beta_2 \cosh \ell\alpha_2 \end{bmatrix}\right) < 4, \tag{4.8}$$

or equivalently

$$\begin{cases} \alpha_1 \sinh \alpha_1 \ell = \alpha_2 \sinh \alpha_2 \ell, \\ \beta_1 \cosh \alpha_1 \ell = \beta_2 \cosh \alpha_2 \ell. \end{cases} \tag{4.9}$$

It is easy to see that there exist two positive numbers u_0, v_0 satisfying

$$u_0 \sin u_0 = v_0 \sin v_0, \quad 0 < v_0 < \frac{\pi}{2} < u_0 < \pi.$$

Set $\alpha_1 = \boldsymbol{i}u_0,\ \alpha_2 = \boldsymbol{i}v_0$, and

$$a_0 = \frac{u_0^2 + v_0^2}{2 - \beta_1 - \beta_2}, \quad b_0 = (1 - \beta_1 - \beta_2)a_0, \quad c_0 = -\beta_1\beta_2 a_0$$

with

$$\beta_1 = \frac{(v_0^2 - u_0^2)\cos v_0}{v_0^2 \cos v_0 - u_0^2 \cos u_0}, \quad \beta_2 = \frac{(v_0^2 - u_0^2)\cos u_0}{v_0^2 \cos v_0 - u_0^2 \cos u_0}.$$

We can easily check that

$$\beta_1 < 0 < \beta_2 < 1,, \quad (1 - \beta_2)u_0^2 = (1 - \beta_1)v_0^2, \quad a_0 > b_0 > c_0 > 0,$$

and (4.3)–(4.6), (4.9) hold.

From the definitions of a, b and c, it is trivial to find $\rho_0,\ K_0,\ I_{\rho_0}, EI_0$ and ω_0, such that $\boldsymbol{i}\omega_0 \in \sigma_\rho(\mathcal{A})$ for the system (1.1) with coefficients $\rho_0,\ K_0,\ I_{\rho_0}$ and EI_0.

(2) $\mu\beta \neq 0,\ \alpha = 0$. By the similar discussion, it follows that the closed loop system (2.5) does not decay asymptotically if and only if

$$\begin{cases} \alpha_1 \sinh \alpha_2 \ell = \alpha_2 \sinh \alpha_1 \ell, \\ \beta_1 \cosh \alpha_2 \ell = \beta_2 \cosh \alpha_1 \ell \end{cases} \tag{4.10}$$

has positive solution (a_0, b_0, c_0). But the solvability of (4.10) can be proven by the similar argument as above.

As for other cases of $\alpha\mu = 0$, it is trivial to prove the desired conclusion. The proof is then complete.

Theorem 4.2. *Assume that $\alpha, \mu > 0$ and $\beta \neq 0$. Then for all ρ, K, I_ρ and EI satisfying Condition S, the energy of the loop system (2.5) decays asymptotically, but not exponentially.*

The proof of the first assertion of Theorem 4.2 is similar to that of Lemma 3.1. Since the control operator is compact, it follow from [4] that the closed loop system can not be exponentially stable.

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