Chin. Ann. of Math.
22B: 4(2001),495-502.

ON UNIFORM ASYMPTOTIC STABILITY OF INFINITE DELAY DIFFERENCE EQUATIONS**

ZHANG Shunian*

Abstract

For the infinite delay difference equations of the general form, two new uniform asymptotic stability criteria are established in terms of the discrete Liapunov functionals.

Keywords Infinite delay difference equations, Uniform asymptotic stability, g-uniform asymptotic stability, Discrete Liapunov functionals
2000 MR Subject Classification 39A11
Chinese Library Classification O175.7 **Document Code** A
Article ID 0252-9599(2001)04-0495-08

§1. Introduction

The aim of this paper is to establish the stability criteria for the infinite delay difference equations of the form

$$x(n+1) = F(n, x_n) \qquad \text{for} \quad n \in Z, \tag{1.1}$$

where $F: Z \times \mathcal{C}_H \to R^k$, Z denotes the integer set, R^k is the n-dimensional Euclidean space, $\mathcal{C}_H = \{\varphi \in \mathcal{C} : \|\varphi\| < H\}$ for some constant $H > 0$, while

$$\mathcal{C} = \{\varphi : \{\dots, -2, -1, 0\} \to R^k \mid \varphi \text{ is bounded}\}$$

with

$$\|\varphi\| = \sup_{s \le 0} |\varphi(s)| \quad \text{for} \quad \varphi \in \mathcal{C},$$

and $x_n(s) = x(n+s)$ for $s \le 0$. Here, and in the sequel, $|\cdot|$ is a norm in R^k, and we always assume the variables n, i, j, and s take integer values and the corresponding intervals and inequalities are discrete ones.

As usual, we assume that $F(n, 0) \equiv 0$ so that (1.1) has the zero solution $x(n) \equiv 0$. Also, we assume that for any given integer $n_0 \in Z$ and a given initial function $\varphi \in \mathcal{C}_H$, there is a unique solution of (1.1) defined for all $n \ge n_0$, denoted by $x(n_0, \varphi)(n)$, such that it satisfies (1.1) for all $n \ge n_0$ and

$$x_{n_0}(n_0, \varphi)(s) = \varphi(s) \quad \text{for} \quad s \le 0.$$

Manuscript received March 6, 2000. Revised October 17, 2000.
*Department of Applied Mathematics, Shanghai Jiaotong University, Shanghai 200030, China.
E-mail: snzhang@online.sh.cn
**Project supported by the National Natural Science Foundation of China (No. 19831030).

For our purpose, we first introduce the following definitions (cf. [1,2]).

Definition 1.1. *The zero solution of* (1.1) *is uniformly stable* (US) *if for each* $\varepsilon > 0$ *and any* $n_0 \in Z$*, there exists a* $\delta > 0$ *independent of* n_0 *such that if* $\|\varphi\| < \delta$*, then*

$$|x(n_0, \varphi)(n)| < \varepsilon \quad \text{for all } n \geq n_0.$$

Definition 1.2. *The zero solution of* (1.1) *is uniformly asymptotically stable* (UAS) *if it is* US *and there is a* $\delta_0 > 0$ *such that for each* $\gamma > 0$*, there exists an integer* $N(\gamma) > 0$ *independent of* n_0 *such that if* $\|\varphi\| < \delta_0$*, then*

$$|x(n_0, \varphi)(n)| < \varepsilon \quad \text{for all } n \geq n_0 + N(\gamma).$$

Definition 1.3. *A continuous, strictly increasing function* $W : [0, \infty) \to [0, \infty)$ *with* $W(0) = 0$ *is called a wedge function.* (*We denote the wedge functions in the sequel by* W *or* W_i*, where* i *is an integer.*)

Definition 1.4. *A semi-norm* $|\cdot|_s$ *on* $\mathcal{C}$ *is said to have a fading memory with respect to the norm* $\|\cdot\|$ *on* $\mathcal{C}$ *if* $|\varphi|_s \leq \|\varphi\|$ *for all* $\varphi \in \mathcal{C}$ *and if for any given* $\varepsilon > 0$ *and* $D > 0$ *there exists an integer* $h_0 > 0$ *such that*

$$|\varphi|_s \leq \max\{\|\varphi(\cdot)\|^{[-h,0]}, \varepsilon\}$$

whenever an integer $h \geq h_0$ *and* $\|\varphi(\cdot)\|^{(-\infty,-h]} \leq D$*, where we define for any two integers* a*,* b *with* $a < b$ (a *may be* $-\infty$) *that*

$$\|\varphi(\cdot)\|^{[a,b]} = \max_{a \leq s \leq b} |\varphi(s)|.$$

More generally, rather than $\mathcal{C}$ we can define the state space (or called the phase space) as follows (cf. [3, 4]).

Let $G = G^0 \cup \{g_0\}$, where $g_0(s) = 1$ for all $s \in Z^-$, and

$$G^0 = \{g : Z^- \to [1, +\infty) \mid g \text{ is nonincreasing, } g(0) = 1, \text{ and } g(s) \to +\infty \text{ as } s \to -\infty\},$$

where Z^- denotes the set of all non-positive integers.

For any given $g \in G$, we define the state space

$$C_g = \{\varphi : Z^- \to R^k \mid |\varphi|_g < +\infty\},$$

where $|\varphi|_g = \sup\limits_{s \leq 0} |\varphi(s)|/g(s)$. Then it is easy to see that $(C_g, |\cdot|_g)$ is a Banach space. Trivially, $C_{g_0} = \mathcal{C}$ is the space of bounded sequences with the supremum norm: $\|\varphi\| = \sup\limits_{s \leq 0} |\varphi(s)|$.

Definition 1.5. *For* $g, g^* \in G$*, by* $g \leq g^*$ *we mean that* $g(s) \leq g^*(s)$ *for all* $s \leq 0$; *while by* $g < g^*$ *we mean that* $g \leq g^*$ *and* $\lim\limits_{n \to \infty} \Big[\sup\limits_{s \leq 0} (g(s)/g^*(s - n))\Big] = 0$.

Corresponding to the state space C_g, we should assume that the right-hand functional $F(n, \varphi)$ of (1.1) is defined on $Z \times C_g$.

Definition 1.6. *The zero solution of* (1.1) *is uniformly stable in* C_g (g-US) *if for each* $\varepsilon > 0$ *and any* $n_0 \in Z$*, there exists a* $\delta > 0$ *independent of* n_0 *such that if* $\|\varphi\|_g < \delta$*, then*

$$|x(n_0, \varphi)(n)| < \varepsilon \quad \text{for all } n \geq n_0.$$

Definition 1.7. *The zero solution of* (1.1) *is uniformly asymptotically stable in* C_g (g-UAS) *if it is* g-US *and there is an* $\delta_0 > 0$ *such that for each* $\gamma > 0$*, there exists an integer*

$N(\gamma) > 0$ *independent of* n_0 *such that if* $\|\varphi\|_g < \delta_0$ *then*

$$|x(n_0, \varphi)(n)| < \varepsilon \quad \textit{for all} \quad n \geq n_0 + N(\gamma).$$

Definition 1.8. *A function* $\beta : Z \to R^+$ *is said to belong to* $\mathcal{B}$, *denoted by* $\beta \in \mathcal{B}$, *if there exist constants* $\alpha,\ L > 0$ *such that* $\sum\limits_{s=n}^{n+L-1} \beta(s) \geq \alpha$ *for all* $n \in Z$.

In [1] and [5] we have established several results on the UAS of (1.1). However, the results we are going to establish in this work are different from them but are the counterparts of the relevant results in [2] and [4] which deal with the infinite delay differential equations.

§2. Main Results

The first result is on UAS of (1.1).

Theorem 2.1. *Suppose that there exists a Liapunov functional* $V : Z \times \mathcal{C}_H \to R^+$, *a semi-norm* $|\cdot|_s$ *having a fading memory with respect to* $\|\cdot\|$, *a function* $\Phi : Z^+ \to R^+$ *with* $\sum\limits_{s=0}^{\infty} \Phi(s) < \infty$, *and a positive constant* η $(\eta < H)$ *such that*

(i) $W_1(|\varphi(0)|) \leq V(n, \varphi) \leq W_2[|\varphi(0)| + \sum\limits_{s=0}^{\infty} \Phi(s) W_3(|\varphi(-s)|)] + W_4(|\varphi|_s)$,

(ii) $\Delta V_{(1)}(n, \varphi) \leq -W_5(|\varphi(0)|)$,

(iii) $W_1(u) - W_4(u) > 0$ *for all* $u \in (0, \eta]$,

where

$$\Delta V_{(1)}(n, \varphi) \equiv V(n + 1, x_{n+1}(n, \varphi)) - V(n, \varphi)$$

with $x_{n+1}(n, \varphi)$ *being a solution of* (1.1). *Then the zero solution of* (1.1) *is* UAS.

Proof. (I) First, we show the US of the zero solution of (1.1).

Let $J = \sum\limits_{s=0}^{\infty} \Phi(s)$. For any given $\varepsilon > 0$ $(\varepsilon < H)$ and $n_0 \in Z$, choose $\delta = \delta(\varepsilon) > 0$ $(\delta < \varepsilon)$ so small that $W_2(\delta + JW_3(\delta)) + W_4(\delta) < W_1(\varepsilon)$. Then let $\varphi \in \mathcal{C}_\delta$, and denote $x(n) = x(n_0, \varphi)(n)$, $x_n = x_n(n_0, \varphi)$, $V(n) = V(n, x_n)$, and $\Delta V(n) = \Delta V_{(1)}(n, x_n)$.

By assumption, $|\cdot|_s$ has a fading memory w.r.t. $\|\cdot\|$, we have $|\varphi|_s \leq \|\varphi\| < \delta$, and thus by (i) and (ii),

$$W_1(|x(n)|) \leq V(n) \leq V(n_0) \leq W_2(\delta + JW_3(\delta)) + W_4(\delta) < W_1(\varepsilon) \quad \text{for all } n \geq n_0,$$

which implies that $|x(n)| < \varepsilon$ for all $n \geq n_0$. This shows that the zero solution of (1.1) is US.

(II) Furthermore, we can assert that the zero solution is UAS.

For $\varepsilon = \eta$, we can find the corresponding $\delta = \delta(\eta) > 0$ $(\delta < \eta)$ by the US. Let $\delta_0 = \delta(\eta)$. Then $[n_0 \in Z, \|\varphi\| < \delta_0, n \geq n_0]$ imply that $V(n) < W_1(\eta)$ and $|x(n)| = |x(n_0, \varphi)(n)| < \eta$.

Now for any given $\gamma > 0$, we will find an integer $N(\gamma) > 0$ such that $[n_0 \in Z, \|\varphi\| < \delta_0, n \geq n_0 + N]$ imply that $|x(n)| = |x(n_0, \varphi)(n)| < \gamma$.

To this end, we first choose a suitable constant μ with $0 < \mu < \eta$ such that

$$W_2(3\mu) + W_4(\mu) < W_1(\gamma). \tag{2.1}$$

By the assumption (iii) there exists a $\sigma > 0$ such that $0 < \sigma < \mu$ and

$$W_1(u) - W_4(u) \geq \sigma + W_2(3\sigma) \quad \text{for} \quad u \in [\mu, \eta]. \tag{2.2}$$

Since W_1 is uniformly continuous on $[\mu,\eta]$, there exists a constant ρ with $0<\rho<\mu-\sigma$ such that

$$W_1(u)-W_1(u-\rho)<\sigma \quad \text{for all} \quad u\in[\mu,\eta]. \tag{2.3}$$

It now follows from (2.3) and (2.2) that

$$W_1(u-\rho)-W_4(u)>W_1(u)-\sigma-W_4(u)\ge W_2(3\sigma) \quad \text{for} \quad u\in[\mu,\eta]. \tag{2.4}$$

Since $|\cdot|_s$ has a fading memory w.r.t. $\|\cdot\|$, for $D\equiv\eta+\delta_0$ and the above $\sigma>0$ there exists an integer $h_0>0$ such that

$$|\psi|_s\le\max\{\|\psi\|^{[-h,0]},\sigma\} \tag{2.5}$$

whenever an integer $h\ge h_0,\ \psi\in\mathcal{C}$ and $\|\psi\|^{(-\infty,-h]}\le D=\eta+\delta_0$. We may choose $h\ge h_0$ so large that

$$W_3(D)\sum_{s=-\infty}^{n-h}\Phi(n-s)=W_3(D)\sum_{s=h}^{\infty}\Phi(s)<\sigma.$$

Noting that $|x(n)|<\eta<D$ for all $n\in Z$, i.e., $\|x_n\|<D$ for all $n\in Z$, which implies $\|x_n\|^{(-\infty,-h]}<D$, we derive from (2.5) that

$$|x_n|_s\le\max\{\|x\|^{[n-h,n]},\sigma\} \quad \text{for any} \quad n\in Z.$$

On the other hand, for any n and $\bar n$ with $n\ge\bar n\ge n_0$ we have

$$V(n)\le V(\bar n)-\sum_{s=\bar n}^{n-1}W_5(|x(s)|)\le W_1(D)-\sum_{s=\bar n}^{n-1}W_5(|x(s)|).$$

It follows that there exists an integer $N_0>0$ such that for each $\bar n\ge n_0$ there exists some $n^*\in[\bar n,\bar n+N_0]$ with $|x(n^*)|<\sigma$.

Hence, we can find a sequence $\{n_i\}$ such that

$$n_{i-1}+h\le n_i\le n_{i-1}+h+N_0, \quad \text{and} \quad |x(n_i)|<\sigma \quad \text{for} \quad i=1,2,\cdots.$$

Then we have for any $n\ge n_0+h$ that

$$\begin{aligned} V(n)&\le W_2[|x(n)|+\sum_{s=n-h}^{n}\Phi(n-s)W_3(|x(s)|)\\ &\quad+\sum_{s=-\infty}^{n-h}\Phi(n-s)W_3(|x(s)|)]+W_4[\max\{\|x\|^{[n-h,n]},\sigma\}]\\ &\le W_2[|x(n)|+\sum_{s=n-h}^{n}\Phi(n-s)W_3(|x(s)|)+\sigma]\\ &\quad+\max\{W_4(\|x\|^{[n-h,n]}),W_4(\sigma)\}.\end{aligned}$$

Thus

$$V(n_i)\le W_2[2\sigma+\sum_{s=n_i-h}^{n_i}\Phi(n_i-s)W_3(|x(s)|)]+\max\{W_4(\|x\|^{[n_i-h,n_i]}),W_4(\sigma)\}. \tag{2.6}$$

Let $Q=\max\limits_{0\le s\le h}\Phi(s)$, and $K>0$ be the least integer such that

$$W_1(\eta)-KW_5[W_3^{-1}(\sigma/(Q(h+1)))]<0.$$

By (ii) we have for any $n \geq n_{i+K} + 1$,

$$V(n) \leq V(n_i) - \sum_{s=n_i}^{n-1} W_5(|x(s)|) \leq W_1(\eta) - \sum_{j=i+1}^{i+K} \sum_{s=n_j-h}^{n_j} W_5(|x(s)|). \tag{2.7}$$

We now claim that there must be some integer $l : i+1 \leq l \leq i+K$ such that

$$Q \sum_{s=n_l-h}^{n_l} W_3(|x(s)|) < \sigma. \tag{2.8}$$

In fact, suppose it is not true, i.e.,

$$Q \sum_{s=n_j-h}^{n_j} W_3(|x(s)|) \geq \sigma \quad \text{for all} \quad j \in [i+1, i+K],$$

which implies that there exists at least one s^* in each interval $[n_j - h, n_j]$ such that

$$W_3(|x(s^*)|) \geq \sigma/(Q(h+1)) \quad \text{and thus} \quad |x(s^*)| \geq W_3^{-1}[\sigma/(Q(h+1))],$$

then it follows from (2.7) that

$$V(n) \leq W_1(\eta) - \sum_{j=i+1}^{i+K} W_5(W_3^{-1}[\sigma/(Q(h+1))]) = W_1(\eta) - KW_5(W_3^{-1}[\sigma/(Q(h+1))]) < 0$$

if $n \geq n_{i+K} + 1$. It is a contradiction. Hence, (2.8) holds.

Therefore, there exists a subsequence $\{s_i\}$ of $\{n_i\}$ such that

$$Q \sum_{s=s_i-h}^{s_i} W_3(|x(s)|) < \sigma \tag{2.9}$$

with $s_{i-1} + h \leq s_i \leq s_{i-1} + K(h + N_0)$ for $i = 1, 2, \cdots$. Hence, we know from (2.6) that

$$V(s_i) \leq W_2(3\sigma) + \max\{W_4(\|x\|^{[s_i-h,s_i]}), W_4(\sigma)\}. \tag{2.10}$$

Let $I_i = [s_i - h, s_i]$. On each I_i we have either

(A) $\|x\|^{[s_i-h,s_i]} \leq \mu$; or

(B) $|x(\tau_i)| > \mu$ for some integer $\tau_i \in I_i$.

Suppose (A) holds. Then for $n \geq s_i$ we have by (i), (ii), (2.10) and (2.1) that

$$W_1(|x(n)|) \leq V(n) \leq V(s_i) \leq W_2(3\sigma) + W_4(\mu) < W_1(\gamma).$$

(Note that $\sigma < \mu$.) This implies that $|x(n)| < \gamma$ for $n \geq s_i$.

Now if (B) holds, then let $M_i = \|x\|^{[s_i-h,s_i]}$, and we claim that

$$|x(n)| < M_i - \rho \quad \text{for all} \quad n \geq s_i, \tag{2.11}$$

where ρ is the one given by (2.3).

In fact, suppose there exists some $n^* \geq s_i$ such that $|x(n^*)| \geq M_i - \rho$. Then we have

$$W_1(M_i - \rho) \leq W_1(|x(n^*)|) \leq V(n^*) \leq V(s_i) \leq W_2(3\sigma) + W_4(M_i).$$

This leads to a contradiction to (2.4). Hence, (2.11) holds. Now choose the least positive integer $\tilde{N}$ such that $\eta - \tilde{N}\rho \leq \mu$.

Suppose (B) holds on I_i for $i = 1, 2, \cdots, \tilde{N}$. Then for $n \geq s_{\tilde{N}}$ we have

$$|x(n)| < M_{\tilde{N}} - \rho < M_{\tilde{N}-1} - 2\rho < \cdots < M_1 - \tilde{N}\rho < \eta - \tilde{N}\rho \leq \mu.$$

This means that (A) must hold on some I_i with $i \le s_{\tilde N+1}$. Thus

$$|x(n)| = |x(n_0,\varphi)(n)| < \gamma \ \text{ for all } \ n \ge s_{\tilde N+1}.$$

Since $s_{\tilde N+1} \le n_0 + K(\tilde N+1)(h+N_0)$, if we let $N = K(\tilde N+1)(h+N_0)$, which is obviously independent of n_0 and φ, then we have $|x(n)| = |x(n_0,\varphi)(n)| < \gamma$ for all $n \ge n_0 + N$. This proves the UAS of the zero solution of (1.1).

In particular, if we choose $|\varphi|_s = |\varphi|_g$ for any $g \in G$, then we immediately obtain the following result.

Corollary 2.1. *Suppose that there exists a Liapunov functional $V : Z \times \mathcal{C}_H \to R^+$, a function $g \in G$, a function $\Phi : Z^+ \to R^+$ with $\sum_{s=0}^{\infty} \Phi(s) < \infty$, and a positive constant η ($\eta < H$) such that*

(i) $W_1(|\varphi(0)|) \le V(n,\varphi) \le W_2[|\varphi(0)| + \sum_{s=0}^{\infty} \Phi(s) W_3(|\varphi(-s)|)] + W_4(|\varphi|_g)$,

(ii) $\Delta V_{(1)}(n,\varphi) \le -W_5(|\varphi(0)|)$,

(iii) $W_1(u) - W_4(u) > 0$ *for all* $u \in (0,\eta]$.

Then the zero solution of (1.1) *is* UAS.

Proof. Trivially, it suffices to show that $|\cdot|_g$ has a fading memory with respect to $\|\cdot\|$.

In fact, since $\varphi \in \mathcal{C}$, we have $|\varphi|_g = \sup_{s\le 0} |\varphi(s)|/g(s) \le \|\varphi\|$. On the other hand, for any given $\varepsilon > 0$ and $D > 0$ there exists an integer $h_0 > 0$ such that $D < \varepsilon g(-h_0)$. Thus, whenever $h \ge h_0$ and $\|\varphi(\cdot)\|^{(-\infty,-h]} \le D$, there holds

$$|\varphi|_g = \max\Big\{ \sup_{-h\le s\le 0} \frac{|\varphi(s)|}{g(s)|}, \sup_{s\le -h} \frac{|\varphi(s)|}{g(s)} \Big\} \le \max\{\|\varphi(\cdot)\|^{[-h,0]}, \varepsilon\}.$$

Hence, $|\cdot|$ has a fading memory with respect to $\|\cdot\|$.

The next result is on g-UAS on (1.1).

Theorem 2.1 *Suppose that for some $g^* \in G$, for each $(n_0,\varphi) \in Z \times \mathcal{C}_{g^*}$, the solution $x(n_0,\varphi)(n)$ of* (1.1) *globally exists, and there exists a Liapunov functional V, a constant $\eta > 0$, and a function $ta \in \mathcal{B}$ such that in $Z \times C_{g^*}$ there hold*

(i) $W_1(|\varphi(0)|) \le V(n,\varphi) \le W_2(|\varphi(0)|) + W_3(|\varphi|_{g^*})$,

(ii) $\Delta V_{(1)}(n,\varphi) \le -ta(n) W_4(|\varphi(0)|)$,

(iii) $W_1(u) - W_3(u) > 0$ *for all* $u \in (0,\eta]$.

Then the zero solution of (1.1) *is g-*US *for any $g \in G$ with $g \le g^*$ and is g-*UAS *for any $g \in G$ with $g < g^*$.*

Remark 2.1. Note that $g_0 < g^*$ for any $g^* \in G^0$ and g_0-UAS is equivalent to UAS. Hence, for any $g^* \in G^0$, under the assumptions (i), (ii), and (iii) we can conclude that the zero solution of (1.1) is UAS.

Proof. (I) First, we claim the g-US. Fix $g \in G$ with $g \le g^*$. Since $C_g \subseteq C_{g^*}$ and $|\varphi|_{g^*} \le |\varphi|_g$, we have from (i) that

$$V(n,\varphi) \le W_2(|\varphi(0)|) + W_3(|\varphi|_g). \tag{2.12}$$

For any given $\varepsilon > 0$ ($\varepsilon < \eta$), choose $\delta = \delta(\varepsilon) > 0$ ($\delta < \varepsilon$) such that $W_2(\delta)+W_3(\delta) < W_1(\varepsilon)$. Now for any $n_0 \in Z$, and $\varphi \in C_g$ with $|\varphi|_g < \delta$, we denote $x(n) = x(n_0,\varphi)(n)$, $V(n) = V(n,x_n)$, and $\Delta V(n) = \Delta V_{(1)}(n,x_n)$. Then it follows from (i), (ii), and (2.12) that

$$W_1(|x(n)|) \le V(n) \le V(n_0) \le W_2(\delta) + W_3(\delta) < W_1(\varepsilon) \ \text{ for all } \ n \ge n_0,$$

which implies that $|x(n)| < \varepsilon$ for all $n \geq n_0$. Hence, the zero solution of (1.1) is g-US.

(II) Next, we show the g-UAS. Fix $g \in G$ with $g < g^*$. By the g-US, for $\varepsilon = \eta$, there exists the corresponding $\delta = \delta(\eta) > 0$. Let $\delta_0 = \delta(\eta)$. Then $[n_0 \in Z, |\varphi|_g < \delta_0, n \geq n_0]$ imply that

$$|x(n)| = |x(n_0, \varphi)(n)| < \eta. \tag{2.13}$$

Let $\gamma > 0$ be any given number. We should find an integer $N(\gamma) > 0$ such that $[n_0 \in Z, |\varphi|_g < \delta_0, n \geq n_0 + N]$ imply that $|x(n)| = |x(n_0, \varphi)(n)| < \gamma$.

First of all, we pick a constant μ with $0 < \mu < \eta$ such that

$$W_2(\mu) + W_3(\mu) < W_1(\gamma). \tag{2.14}$$

By the assumption (iii) there exists a $\sigma > 0$ such that $0 < \sigma < \mu$ and

$$W_1(u) - W_3(u) > \sigma + W_2(\sigma) \quad \text{for} \quad u \in [\mu, \eta]. \tag{2.15}$$

Let $\alpha > 0$ and $L > 0$ be the corresponding numbers in the definition of β. Choose $N^* \in Z$ so that

$$\begin{aligned} &W_2(\delta_0) + W_3(\delta_0) - N^*\alpha W_4(\sigma) < 0, \quad \eta/g^*(-N^*) < \mu, \\ &\sup_{s\leq 0} \delta_0 g(s)/g^*(s - N^*) < \mu, \quad W_2(\delta_0) + W_3(\delta_0) - N^*\sigma < 0. \end{aligned} \tag{2.16}$$

It now follows from (ii) and (2.12) that

$$V(n) \leq V(n_0) \leq W_2(|x(n_0)|) + W_3(|x_{n_0}|_g) \leq W_2(\delta_0) + W_3(\delta_0) \quad \text{for all} \quad n \geq n_0. \tag{2.17}$$

Let $P_0 = W_2(\sigma) + W_3(\mu)$. We assert an important fact that: "If there exists an integer $n^* \geq n_0 + N^*(L+1)$ with $V(n^*) > P_0$, then there must be some integer $\hat{n} \in [n^* - N^*(L+1), n^*]$ such that $V(\hat{n}) > V(n^*) + \sigma$."

In fact, we first can claim that there must be some integer $s^* \in [n^* - N^*L, n^*]$ with $|x(s^*)| \leq \sigma$. On the contrary, we suppose that $|x(s)| > \sigma$ for all $s \in [n^* - N^*L, n^*]$. Then by (ii), (2.17) and (2.16) there holds

$$V(n^*) \leq V(n^* - N^*L) - W_4(\sigma) \sum_{s=n^*-N^*L}^{n^*-1} \beta(s) \leq W_2(\delta_0) + W_3(\delta_0) - N^*\alpha W_4(\sigma) < 0,$$

which leads to a contradiction. Hence, there must be some integer $s^* \in [n^* - N^*L, n^*]$ with $|x(s^*)| \leq \sigma$. Then by assumption, we derive that

$$W_2(\sigma) + W_3(\mu) = P_0 < V(n^*) \leq V(s^*) \leq W_2(\sigma) + W_3(|x_{s^*}|_{g^*}),$$

which implies that $|x_{s^*}|_{g^*} > \mu$.

Thus, in virtue of (2.13) and (2.16), and noting that $n_0 - s^* \leq -N^*$, we arrive at

$$\begin{aligned} \sup_{s\leq -N^*} \frac{|x(s^*+s)|}{g^*(s)} &= \max\Big\{ \sup_{s\leq n_0 - s^*} \frac{|x(s^*+s)|}{g^*(s)}, \sup_{n_0 - s^* \leq s \leq -N^*} \frac{|x(s^*+s)|}{g^*(s)} \Big\} \\ &\leq \max\Big\{ \sup_{s\leq 0} \frac{\delta_0 g(s)}{g^*(s-N^*)}, \frac{\eta}{g^*(-N^*)} \Big\} < \mu. \end{aligned}$$

Since $|x_{s^*}|_{g^*} > \mu$, it follows that

$$|x_{s^*}|_{g^*} = \sup_{-N^* \leq s \leq 0} \frac{|x(s^*+s)|}{g^*(s)} \leq \sup_{s^* - N^* \leq s \leq s^*} |x(s)|.$$

Hence, there exists $\hat{n} \in [s^* - N^*, s^*]$ such that

$$|x(\hat{n})| = \sup_{s^*-N^* \le s \le s^*} |x(s)| \ge |x_{s^*}|_{g^*}. \tag{2.18}$$

Suppose now that $V(\hat{n}) \le V(n^*) + \sigma$. Then we would have

$$W_1(|x(\hat{n})|) \le V(\hat{n}) \le V(n^*) + \sigma \le V(s^*) + \sigma \le W_2(\sigma) + W_3(|x(\hat{n})|) + \sigma,$$

which imply by (2.13) and (2.15) that $|x(\hat{n})| < \mu$, and thus by (2.18) we obtain

$$|x_{s^*}|_{g^*} \le |x(\hat{n})| < \mu.$$

Again, it is a contradiction.

Therefore, we must have $V(\hat{n}) > V(n^*)+\sigma$, where $\hat{n} \in [s^* - N^*, s^*] \subset [n^* - N^*(L+1), n^*]$ as desired. This proves our assertion.

Now suppose there exists an integer $n^* \ge n_0 + N^*N^*(L+1)$ with $V(n^*) > P_0$. By repeatedly applying N^* times, we can conclude that there exists an integer $n_{N^*} \ge n_0$ such that $V(n_{N^*}) > P_0 + N^*\sigma$. It now follows from (2.17) and (2.16) that

$$P_0 < V(n_{N^*}) - N^*\sigma \le W_2(\delta_0) + W_3(\delta_0) - N^*\sigma < 0.$$

It is a contradiction. Hence, we must have $V(n) \le P_0$ for all $n \ge n_0 + N^*N^*(L+1)$. Let $N = N^*N^*(L+1)$. Then there holds

$$W_1(|x(n)|) \le V(n) \le P_0 = W_2(\sigma) + W_3(\mu) < W_1(\gamma) \quad \text{for} \quad n \ge n_0 + N$$

in virtue of $\sigma < \mu$ and (2.14). Hence we arrive at $|x(n)| < \gamma$ for all $n \ge n_0 + N$. Obviously, N is independent of n_0 and φ. Therefore, the zero solution of (1.1) is g-UAS. This completes the proof of Theorem 2.2.

In particular, if $g^* = g_0$, then $\mathcal{C}_{g_0} = \mathcal{C}$ and we have the following

Corollary 2.2. *Suppose that for each* $(n_0, \varphi) \in Z \times \mathcal{C}$, *the solution* $x(n_0, \varphi)(n)$ *of* (1.1) *globally exists, and there exists a Liapunov functional* V, *a constant* $\eta > 0$, *and a function* $ta \in \mathcal{B}$ *such that*

(i) $W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)|) + W_3(\|\varphi\|)$,

(ii) $\Delta V_{(1)}(n, \varphi) \le -ta(n)W_4(|\varphi(0)|)$,

(iii) $W_1(u) - W_3(u) > 0$ *for all* $u \in (0, \eta]$.

Then the zero solution of (1.1) *is* US.

Remark 2.2. Under the assumptions of Corollary 2.2, we cannot assert the UAS or g-UAS since there does not exist any $g \in G$ with $g < g_0$.

References

[1] Zhang Shunian, Stability of infinite delay difference systems [J], *Nonlinear Analysis, TMA,* **22:**9(1994), 1121–1129.

[2] Zhang Bo, Uniform asymptotic satbility in functional differential equations with infinite delay [A], Comparison Method and Stability Theory [C], edited by Xinzhi Liu and David Siegel, Marcel Dekker, Inc., New York, 1994, 349–362.

[3] Zhang Shunian, Periodicity in delay difference systems [A], World Congress of Nonlinear Analysts' 92 [C], Walter de Gruyter, Berlin, 1996, 1861–1870.

[4] Hering, R. H., Uniform asymptotic stability in infinite delay systems [J], *J. Math. Anal. Appl.,* **180** (1993), 160–173.

[5] Zhang Shunian, Razumikhin techniques in delay differnce systems [J], *PanAmerican Math. J.,* **3:**2 (1993), 1–16.