Chin. Ann. of Math.
22B: 4(2001),503-512

ADDITIVE FUNCTIONALS AND PERTURBATION OF SEMIGROUP***

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Abstract

The perturbation of semigroup by a multiplicative functional with bounded variation is investigated in the frame of weak duality. The strong continuity and Schrödinger type equation of the perturbated semigroup are discussed. A few switching identities and formulae conerning dual additive functionals and Revuz measures are given.

Keywords Right processes, Revuz measures, Additive functional
2000 MR Subject Classification 60J45, 60J40
Chinese Library Classification O211.62 **Document Code** A
Article ID 0252-9599(2001)04-0503-10

§1. Introduction and Preliminaries

Suppose that X is a right Markov process with state space E and transition semigroup (P_t). Given a multiplicative functional M of X, we define for any nonnegative measurable function f on E,

$$Q_t f(x) := P^x(f(X_t)M_t), \quad x \in E, \quad t \geq 0.$$

It follows from the multiplicativity of M that (Q_t) is also a semigroup of transition functions on E and usually called a perturbation semigroup of (P_t). The well-known Feynman-Kac semigroup is a special case where X is a Brownian motion on R^d and

$$M_t = \exp \int_0^t \phi(X_s)ds$$

for some bounded measurable function ϕ on R^d. If, in addition, M is also a supermartingale, then (Q_t) gives birth to another nice Markov process which we usually call the transformed process of X by M. There are some interesting questions related to the perturbation semigroup, such as the strong continuity of (Q_t) on L^p-space, the existence of uniqueness of solutions of corresponding Schrödinger's equation, perturbation of the Dirichlet forms (if X is symmetric or nearly symmetric).

Manuscript received February 24, 2000. Revised October 24, 2000.
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***Project supported in part by the National Natural Science Foundation of China (No.19501035).

In the last ten years many authors has investigated those problems from various approaches. The readers who are interested may refer to [9], [4], [1], [2], [5], [10], [11], [12], etc. In this paper, we shall study the similar problems which were discussed in [5]. While the additive functionals involved in [5] are assumed to be continuous, we do not assume the continuity of additive functionals. Though we assume weak duality, the arguments and conclusions are still true without this assumption. The readers may refer to [5] about how to work under a Borel right Markov process with an excessive measure, which owns automatically a moderate Markov process as a dual process.

Let $(E, \mathcal{B})$ be a Lusin space with its Borel σ-algebra and m a σ-finite measure on E. Assume that X and $\widehat{X}$ are Borel right processes in weak duality with respect to m, with common state space E, transition semigroups $(P_t), (\widehat{P}_t)$ and resolvents $(U^q), (\widehat{U}^q)$, respectively. The weak duality simply means that for $f, g \in \mathcal{B}^+$,

$$(P_t f, g)_m = (f, \widehat{P}_t g)_m. \tag{1.1}$$

In this section, we shall introduce basic terminologies and notations used throughout this paper. Most of them are stated for X and the corresponding ones for the dual process $\widehat{X}$ are similar and distinguished by a hat or a suffix "co".

Definition 1.1. *A subset Λ of Ω is called an Ω-equivalent set if there exists an exceptional set N such that $P^x(\Lambda) = 1$ for all $x \notin N$.*

(*Additive functionals*) *We say that A is an additive functional AF of X if $A = (A_t)_{t\geq 0}$ is a $[0, \infty]$-valued adapted process on Ω and there exists an Ω-equivalent set Λ such that for all $\omega \in \Lambda$,*

A.1 $A_t(\omega) < \infty$ *for $t < \zeta(\omega)$;*

A.2 $t \mapsto A_t(\omega)$ *is right continuous;*

A.3 $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ *for all $t, s \geq 0$.*

(*Multiplicative functionals*) *We say that M is a multiplicative functional* (MF) *of X if $M = (M_t)_{t\geq 0}$ is a $[0, \infty]$-valued adapted process on Ω and there exists an Ω-equivalent set Λ such that for all $\omega \in \Lambda$,*

M.1 $0 < M_t(\omega) < \infty$ *for $t < \zeta(\omega)$;*

M.2 $t \mapsto M_t(\omega)$ *is right continuous;*

M.3 $M_{t+s}(\omega) = M_t(\omega) \cdot M_s(\theta_t \omega)$ *for all $t, s \geq 0$.*

(*m-equivalence*) *Let A, B be two additive* (*or mutiplicative*) *functionals of X. We say that A is m-equivalent to B* (*write $A \sim B$*) *if for each $t > 0$, $P^m(A_t \neq B_t, t < \zeta) = 0$.*

While any AF is increasing, an MF here is generally not. Let $\mathcal{A}(X)$ be the set of all additive functionals of X and for $A \in \mathcal{A}(X)$, $\operatorname{Exp} A$ the Stieltjes exponential of A, which is the unique solution Z of equation

$$1 + Z_t = \int_0^t Z_{s-} dA_s. \tag{1.2}$$

Clearly $\operatorname{Exp} A$ is an increasing MF of X. Conversely if Z is an increasing MF, then similarly the Stieltjes logarithm of Z is the additive functional A determined by (2.2). We also use $\mathcal{M}(X)$ to denote the set of all increasing mutiplicative functionals of X.

For any $f \in \mathcal{B}$, define the quasi-supremum norm as $\|f\|_{\mathrm{Q}} := \inf_{N: m-\text{polar}} \sup_{x \notin N} |f(x)|$. f is called quasi-bounded if $\|f\|_{\mathrm{Q}} < \infty$. Clearly the quasi-suprenum norm is no less than the usual L^∞-norm and they are the same when f is finely continuous[12].

Notations and Conventions. We use ':=' as a way of definition, which is always read as 'is defined to be'. For a class $\mathcal{F}$ of functions, we denote by $b\mathcal{F}$ (resp. $p\mathcal{F}(=\mathcal{F}^+)$) the set of bounded (resp. nonnegative) functions in $\mathcal{F}$. We will not distinguish 'nonnegative' from 'positive'. When a number $a > 0$ or a function $f > 0$ everywhere, we say they are strictly positive. For a measure μ and a function f, $\mu(f) := \int f d\mu$. We sometimes write L^p or $L^p(m)$ for $L^p(E, m)$ and $(\cdot, \cdot)$ for the inner product in $L^2(m)$. For $f, g \in \mathcal{B}(E)$, $f \otimes g(x, y) := f(x)g(y)$, $x, y \in E$. Finally we shall use exclusively P^x for both probability measure and expectation.

§2. Switching Identity (I)

In this section we shall present an identity which is a generalized form of what is usually called the Revuz formula. First the following lemma gives a switching identity for energy functionals of dual processes. Let $\mathrm{Exc}^q(X)$ and $S^q(X)$ denote the sets of q-excessive measures and q-excessive functions of X for $q > 0$, respectively. The q-energy functional L^q of X is defined on $\mathrm{Exc}^q(X) \times S^q(X)$ as

$$L^q(\xi, u) := \sup\{\mu(u) : \mu U^q \le \xi\}, \quad \xi \in \mathrm{Exc}^q(X), \quad u \in S^q(X).$$

For more about energy functional, please refer to [6].

Lemma 2.1. *Let L^q (resp. $\widehat{L}^q$) be the q-energy functional of X (resp. $\widehat{X}$) and $q > 0$. Then for any $f \in S^q(X)$ and $\widehat{f} \in S^q(\widehat{X})$,*

$$L^q(\widehat{f}m, f) = \widehat{L}^q(fm, \widehat{f}). \tag{2.1}$$

Proof. Since $\widehat{f}m \in \mathrm{Exc}^q(X)$ and $fm \in \mathrm{Exc}^q(\widehat{X})$, two sides of (2.1) make sense. It follows from the duality of resolvents that $(U^q g) \cdot m = (gm)\widehat{U}^q$. By property of energy functional, we have

$$L^q(\widehat{f}m, U^q g) = (\widehat{f}, g)_m = \widehat{L}q((gm)\widehat{U}^q, \widehat{f}) = \widehat{L}^q((U^q g)m, \widehat{f}).$$

Then (2.1) follows directly from the fact that any q-excessive function is the limit of an increasing sequence of q-potentials.

Now comes the first switching identity.

Theorem 2.1 (Switching identity I). *Let $L \in \mathcal{A}(X)$ and $\widehat{K} \in \mathcal{A}(\widehat{X})$. Then for $f, g \in \mathcal{B}^+$,*

$$\nu_L(\widehat{U}^q_{\widehat{K}} f \otimes g) = \widehat{\nu}_{\widehat{K}}(U^q_L g \otimes f), \tag{2.2}$$

where U^q_L denotes the q-potential operator of L, precisely

$$U^q_L f(x) := E^x \int_0^\infty e^{-qt} f(X_t) dL_t,$$

and ν_L the bivariate Revuz measure of L with respect to m.

Proof. Since $\widehat{U}^q_{\widehat{K}} f$ is q-coexcessive and $U^q_L g$ is q-excessive, by Lemma 2.1 we have $L^q(\widehat{U}^q_{\widehat{K}} f \cdot m, U^q_L g) = \widehat{L}^q(U^q_L g \cdot m, \widehat{U}^q_{\widehat{K}} f)$. However by (8.7) in [7], $L^q(\widehat{U}^q_{\widehat{K}} f \cdot m, U^q_L g) = \nu_L(\widehat{U}^q_{\widehat{K}} f \otimes g)$. That completes the proof.

Let M be a decreasing (or equivalently $[0,1]$-valued) multiplicative functional of X and $\widehat{M}$ the dual of M, which is certainly also decreasing. It is well-known that M gives birth to another right Markov process which is usually called the M-subprocess of X and write as

(X, M). For any $L \in \mathcal{A}(X)$, we define the M-potential of L,

$$V_L^q f(x) := E^x \int_0^\infty e^{-qt} M_{t-} f(X_t) dL_t, \quad x \in E. \tag{2.3}$$

However V_L^q may also be viewed as the potential operator of M-additive functional $M_- \cdot L := \int M_- dL$. When L is continuous, M_{t-} may be replaced by M_t. If $[M]$ is the Stieltjes logarithm of M, then $M_- \cdot d[M] = dM$. Now we state the second switching identity.

Theorem 2.2 (Switching identity II). *Let $L \in \mathcal{A}(X)$ and $\widehat{K} \in \mathcal{A}(\widehat{X})$. Then for $f, g \in \mathcal{B}^+$,*

$$\nu_L(\widehat{V}_{\widehat{K}}^q f \otimes g) = \widehat{\nu}_{\widehat{K}}(V_L^q g \otimes f). \tag{2.4}$$

Proof. Let L_M^q (resp. $\widehat{L}_{\widehat{M}}^q$) be the energy functional of the subprocess (X, M) (resp. $(\widehat{X}, \widehat{M})$). By Lemma 2.1, we find $L_M^q(\widehat{V}_{\widehat{K}}^q f \cdot m, V_L^q g) = \widehat{L}_{\widehat{M}}^q(V_L^q g \cdot m, \widehat{V}_{\widehat{K}}^q f)$. By Lemma I.4.5 in [11], it follows that $L_M^q(\widehat{V}_{\widehat{K}}^q f \cdot m, V_L^q g) = \rho_{M_- \cdot L}^{\widehat{V}_{\widehat{K}}^q f \cdot m}(g)$, where the right hand side is the Revuz measure of $M_- \cdot L$, which is an M-additive functional, taken with respect to the measure $\widehat{V}_{\widehat{K}}^q f \cdot m$, which is q-excessive for the subprocess (X, M). Since $\widehat{V}_{\widehat{K}}^q f$ is cofinely continuous (it can be decomposed into the difference of two q-coexcessive functions), $t \mapsto \widehat{V}_{\widehat{K}}^q f(X_{t-})$ is left continuous a.s. Thus by Theorem II.5.5 in [11] (it is easily seen that this theorem is true for m^* replaced with an excessive measure for (X, M)), we have $\rho_{M_- \cdot L}^{\widehat{V}_{\widehat{K}}^q f \cdot m}(g) = \nu_{M_- \cdot L}(\widehat{V}_{\widehat{K}}^q f \otimes g)$. Finally by Corollary 3.13 in [13], $\nu_{M_- \cdot L} = \nu_L$. That completes the proof.

Remark. We would like to point out several special cases of (2.4):

$$\begin{aligned}
&(\widehat{V}_{\widehat{K}}^q f, g) = \widehat{\nu}_{\widehat{K}}(V^q g \otimes f), \quad \text{if } \ dL_t = dt;\\
&(V_L^q g, f) = \nu_L(\widehat{V}^q f \otimes g), \quad \text{if } \ d\widehat{K}_t = dt;\\
&(\widehat{U}_{\widehat{K}}^q f, g) = \widehat{\nu}_{\widehat{K}}(U^q g \otimes f), \quad \text{if } \ dL_t = dt \text{ and } A = 0;\\
&(U_L^q g, f) = \nu_L(\widehat{U}^q f \otimes g), \quad \text{if } \ d\widehat{K}_t = dt \text{ and } A = 0.
\end{aligned} \tag{2.5}$$

The first two were called the generalized Revuz formulas and proved in [11], and the last two are the classical Revuz formulas and were proved in [7].

§3. Khas'minskii's Lemma

For $A \in \mathcal{A}(X)$, we define

$$k_t(A) := \|E^{\cdot} A_t\|_{\mathrm{Q}}, \quad t > 0; \quad k(A) := \inf_{t>0} k_t(A);$$

$$c_q(A) := \Big\| \int_0^\infty e^{-qt} dA_t \Big\|_{\mathrm{Q}}, \quad q > 0; \quad c(A) := \inf_{q>0} c_q(A).$$

It is easy to check by the additivity of A that the following statements are equivalent to each other

(1) $k_t(A) < \infty$ for some $t > 0$; (2) $k_t(A) < \infty$ for all $t > 0$;

(3) $c_q(A) < \infty$ for some $q > 0$; (4) $c_q(A) < \infty$ for all $q > 0$.

For any bivariate measure ν with left marginal measure λ, we define

$$c_q(\nu) := \inf\{a \in R : \lambda \widehat{U}^q \le a \cdot m\}, \quad q > 0; \quad c(\nu) := \inf_{q>0} c_q(\nu).$$

Lemma 3.1. *If $A \in \mathcal{A}(X)$, then $k(A) = c(A) = c(\nu_A)$.*

Proof. (1) For any $q > 0,\ t > 0$ and $x \in E$, we have

$$E^x \int_0^\infty e^{-qs} dA_s \geq E^x \int_0^t e^{-qs} dA_s \geq e^{-qt} E^x A_t.$$

Hence $k_t(A) \leq e^{qt} c_q(A)$. It follows that $k(A) \leq c_q(A)$ and then $k(A) \leq c(A)$. Now assume that $c(A) < a < \infty$. We may find $q > 0$ and $b < a$ such that $c_q(A) < b < a$ and choose t small enough such that $e^{qt} b < a$. Then $k_t(A) \leq e^{qt} c_q(A) < a$ and $k(A) < a$. It follows that $k(A) \leq c(A)$. Let $k(A) < a < \infty$, then there exists $t > 0$ and $b < a$ such that $k_t(A) < b$. We may find an m-polar set N with $E^x A_t < b$ for $x \notin N$. By the Markov property we have $E^x A_{nt} \leq nb$ for any integer $n \geq 0$. It follows that for $s \geq 0$ and $x \notin N$, $E^x A_s \leq b + \frac{b}{t} s$. Now for any $T > 0$,

$$\begin{aligned} E^x \int_0^T e^{-qs} dA_s &= e^{-qT} E^x A_T + q E^x \int_0^T A_s e^{-qs} ds \\ &\leq e^{-qT} \Big(b + \frac{b}{t} T \Big) + q \int_0^T \Big(b + \frac{b}{t} s \Big) e^{-qs} ds \\ &= e^{-qT} \Big(b + \frac{b}{t} T \Big) + b(1 - e^{-qT}) + \frac{b}{t} \Big[T e^{-qT} + \frac{1}{q} (1 - e^{-qT}) \Big]. \end{aligned}$$

As T goes to infinity, we have

$$E^x \int_0^\infty e^{-qs} dA_s \leq b + \frac{b}{qt}.$$

Hence $c_q(A) < a$ as q is large enough. It implies that $c(A) < a$ and $c(A) \leq k(A)$.

On the other hand, since $c_q(A) = \|U_A^q 1\|_{\mathrm{Q}}$, the equality $c(A) = c(\nu_A)$ follows directly from the Revuz formula $\lambda_A \widehat{U}^q = U_A^q 1 \cdot m$.

The following result generalizes the Khas'minskii's lemma, which was originally stated for Brownian motion (refer to [9]). The same result was proved in Lemma 2.1 of [12] for symmetric case, however the proof may be used here with no modification.

Lemma 3.2. *Let $A \in \mathcal{A}(X)$.*

(1) *If there exist some $t > 0$ and $0 < a < 1$ such that $P^x A_t \leq a$ for all $x \in E$, then for all $x \in E$, $E^x (Exp\, A)_t \leq \frac{1}{1-a}$.*

(2) *If $k(A) < 1$, then there exist constants $c, \beta > 0$ such that for all $t > 0$,*

$$\|E^{\cdot} (Exp\, A)_t\|_{\mathrm{Q}} \leq c \cdot e^{\beta t}.$$

Let us now introduce more notations for $A, L \in \mathcal{A}(X)$ and $q > 0$,

$$U_L^{q-A-} f(x) := E^x \int_0^\infty e^{-qt} (\mathrm{Exp}\, A)_{t-} f(X_t) dL_t.$$

When L is continuous, $(\mathrm{Exp}\, A)_{t-}$ may be replaced by $(\mathrm{Exp}\, A)_t$. Particularly, when $L_t = t$, write U_L^{q-A-} as U^{q-A}.

Corollary 3.1. *Let $A \in \mathcal{A}(X)$. If $k(A) < 1$, then for q large enough, there exists $c > 0$ such that*

$$\|U^{q-A} 1\|_{\mathrm{Q}} + \|U_A^{q-A-} 1\|_{\mathrm{Q}} + \|U_A^q 1\|_{\mathrm{Q}} \leq c,$$

namely, they are all quasi-bounded.

Proof. By Lemma 3.2, it is obvious that $U^{q-A} 1$ is quasi-bounded for q large enough. We need only to show that $U_A^{q-A-} 1$ is bounded for q large enough since $U_A^q 1 \leq U_A^{q-A-} 1$.

However for $t > 0$,

$$\int_0^t e^{-qs}(\operatorname{Exp} A)_{s-}dA_s = \int_0^t e^{-qs}d(\operatorname{Exp} A)_s = e^{-qt}(\operatorname{Exp} A)_t - 1 + q\int_0^t e^{-qs}(\operatorname{Exp} A)_s ds.$$

From Lemma 3.2, the quasi-boundedness follows.

§4. Switching Identity (II)

We shall first present a formula similar to the well-known resolvent equation, which can be shown by a direct computation and readers may refer to Lemma 3.1 (c) in [12].

Lemma 4.1. *Suppose A, $L \in \mathcal{A}(X)$. Then for $q \geq 0$ and $f \in \mathcal{B}^+$ with $U_L^q f(x) < \infty$, it holds that*

$$U_L^{q-A_-} f(x) = U_L^q f(x) + U_A^q U_L^{q-A_-} f(x). \tag{4.1}$$

Therefore $U_L^{q-A_-} f$ is q-excessive.

Now comes another switching identity.

Theorem 4.1 (Switching identity III). *Suppose $A \in \mathcal{A}(X)$ and $\widehat{A}$ is the dual additive functional of A, $L \in \mathcal{A}(X)$ and $\widehat{K} \in \mathcal{A}(\widehat{X})$. If $k(A) < 1$, $k(L) < \infty$, $\widehat{k}(\widehat{A}) < 1$ and $\widehat{k}(\widehat{K}) < \infty$, then*

$$\nu_L(\widehat{U}_{\widehat{K}}^{q-\widehat{A}_-} f \otimes g) = \widehat{\nu}_{\widehat{K}}(U_L^{q-A_-} g \otimes f). \tag{4.2}$$

Proof. By the lemma above, $U_L^{q-A_-} g$ and $\widehat{U}_{\widehat{K}}^{q-\widehat{A}_-} f$ are q-excessive and q-coexcessive respectively. Hence by Lemma 2.1 we have

$$L^q((\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f)\cdot m, U_L^{q-\Lambda_-} g) = \widehat{L}^q((U_L^{q-\Lambda_-} g)\cdot m, \widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-}).$$

However

$$\begin{aligned} L^q((\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f)\cdot m, U_L^{q-\Lambda_-} g) &= L^q((\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f)\cdot m, U_L^q g) + L^q((\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f)\cdot m, U_A^q U_L^{q-\Lambda_-} g) \\ &= \nu_L(\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} \otimes g) + \nu_A(\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f \otimes U_L^{q-\Lambda_-} g), \\ L^q((U_L^{q-\Lambda_-} g)\cdot m, \widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f) &= \widehat{\nu}_{\widehat{K}}(U_L^{q-\Lambda_-} g \otimes f) + \widehat{\nu}_{\widehat{A}}(U_L^{q-\Lambda_-} g \otimes \widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f). \end{aligned}$$

Since ν_A is in duality with $\widehat{\nu}_{\widehat{A}}$, we see that (4.2) holds as soon as

$$\nu_A(\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} \otimes U_L^{q-\Lambda_-} g)) < \infty.$$

However ν_A is a σ-finite measure and hence we may assume that it is a finite measure. In this case we need only to verify that $\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f \otimes U_L^{q-\Lambda_-} g$ is quasi-bounded or $\widehat{U}_{\widehat{K}}^{q-\widehat{\Lambda}_-} f$, $U_L^{q-\Lambda_-} g$ are quasi-bounded. It suffices to show that $U_L^{q-A_-} 1$ is bounded. There exists $\alpha > 0$ such that $k(A + \alpha L) < 1$. By Corollary 3.1, there exists a large enough s such that

$$\alpha \| U_L^{s-A_-} 1 \|_{\mathrm{Q}} = \| U_{\alpha L}^{s-A_-} 1 \|_{\mathrm{Q}} \leq \| U^{s-(A+\alpha L)_-} \|_{\mathrm{Q}} < \infty.$$

Therefore $U_L^{s-A_-} 1$ is bounded. From a generalized resolvent equation,

$$U_L^{q-A_-} = U_L^{s-A_-} + (s-q)U^{q-A} U_L^{s-A_-},$$

it follows that $U_L^{q-A_-} 1$ is bounded.

If we take $\widehat{K}_t = L_t = t$, then ν_L and $\widehat{\nu}_{\widehat{K}}$ concentrate on the diagonal of $E \times E$ as m. Therefore we have

Corollary 4.1. *Let $A \in \mathcal{A}(X)$ and $\widehat{A} \in \mathcal{A}(\widehat{X})$ with $k(A) < 1$, $k(\widehat{A}) < 1$. If they are dual, then for q large enough, U^{q-A} and $\widehat{U}^{q-\widehat{A}}$ are dual, i.e., for $f, g \in \mathcal{B}^+$,*

$$(U^{q-A}f, g)_m = (f, \widehat{U}^{q-\widehat{A}}g)_m.$$

§5. Strong Continuity of Perturbated Semigroup

Let $A \in \mathcal{A}(X)$ and $\widehat{A}$ be the dual of A. We define for any nonnegative measurable function f,

$$Q_t f(x) := E^x[(\operatorname{Exp} A)_t f(X_t)], \quad \widehat{Q}_t f(x) := \widehat{E}^x[(\operatorname{Exp} \widehat{A})_t f(\widehat{X}_t)]. \tag{5.1}$$

Obviously (Q_t) is a semigroup of kernels on E and is usually called the perturbated semigroup of (P_t) by A. It follows from the duality of A and $\widehat{A}$ that (Q_t) and $(\widehat{Q}_t)$ are dual, i.e., for nonnegative measurable functions f, g, and $t > 0$, $(Q_t f, g) = (f, \widehat{Q}_t g)$.

Lemma 5.1. *Let $A \in \mathcal{A}(X)$. (1) If $\widehat{k}(\widehat{A}) < 1$, then $f = 0$ a.e. implies that $Q_t f = 0$ a.e. for each $t \geq 0$. (2) If $k(A) < 1$ and $\widehat{k}(\widehat{A}) < 1$, then for $1 \leq p \leq \infty$, (Q_t) is a semigroup of bounded operators on $L^p(m)$ for each $t \geq 0$ and for $p \in]0, \infty[$,*

$$\|Q_t\|_{L^p} \leq \|Q_t 1\|_{\mathrm{Q}}^{\frac{1}{p'}} \|\widehat{Q}_t 1\|_{\mathrm{Q}}^{\frac{1}{p}}, \tag{5.2}$$

where p' is the conjugate number of p: $\frac{1}{p'} + \frac{1}{p} = 1$.

Proof. (1) For any $f \in \mathcal{B}$, we have

$$m(|Q_t f|) \leq m(Q_t|f|) = m(|f| \cdot \widehat{Q}_t 1) \leq \|\widehat{Q}_t 1\|_{\mathrm{Q}} m(|f|).$$

Thus $f = 0$ a.e. implies that $Q_t f = 0$ a.e. and Q_t is a bounded operator on $L^1(m)$ and $\|Q_t\|_{L^1} \leq \|\widehat{Q}_t 1\|_{\mathrm{Q}}$.

(2) If $k(A) < 1$ and $\widehat{k}(\widehat{A}) < 1$, then it is easily seen that Q_t is a bounded operator on $L^p(m)$ for each $t \geq 0$ and $1 \leq p \leq \infty$. In particular if $1 < p < \infty$ and $f \in \mathcal{B}$, we find

$$m(|Q_t f|^p) \leq m((Q_t 1)^{\frac{p}{p'}} (Q_t |f|^p)) \leq \|Q_t 1\|_{\mathrm{Q}}^{\frac{p}{p'}} \cdot m(|f| \cdot \widehat{Q}_t 1) \leq \|Q_t 1\|_{\mathrm{Q}}^{\frac{p}{p'}} \|\widehat{Q}_t 1\|_{\mathrm{Q}} m(|f|^p).$$

The conclusion follows.

The following theorem gives a sufficient condition that guarantees that the perturbation semigroup is a strongly continuous semigroup of bounded operators on L^p-spaces. The proof is basically similar to that in §3 in [5].

Theorem 5.1. *Let $A \in \mathcal{A}(X)$ with $k(A) < 1$ and $\widehat{k}(\widehat{A}) < 1$. Then (Q_t) is a strongly continuous semigroup of bounded operators on $L^p(m)$ for $1 \leq p \leq \infty$.*

Proof. To show (Q_t) is strongly continuous on $L^p(m)$ it suffices to show that it is weakly continuous, that is, if $g \in L^{p'}(m)$ and $f \in L^p(m)$, then as $t \downarrow 0$,

$$(g, Q_t f) \to (g, f). \tag{5.3}$$

However $L^1(m) \cap L^\infty(m)$ is dense in $L^{p'}(m)$ and $U^1(L^1(m) \cap L^\infty(m))$ is dense in $L^p(m)$ (see [5]). Hence we need only to verify (5.3) for $g \in L^1(m) \cap L^\infty(m)$ and $f = U^1 h$ with $h \in L^1(m) \cap L^\infty(m)$. From the dominated convergence theorem and the fact that h is finely continuous, it follows that $Q_t U^1 h$ converges to $U^1 h$ q.e. as $t \downarrow 0$ and then $(g, Q_t U^1 h) \to (g, U^1 h)$ as $t \downarrow 0$. That completes the proof.

By the theorem above, it is reasonable to discuss the class of additive functionals A which satisfy the Kato type condition: $k(A) < 1$ and $\widehat{k}(\widehat{A}) < 1$. We call this class the extended Kato class and denote it by $\mathcal{A}_K$.

§6. Schrödinger Type Equations

A multiplicative functional M is said to be of bounded variation if $t \mapsto M_t$ is of bounded variation on any compact interval a.s. Let $\overline{\mathcal{M}}(X)$ be the set of multiplicative functionals with bounded variation. For $M \in \overline{\mathcal{M}}(X)$, we may (and do) define

$$A_t := \int_0^t \frac{dM_s}{M_{s-}}, \quad t > 0. \tag{6.1}$$

Then A is an adapted process which is of bounded variation and has additivity and clearly $\Delta A = \frac{M}{M_-} - 1 > -1$. Therefore A may be written into $A^1 - A^2$ with $A^1, A^2 \in \mathcal{A}(X)$ and $\Delta A^2 < 1$ a.s.

Natually we denote by $\overline{\mathcal{A}}(X)$ the totality of processes $A^1 - A^2$ with $A^1, A^2 \in \mathcal{A}(X)$ and $\Delta A^2 < 1$ a.s. Clearly the Stieltjes logarithm establishes a one-to-one correspondence between $\overline{\mathcal{M}}(X)$ and $\overline{\mathcal{A}}(X)$. Let $A = A^1 - A^2 \in \mathcal{A}(X)$ and $\nu := \nu_A := \nu_{A^1} - \nu_{A^2}$. Then ν may be decomposed as $\nu = \nu^+ - \nu^-$. It is clear that $\nu^+ \leq \nu_{A^1}$ and $\nu^- \leq \nu_{A^2}$. Hence ν^+ and ν^- are also bivariate smooth measures and there exists $A^+, A^- \in \mathcal{A}(X)$ such that $\nu^+ = \nu_{A^+}$, $\nu^- = \nu_{A^-}$, $A = A^+ - A^-$ and $\Delta A^+ \cdot \Delta A^- = 0$. Hence

$$\operatorname{Exp} A = \operatorname{Exp} A^+ \cdot \operatorname{Exp}(0 - A^-).$$

We write $M^- := \operatorname{Exp}(0 - A^-)$, which is a decreasing multiplicative functional of X. We say that $A \in \overline{\mathcal{A}}(X)$ is an additive functional of extended Kato class if there exists a representation $A = A^1 - A^2$ with $A^1 \in \mathcal{A}_K$, and we write as $A \in \overline{\mathcal{A}}_K$ in that case. We also denote by (Q_t) and (V^q) the transition semigroup and resolvent of the M^--subprocess of X. Then $V_L^{q-A_-^+} = U_L^{q-A_-}$ for generalized potential operators. The following identities are easy to verify.

Proposition 6.1. *Let $A \in \overline{\mathcal{A}}(X)$ and q be large enough. Then the following identities hold*:

(1) $U^{q-A} + U^q_{A^-} U^{q-A} = U^q + U^q_{A^+} U^{q-A}$.

(2) $U^{q-A} = V^q + V^q_{A^+} U^{q-A}$.

(3) *If* $L \in \mathcal{A}(X)$, $U_L^{q-A_-} + U^{q-A}_{A^-} U^q_L = U^q_L + U^{q-A_-}_{A^+} U^q_L$.

In this section we assume that $A \in \overline{\mathcal{A}}_K$, $\nu = \nu_A$. The classical Schrödinger equation and theory of Feynman-Kac semigroup suggest that $L + \nu$ should be the generator in some sense for the strongly continuous semigroup (P_t^{-A}) where L is the generator of (P_t). We shall first make it clear what our Schrödinger type equation

$$(q - L - \nu)u = f \tag{6.2}$$

means exactly.

Suppose $1 < p < \infty$ and $f \in L^p$. We say that a function u on E is a solution of (6.2) with parameter p provided that (1) u is quasi-continuous; (2) for all $g \in D(\widehat{L})$, we have

$$\begin{aligned} &\nu^+(|g| \otimes |u|) < \infty, \quad \nu^-(|g| \otimes |u|) < \infty, \\ &((q - \widehat{L})g, u) - \nu^+(g \otimes u) + \nu^-(g \otimes u) = (g, f), \end{aligned} \tag{6.3}$$

where $\widehat{L}$ is the generator of $(\widehat{P}_t)$ on $L^{p'}(m)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Clearly if u is a solution with parameter p of (6.2), then $u \in L^p(m)$. Thus we sometimes say a solution in $L^p(m)$ instead of a solution with parameter p.

Theorem 6.1. *Let $f \in L^p$. Then for q large enough, $u := U^{q-A}f$ is a solution of* (6.2) *in $L^p(m)$.*

Proof. Without loss of generality, we assume that f is nonnegative. First u is a difference of two q-excessive functions and is certainly quasi-continuous. We know that $U^{q-A^+}f \in L^p(m)$. By Proposition 6.1,

$$U^{q-A}f + U^q_{A^-}U^{q-A}f = U^q f + U^q_{A^+}U^{q-A}f \le U^q f + U^q_{A^+}U^{q-A^+}f = U^{q-A^+}f.$$

Therefore for $g = \widehat{U}^q v \in D(\widehat{L})$ with $v \in L^{p'}(m)$, we have

$$\nu^+(|g| \otimes U^{q-A}f) \le \nu^+(U^q|v| \otimes U^{q-A}f) = (U^q_{A^+}U^{q-A}f, |v|)_m \le (U^{q-A^+}f, |v|)_m < \infty;$$
$$\nu^-(|g| \otimes U^{q-A}f) < \infty.$$

Now using the switching identity again,

$$\begin{aligned}((q-\widehat{L})g, u)_m &= (v, U^{q-A}f)_m \\ &= (v, U^q f)_m + (v, U^q_{A^+}U^{q-A}f)_m - (v, U^q_{A^-}U^{q-A}f) \\ &= (\widehat{U}^q v, f) + \nu^+(\widehat{U}^q v \otimes U^{q-A}f) - \nu^-(\widehat{U}^q v \otimes U^{q-A}f) \\ &= (g, f) + \nu^+(g \otimes u) - \nu^-(g \otimes u).\end{aligned}$$

Hence u is a solution of (6.2) in $L^p(m)$.

Now we turn to the uniqueness of solution. Let $\nu^* := \nu^+ + \nu^-$ and $\rho^* := \nu^*(1 \otimes \cdot)$.

Lemma 6.1. *Let $A \in \overline{\mathcal{A}}_K(X)$ and q be large enough. Then $U^{q-A}(L^p(m)) \subset L^p(\rho^*)$ for $1 < p < \infty$.*

Proof. Let p' be the conjugate of p and $f \in L^p(m)$. First

$$\begin{aligned}\rho^+(|U^{q-A}f|^p) &\le \rho^+((U^{q-A}1)^{\frac{p}{p'}}(U^{q-A}|f|^p)) \le \|U^{q-A}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\widehat{\nu}_{\widehat{A}^+}(U^{q-A}|f|^p \otimes 1) \\ &= \|U^{q-A}1\|_{\mathrm{Q}}^{\frac{p}{p'}}(\widehat{U}^{q-\widehat{A}^-}_{\widehat{A}^+}1, |f|^p)_m \le \|U^{q-A}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\|\widehat{U}^{q-\widehat{A}^+_-}_{\widehat{A}^+}1\|_{\mathrm{Q}} \cdot m(|f|^p).\end{aligned}$$

Hence $U^{q-A} : L^p(m) \to L^p(\rho^+)$ and

$$\|U^{q-A}\|_{L^p(m)\to L^p(\rho^+)} \le \|U^{q-A}1\|_{\mathrm{Q}}^{\frac{1}{p'}} \cdot \|\widehat{U}^{q-\widehat{A}^+}_{\widehat{A}^+}1\|_{\mathrm{Q}}^{\frac{1}{p}}.$$

Similarly in order to show that U^{q-A} carries $L^p(m)$ into $L^p(\rho^-)$, it suffices to show that $\|\widehat{U}^{q-\widehat{A}^-}_{\widehat{A}^-}1\|_{\mathrm{Q}} < \infty$. As a matter of fact,

$$\begin{aligned}0 &\le U^{q-A}_{A^-}1 \le P^x \int_0^\infty e^{-qt}[\mathrm{Exp}\,(A^+ - A^-)]_{t-}dA^-_t \\ &= -P^x \int_0^\infty e^{-qt}(\mathrm{Exp}\,A^+)_{t-}d(\mathrm{Exp}\,(-A^-))_t \\ &= 1 + P^x \int_0^\infty e^{-qt}(\mathrm{Exp}\,(-A^-))_t(\mathrm{Exp}\,A^+)_{t-}dA^+_t - qP^x \int_0^\infty e^{-qt}(\mathrm{Exp}\,(A^+ - A^-))_t dt \\ &\le 1 + U^{q-A^+}_{A^+}1.\end{aligned}$$

That completes the proof.

Theorem 6.2. *Let $f \in L^p(m)$. Then $U^{q-A}f$ is the unique solution in $L^p(m + \rho^*)$ of* (6.2)*; more precisely if u is a solution of* (6.2) *and $u \in L^p(m + \rho^*)$, then $u = U^{q-A}f$.*

Proof. It suffices to show that if u is a solution of (6.2) for $f = 0$ and $u \in L^p(m + \rho^*)$, then $u = 0$. First we shall verify that $U^q_{A^+}u$ and $U^q_{A^-}u$ are finite q.e. In fact, by the switching identity,

$$m(|U^q_{A^+}u|^p) \le m((U^q_{A^+}1)^{\frac{p}{p'}} \cdot U^q_{A^+}|u|^p) \le \|U^q_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}}(U^q_{A^+}|u|^p, 1)_m$$
$$= \|U^q_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\nu_{A^+}(\widehat{U}^q 1 \otimes |u|^p) \le \|U^q_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\frac{1}{q}\rho^+(|u|^p).$$

Thus that $u \in L^p(\rho^*)$ implies that $|U^q_{A^+}u| + |U^q_{A^-}u| < \infty$ q.e.

Now since $f = 0$, for any $v \in L^{p'}(m)$ we have

$$(v, u) = \nu^+(\widehat{U}^q v \otimes u) - \nu^-(\widehat{U}^q \otimes u) = (v, U^q_{A^+}u - U^q_{A^-}u).$$

Hence $u = U^q_A u$ q.e. Using the switching identity again,

$$m(|U^{q-A_-}_{A^+}u|^p) \le \|U^{q-A_-}_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}} m(U^{q-A_-}_{A^+}|u|^p)$$
$$\le \|U^{q-A_-}_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\nu_{A^+}(\widehat{U}^{q-\widehat{A}}1 \otimes |u|^p)$$
$$\le \|U^{q-A_-}_{A^+}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\|\widehat{U}^{q-\widehat{A}}1\|_{\mathrm{Q}}\rho^+(|u|^p),$$
$$m(|U^{q-A_-}_{A^-}u|^p) \le \|U^{q-A_-}_{A^-}1\|_{\mathrm{Q}}^{\frac{p}{p'}}\|\widehat{U}^{q-\widehat{A}}1\|_{\mathrm{Q}}\rho^-(|u|^p).$$

It follows that $|U^{q-A_-}_{A^+}u| + |U^{q-A_-}_{A^-}u| < \infty$ q.e. By Lemma 6.1 (3), we have

$$U^{q-A_-}_A u = U^q_A u + U^{q-A_-}_A U^q_A u = u + U^{q-A_-}_A u \quad \text{q.e.}$$

Therefore $u = 0$ q.e. That completes the proof.

References

[1] Albeverio, S. & Ma, Z., Perturbation of Dirichlet forms, lower semiboundedness, closability and form cores [J], *J. Funct. Anal.*, **99**(1991), 332–356.

[2] Albeverio, S. & Ma, Z., Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms [J], *Osaka J. Math.*, **29**(1992), 247–265.

[3] Blumenthal, R. M. & Getoor, R. K., Markov processes and potential theory [M], Academic Press, 1968.

[4] Fitzsimmons, P. J., Time change of symmetric Markov processes and a Feynman-Kac formula [J], *J. Theor. Prob.*, **2**(1989), 487–501.

[5] Getoor, R. K., Measure perturbation of Markovian semigroups (to appear).

[6] Getoor, R. K., Excessive measures [M], Birkhauser, 1990.

[7] Getoor, R. K. & Sharpe, M. J., Naturality, standardness, and weak duality for Markov processes [J], *Z. W.*, **67**(1984), 1–62.

[8] Sharpe, M. J., General theory of Markov processes [M], Academic Press, 1988.

[9] Simon, B., Schrödinger semigroup [J], *Bull. Amer. Math. Soc.*, **7**:3(1982), 477–526.

[10] Stollmann, P. & Voigt, J., Perturbation of Dirichlet forms by measures [J], *Potential Analysis*, **5**(1996), 109–138.

[11] Ying, J., Bivariate Revuz measures and the Feynman-Kac formula [J], *Ann. Inst. Henri Poincare*, **32**:2(1996), 251–287.

[12] Ying, J., Dirichlet forms perturbated by AF's of extended Kato class [J], *Osaka J. Math.*, **34**(1997), 933–952.

[13] Ying, J., Revuz measures and the related formulas [J], *Potential Analysis*, **8**:1(1998), 1–19.