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THE EXISTENCE OF RADIAL LIMITS OF ANALYTIC FUNCTIONS IN BANACH SPACES**

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Abstract

Let X be a complex Banach space without the analytic Radon-Nikodym property. The author shows that $G = \{f \in H^\infty(\mathbf{D}, X) :$ there exists $\epsilon > 0$, such that for almost all $\theta \in [0, 2\pi]$, $\limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| \geq \epsilon \}$ is a dense open subset of $H^\infty(\mathbf{D}, X)$. It is also shown that for every open subset B of $\mathbf{T}$, there exists $F \in H^\infty(\mathbf{D}, X)$, such that F has boundary values everywhere on B^c and F has radial limits nowhere on B. When A is a measurable subset of $\mathbf{T}$ with positive measure, there exists $f \in H^\infty(\mathbf{D}, X)$, such that f has nontangential limits almost everywhere on A^c and f has radial limits almost nowhere on A.

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Let X be a complex Banach space and let $\mathbf{D}$ be the open unit disc in the complex plane. We shall denote by $H^\infty(\mathbf{D}, X)$ the Banach space consisting of all uniformly bounded X-valued analytic functions defined on $\mathbf{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbf{D}} \|f(z)\|$. A complex Banach space X is said to have the analytic Radon-Nikodym property, if each element $f \in H^\infty(\mathbf{D}, X)$ has radial limits almost everywhere on the torus $\mathbf{T} = \{e^{i\theta} : \theta \in [0, 2\pi]\}$ (see [1]), this means that for almost all $\theta \in [0, 2\pi]$, $\lim_{r\uparrow 1} f(re^{i\theta})$ exists in X. The analytic Radon-Nikodym property has been introduced by A. V. Bukhvalov and A. A. Danilevich in [1] and it has been extensively studied in the latest years, we refer the reader to [2–6] for more information about this property.

We have shown that if X is a complex Banach space without the analytic Radon-Nikodym property, then there exists an element $F \in H^\infty(\mathbf{D}, X)$, such that $\|F\|_\infty \leq 1$ and that for almost all $\theta \in [0, 2\pi]$, $\limsup_{r,s\uparrow 1} \|F(re^{i\theta}) - F(se^{i\theta})\| \geq 3/4$ (see [2]). We have also shown that when a complex Banach space X has no the analytic Radon-Nikodym property, then there exists an element $F \in H^\infty(\mathbf{D}, X)$ and there exists $r_n \uparrow 1$, such that for all $\alpha, \beta \in [0, 2\pi]$ and for all $n, m \in \mathbf{N}$, $n \neq m$, we have $\|F(r_n e^{i\alpha}) - F(r_m e^{i\beta})\| \geq 1$ (see [3]). In this paper, we are interested in the density of such functions in $H^\infty(\mathbf{D}, X)$, and we shall show that when a complex Banach space X has no the analytic Radon-Nikodym property, the subset of $H^\infty(\mathbf{D}, X)$ consisting of all f which has radial limits almost nowhere on the torus $\mathbf{T}$ is a

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dense open subset in $H^\infty(\mathbf{D}, X)$. In view of this result, it is natural to ask, if $f \in H^\infty(\mathbf{D}, X)$ has radial limits almost nowhere on some measurable subset of $\mathbf{T}$ with positive measure, whether f has radial limits almost nowhere on $\mathbf{T}$, and we shall see that this is not the case. We shall see that for every open subset A of $\mathbf{T}$, there exists $f \in H^\infty(\mathbf{D}, X)$ so that f has radial limits nowhere on A and f has boundary values everywhere on A^c (this means that for each $e^{i\theta} \in A^c$, for each sequence z_n in $\mathbf{D}$ converging to $e^{i\theta}$, the limit $\lim\limits_{n\to\infty} f(z_n)$ exists in X). The proof of this result will be devided into several steps: first we shall show that a stronger result actually is true for closed intervals on $\mathbf{T}$; then for open intervals of $\mathbf{T}$, this will enable us to establish the desired result since each open subset of $\mathbf{T}$ is the union of a sequence of disjoint open intervals. We do not know whether this result remains true if A is replaced by an arbitrary measurable subset of $\mathbf{T}$ with positive measure. We shall only establish the following result: when X lacks the analytic Radon-Nikodym property, for each measurable subset B of $\mathbf{T}$ with positive measure, there exists $g \in H^\infty(\mathbf{D}, X)$, such that g has nontangential limits almost everywhere on B and g has radial limits almost nowhere on B^c.

First we recall the notion of nontangential limit for element in $h^\infty(\mathbf{D}, X)$. Let $e^{i\theta} \in \mathbf{T}$ and let $0 \le \alpha < \pi$. We denote by $S_\alpha(\theta)$ the subset of $\mathbf{D}$ consisting of all z verifying $|\mathrm{Arg}((z - e^{i\theta})/e^{i\theta})| < \alpha$. A function $f \in h^\infty(\mathbf{D}, X)$ is said to have nontangential limit on $e^{i\theta}$, if for every $0 \le \alpha < \pi$, for every sequence $(z_n)_{n\ge1}$ in $S_\alpha(\theta)$, $z_n \to e^{i\theta}$, the sequence $(f(z_n))_{n\ge1}$ converges in X. It is clear that when f has nontangential limit on $e^{i\theta}$, f has radial limit on $e^{i\theta}$. Inversely when f has radial limits almost everywhere on the torus, if we denote by the same letter $f(e^{i\theta})$ for its radial limit on $e^{i\theta}$, then the harmonic extension of f via the Poisson kernel $P_{re^{i\alpha}}(e^{i\theta}) = \frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}$ coincides with f on $\mathbf{D}$. Using the same argument as in the scalar case, we can show that when $f \in h^\infty(\mathbf{D}, X)$ has radial limits almost everywhere on the torus, f has also nontangential limits almost everywhere on the torus.

We shall use the following well-known fact about conformal mapping between simply conneted regions in the complex plane. Let γ be a Jordan cuvre consisting of two disjoint intervals of circles I and J. We suppose that I is an open interval of the unit circle $\mathbf{T}$. Let E be the bounded simply conneted region determined by γ. Then by the Riemann mapping theorem, there exists a conformal mapping h which maps $\mathbf{D}$ onto E; h can be continuously extented to $\mathbf{T}$ so that h is one to one from $\mathbf{T}$ onto $I \cup J$, the boundary of E. As I is an open interval of the unit circle, h can be analytically extented on $h^{-1}(I)$, so for every $e^{i\theta} \in h^{-1}(I)$, $h(e^{i\theta}) = e^{i\alpha} \in I$, for every $0 \le s < \pi$, there exists $0 \le t < \pi$, such that if $(z_n)_{n\ge1}$ is a sequence in $S_s(\alpha)$, $z_n \to e^{i\alpha}$, then the sequence $(h^{-1}(z_n))_{n\ge1}$ belongs to $S_t(e^{i\theta})$ when n is big enough and $\lim\limits_{n\to\infty} h^{-1}(z_n) = e^{i\theta}$. Hence if $f \in H^\infty(\mathbf{D}, X)$ has no nontangential limit on $e^{i\alpha}$, then the function $f(h(z))$ has no nontangential limit on $e^{i\theta}$.

One of the main results in this paper is the following

Theorem 1. *Let X be a complex Banach space without the analytic Radon-Nikodym property. Then the set*

$$G = \Big\{ f \in H^\infty(\mathbf{D}, X) : \quad \text{There exists } \epsilon > 0, \text{ such that for almost all } \theta \in [0, 2\pi], \ \limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| \ge \epsilon \Big\}$$

is a dense open subset of $H^\infty(\mathbf{D}, X)$.

Proof. In [2], we have shown that if X is a complex Banach space without the analytic Radon-Nikodym property, then there exists $F \in H^\infty(\mathbf{D}, X)$, such that $\|F\|_\infty \le 1$ and for almost all $\theta \in [0, 2\pi]$,

$$\limsup_{r,s\uparrow 1} \|F(re^{i\theta}) - F(se^{i\theta})\| \ge 3/4.$$

First let us show that the subset G of $H^\infty(\mathbf{D}, X)$ is open. If $f \in G$, then there exists

$\epsilon > 0$, such that for almost all $\theta \in [0, 2\pi]$, we have $\limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| \geq \epsilon$. Let $g \in H^\infty(\mathbf{D}, X)$ be such that $\|f-g\|_\infty < \epsilon/4$. For almost all $\theta \in [0, 2\pi]$, we have $\limsup_{r,s\uparrow 1} \|g(re^{i\theta}) - g(se^{i\theta})\| \geq \limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| - \limsup_{r,s\uparrow 1} \|(f-g)(re^{i\theta}) - (f-g)(se^{i\theta})\| \geq \epsilon - 2\|f-g\|_\infty \geq \epsilon/2$, this implies that $g \in G$, therfore G is an open subset of $H^\infty(\mathbf{D}, X)$.

Now let us show that G is also dense in $H^\infty(\mathbf{D}, X)$. Let $f \in H^\infty(\mathbf{D}, X)$, $\epsilon > 0$ and $A = \{e^{i\theta} : \limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| \geq \epsilon/4\}$. Define the function h on $\mathbf{T}$ by $h(e^{i\theta}) = \epsilon/16$ if $e^{i\theta} \in A$ and $h(e^{i\theta}) = \epsilon$ if $e^{i\theta} \in A^c$. As $ln|h| \in L^\infty(\mathbf{T})$, there exists an outer function $h' \in H^\infty$, such that $|h'(e^{i\theta})| = h(e^{i\theta})$ for almost all $\theta \in [0, 2\pi]$. Considering the function $g = f + h'F \in H^\infty(\mathbf{D}, X)$, we have $\|f - g\|_\infty = \|h'F\|_\infty \leq \epsilon$. For almost all $e^{i\theta} \in A$,

$$\begin{aligned}&\limsup_{r,s\uparrow 1} \|g(re^{i\theta}) - g(se^{i\theta})\| \\ &\geq \limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| - \limsup_{r,s\uparrow 1} \|(h'F)(re^{i\theta}) - (h'F)(se^{i\theta})\| \\ &\geq \epsilon/4 - 2\|F\|_\infty \|h'\|_\infty \geq \epsilon/4 - \epsilon/8 = \epsilon/8.\end{aligned}$$

For $e^{i\theta} \in A^c$, we have that $\lim_{r\uparrow 1} h'(re^{i\theta})$ exists, $\lim_{r\uparrow 1} |h'(re^{i\theta})| = \epsilon$ and so

$$\begin{aligned}&\limsup_{r,s\uparrow 1} \|g(re^{i\theta}) - g(se^{i\theta})\| \\ &\geq \limsup_{r,s\uparrow 1} \|(h'F)(re^{i\theta}) - (h'F)(se^{i\theta})\| - \limsup_{r,s\uparrow 1} \|f(re^{i\theta}) - f(se^{i\theta})\| \\ &\geq \limsup_{r,s\uparrow 1} \|F(re^{i\theta}) - F(se^{i\theta})\|\epsilon - \epsilon/4 \geq 3\epsilon/4 - \epsilon/4 = \epsilon/2.\end{aligned}$$

Hence for almost all $e^{i\theta} \in \mathbf{T}$, we have $\limsup_{r,s\uparrow 1} \|g(re^{i\theta}) - g(se^{i\theta})\| \geq \epsilon/8$ and so $g \in G$. This shows that G is dense in $H^\infty(\mathbf{D}, X)$ and completes the proof.

In view of Theorem 1, we may hope to show that when $f \in H^\infty(\mathbf{D}, X)$ has no radial limits on a subset of $\mathbf{T}$ with positive measure, f should have radial limits almost nowhere on the torus. The following result shows that this is not the case. We shall say that $F \in H^\infty(\mathbf{D}, X)$ has boundary value on $e^{i\theta} \in \mathbf{T}$, if for every sequence $z_n \in \mathbf{D}$, $\lim_{n\to\infty} z_n = e^{i\theta}$, the limit $\lim_{n\to\infty} F(z_n)$ exists in X; F is said to have boundary values everywhere on some measurable subset A of $\mathbf{T}$, if F has boundary value on each point of A. It is clear that when F has boundary values on $e^{i\theta}$, F has radial limit on $e^{i\theta}$.

Theorem 2. *Let X be a complex Banach space without the analytic Radon-Nikodym property and let I be an open interval of $\mathbf{T}$. Then there exists $F \in H^\infty(\mathbf{D}, X)$, such that F has boundary values everywhere on I and F has radial limits nowhere on I^c.*

Proof. Let γ be the Jordan cuvre consisting of the closed interval $A = \{e^{i\theta} : \theta \in [-1, 1]\}$ of $\mathbf{T}$ and the open segment $B = \{\lambda e^i + (1-\lambda)e^{-i} : 0 < \lambda < 1\}$ in $\mathbf{D}$. Let E be the simply connected region inside the cuvre γ, we have $E \subset \overline{\mathbf{D}}$. From the Riemann mapping theorem, there exists a conformal mapping h which maps $\mathbf{D}$ onto the interior of E. By Caratheodory's theorem, h can be extended to a continuous function from $\overline{\mathbf{D}}$ onto E and h is one to one from $\mathbf{T}$ onto the boundary of E. Let $T_1 = h^{-1}(B)$ and $T_2 = h^{-1}(A)$. We have $\mathbf{T} = T_1 \cup T_2$, T_1 is an open interval of $\mathbf{T}$ and T_2 is a closed interval of $\mathbf{T}$ since h is continuous and one to one from $\mathbf{T}$ onto $A \cup B$, the boundary of E.

Now let $I = \{e^{i\theta} : \theta \in (a, b)\}$ be any open interval of $\mathbf{T}$ and let $J = \mathbf{T} \setminus I$. There exists a conformal mapping g from $\overline{\mathbf{D}}$ onto $\overline{\mathbf{D}}$ so that the image of I by g is T_1, and the image of J by g is T_2. Recall that g is continuous and one to one from $\mathbf{T}$ onto $\mathbf{T}$.

As X lacks the analytic Radon-Nikodym property, there exists $f \in H^\infty(\mathbf{D}, X)$ and $r_n \uparrow 1$ so that for all $\alpha, \beta \in [0, 2\pi]$, and for all $m, n \in \mathbf{N},\ n \neq m$, we have $\|f(r_n e^{i\alpha}) - f(r_m e^{i\beta})\| \geq 1$ (see [2]). Let $F(z) = f(h(g(z)))$ be the composition of f, h and g, it is easy to see that F is analytic and $F \in H^\infty(\mathbf{D}, X)$. The function f is continuous on the open segment B since f is analytic on $\mathbf{D}$ and $B \subset \mathbf{D}$. Hence F is also continuous on $I = g^{-1}(T_1) = g^{-1}(h^{-1}(B))$ and so F has boundary value on $e^{i\theta}$ for every $e^{i\theta} \in I$. Hence F has boundary values everywhere on I. We shall see that F has radial limits nowhere on J. Let $e^{i\alpha} \in J$ be fixed and $h(g(e^{i\alpha})) = e^{i\theta} \in A$. As $r_n \uparrow 1$, there exists $s_n \uparrow 1$ such that $|h(g(s_n e^{i\alpha}))| = r_n$ when n is big enough. Hence if $m, n \in \mathbf{N}$ is big enough and if $n \neq m$, $\|F(s_m e^{i\alpha}) - F(s_n e^{i\alpha})\| = \|f(h(g(s_m e^{i\alpha}))) - f(h(g(s_n e^{i\alpha})))\| \geq 1$. This shows that F has no radial limit on $e^{i\theta}$, and so F has radial limits nowhere on J. This finishes the proof of Theorem 2.

It is interesting to know what are the subsets I of $\mathbf{T}$ which verify the same conclusion of Theorem 2. First we have the following

Theorem 3. *Let X be a complex Banach space without the analytic Radon-Nikodym property. Let E_n be a sequence of disjoint subsets of $\mathbf{T}$ with positive measure so that for each $n \in \mathbf{N}$, there exists an element $F_n \in H^\infty(\mathbf{D}, X)$, such that F_n has radial limits nowhere on E_n and F_n has boundary values everywhere on E_n^c. Then there exists $F \in H^\infty(\mathbf{D}, X)$ such that F has radial limits nowhere on $E = \bigcup_{n=1}^{\infty} E_n$ and F has boundary values everywhere on E^c.*

Proof. Without loss of generality, we can suppose that $\|F_n\|_\infty \leq 1$ for each $n \in \mathbf{N}$. Define $F = \sum_{n=1}^{\infty} \frac{F_n}{2^n}$. Then $F \in H^\infty(\mathbf{D}, X)$ and $\|F\|_\infty \leq 1$. We shall show that F verifies the conclusion of the theorem. Let $e^{i\theta} \in E$. There exists $n \in \mathbf{N}$ such that $e^{i\theta} \in E_n$. As F_m has boundary values everywhere on E_n for every $m \neq n$, we have $\lim_{r \uparrow 1} F_m(re^{i\theta})$ exists in X for every $m \neq n$. $\lim_{r \uparrow 1} F_n(re^{i\theta})$ does not exist in X since F_n has radial limit nowhere on E_n. Let $\epsilon = \limsup_{r,s \uparrow 1} \|F_n(re^{i\theta}) - F_n(se^{i\theta})\| > 0$. There exists $n_0 \in \mathbf{N}$ such that $\sum_{j=n_0}^{\infty} \frac{1}{2^j} \leq \epsilon / 2^{n+2}$, hence

$$
\begin{aligned}
&\limsup_{r,s \uparrow 1} \|F(re^{i\theta}) - F(se^{i\theta})\| \geq \frac{1}{2^n} \limsup_{r,s \uparrow 1} \|F_n(re^{i\theta}) - F_n(se^{i\theta})\| \\
&- \sum_{1 \leq j < n_0,\ j \neq n} \frac{1}{2^j} \limsup_{r,s \uparrow 1} \|F_j(re^{i\theta}) - F_j(se^{i\theta})\| - \sum_{j=n_0}^{\infty} \frac{1}{2^j} \limsup_{r,s \uparrow 1} \|F_j(re^{i\theta}) - F_j(se^{i\theta})\| \\
&\geq \frac{\epsilon}{2^n} - 2 \sum_{j=n_0}^{\infty} \frac{1}{2^j} \|F_j\|_\infty \geq \frac{\epsilon}{2^n} - \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2^{n+1}}.
\end{aligned}
$$

Now let $e^{i\theta} \in E^c$. As each F_i has boundary values everywhere on E^c, it is easy to see that F has boundary value on $e^{i\theta}$. This finishes the proof.

It is useful to note that in the proof of Theorem 3, the factor $\frac{1}{2^n}$ in the definition of the function F can be replaced by any sequence of positive numbers ϵ_n verifying $\sum_{n=1}^{\infty} \epsilon_n < \infty$. The following result is one of the main results of this paper which states that one can take every open subset of $\mathbf{T}$ instead of a closed interval I^c of $\mathbf{T}$.

Theorem 4. *Let X be a complex Banach space without the analytic Radon-Nikodym property. Then for each open subset A of the torus, there exists $F \in H^\infty(\mathbf{D}, X)$ such that F has boundary values everywhere on A^c and F has radial limits nowhere on A.*

Proof. As each open subset A of $\mathbf{T}$ is the union of a sequence of disjoint open intervals of $\mathbf{T}$, by Theorem 3, to show the theorem, it will suffice to show that the same conclusion is true for open interval I of $\mathbf{T}$. As each open interval of $\mathbf{T}$ is the union of a sequence of disjoint intervals J of $\mathbf{T}$ of the form $\{e^{i\theta} : \theta \in (a,b]\}$ for some a, b, it will suffice to show that the same conclusion is true for intervals I of the form $I = \{e^{i\theta} : \theta \in (a,b]\}$. We shall only give the proof for the interval $I = \{e^{i\theta} : \theta \in (0,1]\}$, the proof for the general case is similar.

First we divide the interval I into a sequence of closed intervals. Let $I_n = \{e^{i\theta} : \theta \in [\frac{1}{n+1}, \frac{1}{n}]\}$ for $n \in \mathbf{N}$, we have $I = \bigcup\limits_{n=1}^{\infty} I_n$. It will be useful to note that for $n, m \in \mathbf{N},\ n \neq m$, $I_n \cap I_m = \emptyset$ except for $m = n-1$ or $m = n+1$. By Theorem 2, there exists for each $n \in \mathbf{N}$, an element $F_n \in H^\infty(\mathbf{D}, X)$, so that F_n has boundary values on I_n^c and F_n has radial limits nowhere on I_n. We may suppose that $\|F_n\|_\infty \leq 1$. The function $F \in H^\infty(\mathbf{D}, X)$ which verifies the conclusion of the theorem for $I = \{e^{i\theta} : \theta \in (0,1]\}$ will be of the form $F = \sum\limits_{n=1}^{\infty} \epsilon_n F_n$ and the sequence $(\epsilon_n)_{n\geq 0}$ will be chosen by induction on n. The condition $0 < \epsilon_n \leq \frac{1}{2^n}$ for each n will be imposed, so F is analytic and $F \in H^\infty(\mathbf{D}, X)$.

First note that with the same proof as that of Theorem 3, it is easy to verify that F has boundary values on $I^c = \bigcap\limits_{n=1}^{\infty} I_n^c$ and F has radial limits nowhere on $\bigcup\limits_{n=1}^{\infty} \{e^{i\theta} : \theta \in (\frac{1}{n+1}, \frac{1}{n})\}$. Therefore it will suffice to ensure that F has no radial limit on $e^{\frac{i}{n}}$ for every $n \in \mathbf{N}$.

Let $\epsilon_1 = \frac{1}{2}$. By the choice of F_n, F_n has boundary value on $e^{\frac{i}{2}}$ for $n \geq 3$, hence

$$\begin{aligned}&\limsup_{r,s\uparrow 1} \|F(re^{\frac{i}{2}}) - F(se^{\frac{i}{2}})\| \\ \geq\ & \frac{1}{2} \limsup_{r,s\uparrow 1} \|F_1(re^{\frac{i}{2}}) - F_1(se^{\frac{i}{2}})\| - \epsilon_2 \limsup_{r,s\uparrow 1} \|F_2(re^{\frac{i}{2}}) - F_2(se^{\frac{i}{2}})\| \\ \geq\ & \frac{1}{2} \limsup_{r,s\uparrow 1} \|F_1(re^{\frac{i}{2}}) - F_1(se^{\frac{i}{2}})\| - 2\epsilon_2 \|F_2\|_\infty.\end{aligned}$$

As $\limsup\limits_{r,s\uparrow 1} \|F(re^{\frac{i}{2}}) - F(se^{\frac{i}{2}})\| > 0$ and $\|F_2\|_\infty \leq 1$, there exists $0 < \epsilon_2 \leq \frac{1}{4}$ so that $\limsup\limits_{r,s\uparrow 1} \|F(re^{\frac{i}{2}}) - F(se^{\frac{i}{2}})\| > 0$. Suppose that $0 < \epsilon_n \leq \frac{1}{2^n}$ has been chosen for $n \leq k$. We know that F_n has boundary value on $e^{\frac{i}{k+1}}$ for $n \geq k+2$ and $n \leq k-1$, hence it will suffice to study the behavior of $F_k(re^{\frac{i}{k+1}})$ and $F_{k+1}(se^{\frac{i}{k+1}})$ when $r \uparrow 1$. As $\limsup\limits_{r,s\uparrow 1} \|F_k(re^{\frac{i}{k+1}}) - F_k(se^{\frac{i}{k+1}})\| > 0$ and $\|F_{k+1}\|_\infty \leq 1$, the same method as in the previous cases shows that there exists $0 < \epsilon_{k+1} \leq \frac{1}{2^{k+1}}$ so that $\limsup\limits_{r,s\uparrow 1} \|F(re^{\frac{i}{k+1}}) - F(se^{\frac{i}{k+1}})\| > 0$. In this way we can choose $0 < \epsilon_n \leq \frac{1}{2^n}$ for $n \in \mathbf{N}$ so that F has no radial limit on $e^{\frac{i}{n}}$ for each $n \in \mathbf{N}$. This finishes the proof.

In the proof of Theorem 2, as for all $\alpha, \beta \in [0, 2\pi]$, for all $m, n \in \mathbf{N},\ n \neq m,\ \|F(r_n e^{i\alpha}) - F(r_m e^{i\beta})\| \geq 1$, the function $f \in H^\infty(\mathbf{D}, X)$ actually has the following stronger property: for each $e^{i\theta} \in I^c$, for each Jordan curve γ inside $\mathbf{D}$ with parameterization $\phi : [0,1) \to \mathbf{D}$ such that $\lim\limits_{t\uparrow 1} \phi(t) = e^{i\theta}$, the limit $\limsup\limits_{z,z'\to e^{i\theta}, z,z'\in\gamma} \|f(z) - f(z')\| \geq 1$. This enables us to show that in Theorem 4, the function $f \in H^\infty(\mathbf{D}, X)$ actually has the following stronger property: for each $e^{i\theta} \in A$, for each Jordan curve γ inside $\mathbf{D}$ with parameterization $\phi : [0,1) \to \mathbf{D}$ such that $\lim\limits_{t\uparrow 1} \phi(t) = e^{i\theta}$, we have $\limsup\limits_{z,z'\to e^{i\theta}, z,z'\in\gamma} \|f(z) - f(z')\| > 0$. It is interesting to note that $f \in H^\infty(\mathbf{D}, X)$ has boundary values everywhere on A^c if and only if for each $e^{i\theta} \in A^c$, for each Jordan curve γ inside $\mathbf{D}$ with parameterization $\phi : [0,1) \to \mathbf{D}$ such that $\lim\limits_{t\uparrow 1} \phi(t) = e^{i\theta}$, we have $\limsup_{z,z'\to e^{i\theta},z,z'\in\gamma} \|f(z)-f(z')\|=0.$

We do not know whether Theorem 4 remains true when A is an arbitrary measurable subset of $\mathbf{T}$ with positive measure. We have only the following

Theorem 5. *Let X be a complex Banach space without the analytic Radon-Nikodym property and let $A\subset\mathbf{T}$ with positive measure. There exists $F\in H^\infty(\mathbf{D},X)$, such that F has radial limits almost nowhere on A and F has nontangential limits almost everywhere on A^c.*

Proof. Without loss of generality, we can suppose that the measure of A^c is positive. Let g_1 be the bounded measurable function on $\mathbf{T}$ defined by $g_1(t)=0$ if $t\in A$ and $g_1(t)=-1$ if A^c. g_1 can be extended via the Poisson kernel in $\mathbf{D}$ so that g_1 becomes a bounded harmonic function on $\mathbf{D}$. Let g_2 be the harmonic conjugate of g_1 satisfying $g_2(0)=0$, consider the analytic functions $g(z)=g_1(z)+ig_2(z)$ and $h(z)=e^{g(z)}$ on $\mathbf{D}$. We have $|h(z)|=e^{g_1(z)}\le 1$ as $g_1(z)\le 0$. By principle of maximum, h is analytic on $\mathbf{D}$ with values in $\mathbf{D}$.

As X lacks the analytic Radon-Nikodym property, there exists $F\in H^\infty(\mathbf{D},X)$ and $r_n\uparrow 1$, such that for all $n\in\mathbf{N}$, $0\le r\le r_n$ and $(\alpha,\beta)\in[0,2\pi]^2$, we have $\|f(re^{i\alpha})-f(r_{n+1}e^{i\beta})\|\ge 1$ (see [3]). Let $f(z)=F(h(z))$ for $z\in\mathbf{D}$. f is uniformly bounded and analytic. We shall show that f verifies the conlcusion of the theorem.

Let $e^{i\theta}\in A^c$ be fixed. As g_1 and h have nontangential limits almost everywhere on $\mathbf{T}$, we can suppose that g_1 and h have nontangential limits on $e^{i\theta}$ and the nontangential limit of g_1 on $e^{i\theta}$ is -1. Let $0\le\alpha<\pi$ and let z_n be a sequence inside the region $S_\alpha(\theta)$ converging to $e^{i\theta}$. The limit $\lim_{n\to\infty}h(z_n)=a$ exists and $|a|=\lim_{n\to\infty}|h(z_n)|=e^{\lim_{n\to\infty}g_1(z_n)}=e^{-1}$. This means that $h(z_n)$ converges to some point $a\in\mathbf{D}$ when n tends to ∞, hence $f(z_n)=F(h(z_n))$ converges to $F(a)$ since F is continuous on a. This shows that f has nontangential limits almost everywhere on A^c.

Now let $e^{i\theta}\in A$ be fixed. As g_1 and h have nontangential limits almost everywhere on $\mathbf{T}$, we can suppose that g_1 and h have nontangential limits on $e^{i\theta}$ and the nontangential limit of g_1 on $e^{i\theta}$ is 0. We get that the limit $\lim_{s\uparrow 1}h(se^{i\theta})=b$ exists and $|b|=\lim_{s\uparrow 1}|h(se^{i\theta})|=e^{\lim_{s\uparrow 1}g_1(se^{i\theta})}=1$. This means that $h(se^{i\theta})$ converges to some point $e^{i\alpha}\in\mathbf{T}$ when $s\uparrow 1$. As $r_n\uparrow 1$ and $h(se^{i\theta})$ is continuous on the variable $s\in[0,1)$, there exists $0\le s_n<1$ such that $|h(s_ne^{i\theta})|=r_n$ when n is big enough and $s_n\uparrow 1$. We get that $\lim_{n\to\infty}f(s_ne^{i\theta})$ does not exist since $\|F(r_ne^{i\alpha})-F(r_me^{i\beta})\|\ge 1$ for every $\alpha,\beta\in[0,2\pi]$ and $m,n\in\mathbf{N},\ m\ne n$. Hence $\|f(r_ne^{i\theta})-f(r_me^{i\theta})\|\ge 1$ when $m\ne n$ are big enough. This shows that f has radial limits almost nowhere on A. The claim is proved.

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