Chin. Ann. of Math.
22B: 4(2001),519-528.

ON HARMONIC MAPS INTO SYMPLECTIC GROUPS $Sp(N)$***

HE Qun* SHEN Yibing**

Abstract

By means of the theory of harmonic maps into the unitary group $U(N)$, the authors study harmonic maps into the symplectic group $Sp(N)$. The symplectic uniton and symplectic extended uniton are introduced. The method of the symplectic Bäcklund transformation and the Darboux transformation is used to construct new symplectic unitons from a known one.

Keywords Harmonic map, Symplectic group, Symplectic uniton, Darboux transformation
2000 MR Subject Classification 53C20, 37J35
Chinese Library Classification O186.16 **Document Code** A
Article ID 0252-9599(2001)04-0519-10

§0. Introduction

The construction and the factorization of harmonic maps from R^2 (or its simply-connected domain) into the unitary group $U(N)$ were firstly solved by K.Uhlenbeck in [1], where the conception of unitons was introduced. Since then various developments have been contributed[2−5]. Recently, by introducing (singular) Darboux transformations, a purely algebraic method to construct harmonic maps and unitons into $U(N)$ has been shown in [6,7]. This method can be also applied to the case of harmonic maps into complex Grassmannians[8].

The purpose of this paper is to study harmonic maps from R^2 into the symplectic group $Sp(N)$ which is a totally geodesic subgroup of $U(2N)$. It is different from the case of the unitary group that there is no nontrivial single factor of simplest type acting on harmonic maps into $Sp(N)$ (Proposition 2.1). We introduce the factor of symplectic simplest type which consists of double factors of simplest type (§2). The conception of symplectic unitons and symplectic extended unitons is defined naturally. A theorem for the description of the action on the space $\mathcal{M}$ of symplectic extended unitons is proved (Theorem 2.2). Then,

Manuscript received November 17, 1999. Revised July 18, 2000.

*Department of Applied Mathematics, Tongji University, Shanghai 200092, China.
E-mail: qun-he@ 163.com

**Department of Mathematics, West-Brook, Zhejiang University, Hangzhou 310028, China.
E-mail: ybsh@ mail.hz.zj.cn.

***Project supported by the National Natural Science Foundation of China (No.19531050), the Scientific Foundation of the Minnstry of Education of China (No.1999033554), and the Natural Science Foundation of Zhejiang Province, China (No.198040).

the symplectic Bäcklund transformation as well as the singular Bäcklund transformation for harmonic maps into $Sp(N)$ is given (Propositions 3.1 and 3.3). Finally, the Darboux transformation method described as in [6,7] is used to construct new symplectic unitons from a known one via a purely algebraic algorithm (Theorem 4.2).

Throughout this paper we are going to work on a simply-connected domain $\Omega \subseteq \mathbb{C} \cap \{\infty\}$. All results also hold on S^2. The notations used here will follow those in [1, 8].

§1. Preliminaries

Let $\mathbb{C}^{2N}$ be endowed with the usual Hermitian metric and $\{e_1, \cdots, e_{2N}\}$ the canonical basis of $\mathbb{C}^{2N}$. The complex structure J in $\mathbb{C}^{2N}$ may be expressed as

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

with respect to the canonical basis, where I_N is the identity matrix of degree N. For the sake of simplicity, we introduce the following notation of the algabraic operation:

$$A_* := J^{-1}\overline{A}J = -J\overline{A}J \quad \text{for} \quad A \in gl(2N,\mathbb{C}). \tag{1.1}$$

Thus, the symplectic group $Sp(N)$ is defined as

$$Sp(N) = \{A \in U(2N) \mid A_* = A\}, \tag{1.2}$$

which is a totally geodesic closed subgroup of the unitary group $U(2N)$. The Lie algebra of $Sp(N)$ is then

$$sp(N) = \{X \in u(2N) \mid X_* = X\}, \tag{1.3}$$

where $u(2N)$ is the Lie algebra of $U(2N)$.

By direct verfication, the operator defined by (1.1) satisfies the following propertities.

Lemma 1.1. *For $A, B \in gl(2N,\mathbb{C})$ and $\lambda \in \mathbb{C}$, we have* (1) $(A_*)_* = A$; (2) $(A + B)_* = A_* + B_*$; (3) $(AB)_* = A_*B_*$; (4) $(\lambda A)_* = \overline{\lambda}A_*$; (5) $(A_*)^{-1} = (A^{-1})_*$ *for* $A \in GL(2N,\mathbb{C})$; (6) $J_* = J$, $I_* = I$; (7) $(A^*)_* = (A_*)^*$, *where* $A^* = \overline{A}^t$.

Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be a simmly-connected domain and z the complex coordinate on Ω. Consider a smooth map $\varphi : \Omega \to Sp(N)$ and set $A = \frac{1}{2}\varphi^{-1}d\varphi$ which is a 1-form valued in $sp(N)$ and can be decomposed as $A = A_z dz + A_{\overline{z}} d\overline{z}$ satisfying the following conditions:

$$\varphi^* = \varphi^{-1}, \quad \varphi_* = \varphi; \tag{1.4}$$

$$A_{\overline{z}}^* = -A_z, \quad (A_{\overline{z}})_* = A_z; \tag{1.5}$$

$$\overline{\partial}A_z - \partial A_{\overline{z}} + 2[A_{\overline{z}}, A_z] = 0, \tag{1.6}$$

where $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$. It is known that φ is harmonic if and only if

$$\overline{\partial}A_z + \partial A_{\overline{z}} = 0. \tag{1.7}$$

The Lax pair of the harmonic maps is

$$\overline{\partial}\Phi_\lambda = (1 - \lambda)\Phi_\lambda A_{\overline{z}}, \quad \partial\Phi_\lambda = (1 - \lambda^{-1})\Phi_\lambda A_z, \tag{1.8}$$

whose integrability condition is just (1.6) and (1.7). Thus, if φ is harmonic and $\varphi(p) = I$ for a fixed point $p \in \Omega$, then there exists a unique Φ_λ satisfying the equations (1.8) with $\Phi_1 \equiv I$; $\Phi_{-1} = \varphi$; $\Phi_\lambda(p) = I$. From (1.5) and (1.8) it follows that

$$d\Big(\Phi_\lambda \Phi_{\sigma(\lambda)}^*\Big) = 0 \quad \text{and} \quad d\Big(\Phi_\lambda^{-1}(\Phi_{\sigma(\lambda)})_*\Big) = 0,$$

where $\sigma(\lambda) = (\overline{\lambda})^{-1}$. Thus, Φ_λ can be normalized so that

$$\Phi^*_{\sigma(\lambda)} = \Phi_\lambda^{-1} \quad \text{and} \quad (\Phi_{\sigma(\lambda)})_* = \Phi_\lambda. \tag{1.9}$$

Conversely, if Φ_λ is a solution to (1.8) satisfying (1.9), then $\Phi_{-1} : \Omega \to Sp(N)$ is harmonic. Such $\Phi_\lambda : \mathbb{C}^* \times \Omega \to GL(2N,\mathbb{C}), \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, will be called a symplectic extended solution, or a symplectic extended harmonic map.

Clearly, a harmonic map into $Sp(N)$ can be viewed as a harmonic map into $U(2N)$, while the inversion is not true, in general.

Lemma 1.2. *If Φ'_λ is an extended solution of a harmonic map $\varphi' : \Omega \to U(N)$, then*

$$\Phi_\lambda = \begin{pmatrix} \Phi'_\lambda & 0 \\ 0 & \overline{\Phi}'_{\sigma(\lambda)} \end{pmatrix} \tag{1.10}$$

is a symplectic extended solution of the harmonic map $\varphi = \Phi_{-1} : \Omega \to Sp(N)$.

Proof. Let $A' = \frac{1}{2}\varphi'^{-1}d\varphi' = A'_z dz + A'_{\overline{z}} d\overline{z}$. It follows that

$$(1-\lambda)^{-1}\Phi_\lambda^{-1}\overline{\partial}\Phi_\lambda = \begin{pmatrix} A'_{\overline{z}} & 0 \\ 0 & -A'^{t}_{\overline{z}} \end{pmatrix}$$

is independent of λ, as well as $(1-\lambda^{-1})^{-1}\Phi_\lambda^{-1}\partial\Phi_\lambda$. By virtue of Theorem 2.3 in [1], Φ_{-1} is harmonic. It is easy to check that Φ_λ defined by (1.10) satisfies the condition (1.9). Hence, Φ_{-1} maps Ω into $Sp(N)$.

This lemma provides a way to construct harmonic maps into $Sp(N)$ via harmonic maps into $U(N)$.

If the Laurent series of Φ_λ is $\Phi_\lambda = \sum\limits_{\alpha=-\infty}^{\infty} T_\alpha \lambda^\alpha$ where $T_\alpha : \Omega \to gl(2N,\mathbb{C})$, then the symplectic condition (1.9) is equivalent to

$$\Phi_\lambda^{-1} = \sum_{\alpha=-\infty}^{\infty} T^*_{-\alpha}\lambda^\alpha, \quad (T_\alpha)_* = T_{-\alpha}. \tag{1.11}$$

§2. Dressing Actions on Symplectic Extended Unitons

Definition 2.1. *A symplectic n-uniton is a harmonic map $\varphi : \Omega \to Sp(N)$ which has a symplectic extended solution $\Phi_\lambda : \mathbb{C}^* \times \Omega \to GL(2N,\mathbb{C})$ with*

(a) $\Phi_\lambda = \sum\limits_{\alpha=-n}^{n} T_\alpha \lambda^\alpha$ *for* $T_\alpha : \Omega \to gl(2N,\mathbb{C})$, (b) $\Phi_1 = I$,

(c) $\Phi_{-1} = Q\varphi$ *for* $Q \in Sp(N)$ *constant,* (d) Φ_λ *satisfies* (1.9).

In such a case, Φ_λ is also called the symplectic extended uniton. Clearly, a symplectic n-uniton is a $2n$-uniton for $U(2N)$.

Let

$$\begin{aligned} \mathcal{A}_R(S^2, G) = \{f : S^2\backslash\{p_1, \cdots, p_l\} \to G \quad &\text{meromorphic with no zeros} \\ &\text{or poles at } (0,\infty) \text{ and } f(1) = I, \quad f(\lambda)^{-1} = f(\sigma(\lambda))^*\}, \\ \mathcal{B}(S^2, G) = \{f \in \mathcal{A}_R(S^2, G) \mid f(\lambda) = f(\sigma(\lambda))_*\}&, \end{aligned} \tag{2.1}$$

where $G = GL(2N,\mathbb{C})$. Define

$$\mathcal{M}^n(G) = \{\text{all of the symplectic extended } n\text{-unitons}\}, \quad \mathcal{M}(G) = \bigcup_{n=0}^{\infty} \mathcal{M}^n(G).$$

We write $f^\# \Phi_\lambda = f(\lambda)\cdot\Phi_\lambda\cdot R_\lambda$ for $f \in \mathcal{A}_R(S^2, G)$, $R : \Omega \to \mathcal{A}_R(S^2, G)$. $f^\#$ is the so-called dressing action by f. By [1], R_λ is determined uniquely by $f(\lambda)$ and Φ_λ.

Theorem 2.1. *Given $f \in \mathcal{B}(S^2, G)$, then we have $f^{\#} : \mathcal{M}^n(G) \to \mathcal{M}^n(G)$, and there is a unique $R : \Omega \to \mathcal{B}(S^2, G)$.*

Proof. It is known in [1] that there exists a unique $R_\lambda : \Omega \to \mathcal{A}_R(S^2, G)$ such that $f^{\#}\Phi_\lambda = f(\lambda)\Phi_\lambda R_\lambda$ is still an extended $2n$-uniton when Φ_λ is an extended $2n$-uniton. We now assume that Φ_λ is a symplectic extended n-uniton. From (1.9) we see that $f^{\#}\Phi_\lambda$ is still a symplectic extended n-uniton if and only if

$$f(\lambda)\Phi_\lambda R_\lambda = \Big(f(\sigma(\lambda))\Phi_{\sigma(\lambda)} R_{\sigma(\lambda)}\Big)_* = f(\sigma(\lambda))_* \Phi_\lambda (R_{\sigma(\lambda)})_*. \tag{2.2}$$

Since the representation of $f(\lambda)\Phi_\lambda = (f^{\#}\Phi_\lambda)R_\lambda^{-1}$ is unique (cf. [1, Lemma 5.1]), (2.2) holds if only $f(\lambda) = f(\sigma(\lambda))_*$. Moreover, from (2.2) it follows that $R_\lambda = (R_{\sigma(\lambda)})_*$, namely, $R : \Omega \to \mathcal{B}(S^2, G)$.

On putting $\mathcal{P}(2N) = \{\pi \in L(\mathbb{C}^{2N}, \mathbb{C}^{2N}) \mid \pi^2 = \pi \text{ and } \pi^* = \pi\}$, we let $f(\lambda) = \pi + \xi_\alpha(\lambda)\pi^\perp$ where

$$\xi_\alpha(\lambda) = \frac{(\lambda - \alpha)}{(\overline{\alpha}\lambda - 1)} \frac{(\overline{\alpha} - 1)}{(1 - \alpha)} \quad \text{for} \quad \alpha \in \mathbb{C}^*. \tag{2.3}$$

Such $f(\lambda)$ is called the factor of simplest type in [1]. As distingushed from the case of unitary groups, we have

Proposition 2.1. *Let $\pi \in \mathcal{P}(2N)$ and $\xi_\alpha(\lambda)$ be defined by* (2.3). *Then $\pi + \xi_\alpha(\lambda)\pi^\perp$ belongs to $\mathcal{B}(S^2, G)$ if and only if $|\alpha| = 1$, i.e., $\xi_\alpha(\lambda) = 1$. In other words, there is no nontrivial factor of simplest type in $\mathcal{B}(S^2, G)$.*

Proof. Assume that $f(\lambda) = \pi + \xi_\alpha(\lambda)\pi^\perp \in \mathcal{B}(S^2, G)$ and $\operatorname{rank}\pi = k$. We can choose $Q \in U(2N)$ such that

$$Q^* f(\lambda) Q = \begin{pmatrix} I_k & 0 \\ 0 & \xi_\alpha(\lambda) I_{2N-k} \end{pmatrix}, \quad \overline{Q}^* \overline{f(\sigma(\lambda))}\,\overline{Q} = \begin{pmatrix} I_k & 0 \\ 0 & \xi_\alpha(\lambda)^{-1} I_{2N-k} \end{pmatrix}.$$

By writting $\overline{Q}^* J Q = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$, where J_1 and J_4 are respectively $k \times k$ and $(2N-k) \times (2N-k)$ matrices, we see that the condition $f(\lambda) = (f(\sigma(\lambda)))_*$ yields that $\xi_\alpha(\lambda) J_2 = J_2$, $J_3 = \xi_\alpha(\lambda)^{-1} J_3$ and $(\xi_\alpha(\lambda))^2 J_4 = J_4$. Noting that J_2, J_3 and J_4 can not vanish simultaneously, we have either $\xi_\alpha(\lambda) = 1$ or $(\xi_\alpha(\lambda))^2 = 1$. In the latter case, from (2.3) it follows that $|\alpha| = 1$, which implies that $\xi_\alpha(\lambda) = 1$.

Example 2.1. Let $G' = GL(N, \mathbb{C})$ and $f' \in \mathcal{A}_R(S^2, G')$. By Lemma 1.2, we then have

$$f(\lambda) = \begin{pmatrix} f'(\lambda) & 0 \\ 0 & \overline{f'(\sigma(\lambda))} \end{pmatrix} \in \mathcal{B}(S^2, G), \quad G = GL(2N, \mathbb{C}). \tag{2.4}$$

If $f'(\lambda) \in \mathcal{A}_R(S^2, G')$ is the factor of simplest type, i.e., $f'(\lambda) = \pi' + \xi_\alpha(\lambda)\pi'^\perp$ for $\pi' \in \mathcal{P}(N)$, then from (2.4) we see that

$$\begin{aligned} f(\lambda) &= \begin{pmatrix} \pi' + \xi_\alpha(\lambda)\pi'^\perp & 0 \\ 0 & \overline{\pi}' + \xi_\alpha(\lambda)^{-1}\overline{\pi}'^\perp \end{pmatrix} \\ &= (\pi + \xi_\alpha(\lambda)\pi^\perp)(\pi_* + \xi_\alpha(\lambda)^{-1}\pi_*^\perp) \in \mathcal{B}(S^2, G), \end{aligned} \tag{2.5}$$

where $\pi = \begin{pmatrix} \pi' & 0 \\ 0 & I_N \end{pmatrix} \in \mathcal{P}(2N)$, $\pi_*^\perp = (\pi^\perp)_* = (\pi_*)^\perp$.

In general, we have the following

Lemma 2.1. *Let π_1, $\pi_2 \in \mathcal{P}(2N)$, and $f(\lambda) = (\pi_1 + \xi_\alpha(\lambda)\pi_1^\perp)(\pi_2 + \xi_\alpha(\lambda)^{-1}\pi_2^\perp)$. Then, $f(\lambda) \in \mathcal{B}(S^2, G)$ if and only if $\pi_1\pi_2^\perp = (\pi_1^\perp\pi_2)_*$.*

Proof. By (2.1), we have $f(\lambda) = f(\sigma(\lambda))_*$, from which it follows that

$$\pi_1\pi_2 + \pi_1^\perp\pi_2^\perp = \pi_{1*}\pi_{2*} + \pi_{1*}^\perp\pi_{2*}^\perp, \quad \pi_1\pi_2^\perp = \pi_{1*}^\perp\pi_{2*}, \quad \pi_1^\perp\pi_2 = \pi_{1*}\pi_{2*}^\perp.$$

By Lemma 1.1, these are equivalent to the condition that $\pi_1\pi_2^\perp = (\pi_1^\perp\pi_2)_*$.

Lemma 2.2. *Let $\pi_i \in \mathcal{P}(2N)$, $i = 1.2$. If $\pi_1\pi_2^\perp = (\pi_1^\perp\pi_2)_*$, then* $\operatorname{rank}\pi_1 = \operatorname{rank}\pi_2$.

Proof. Let $\operatorname{rank}\pi_i = k_i$, $i = 1, 2$. Since

$$\begin{aligned}
\operatorname{null}(\pi_2^\perp\pi_1) &= \operatorname{null}(\pi_1\pi_2^\perp) = \operatorname{null}(\pi_1\pi_2^\perp)_* = \operatorname{null}(\pi_1^\perp\pi_2),\\
\ker(\pi_2^\perp\pi_1) &= \ker\pi_1 \oplus (\operatorname{Im}\pi_1 \cap \ker\pi_2^\perp) = \ker\pi_1 \oplus (\operatorname{Im}\pi_1 \cap \operatorname{Im}\pi_2),\\
\ker(\pi_1^\perp\pi_2) &= \ker\pi_2 \oplus (\ker\pi_1^\perp \cap \operatorname{Im}\pi_2) = \ker\pi_2 \oplus (\operatorname{Im}\pi_1 \cap \operatorname{Im}\pi_2),
\end{aligned}$$

we have

$$2N - k_1 + \dim(\operatorname{Im}\pi_1 \cap \operatorname{Im}\pi_2) = 2N - k_2 + \dim(\operatorname{Im}\pi_1 \cap \operatorname{Im}\pi_2), \tag{2.6}$$

i.e., $k_1 = k_2$.

The factor $f(\lambda)$ in Lemma 2.1 with $(\pi_1^\perp\pi_2)_* = \pi_1\pi_2^\perp$ will be called the factor of symplectic simplest type. By Lemma 2.2, $\operatorname{rank}\pi_1$ $(= \operatorname{rank}\pi_2)$ will be called the rank of the factor $f(\lambda)$. In particular, for $\pi \in \mathcal{P}(2N)$, if $\pi^\perp\pi_*^\perp = 0$ (or $\pi\pi_* = 0$), then it is easy to verify that the factor $(\pi + \xi_\alpha(\lambda)\pi^\perp)(\pi_* + \xi_\alpha(\lambda)^{-1}\pi_*^\perp)$ is of symplectic simplest type, where $\xi_\alpha(\lambda)$ is defined by (2.3).

Theorem 2.2. *Let $\pi_i : \mathbb{C}^{2N} \to V_i \subseteq \mathbb{C}^{2N}$ $(i = 1, 2)$ be Hermitian projections with $\pi_2\pi_1^\perp = (\pi_2^\perp\pi_1)_*$. Let $f(\lambda) = (\pi_2 + \xi_\alpha(\lambda)\pi_2^\perp)(\pi_1 + \xi_\alpha(\lambda)^{-1}\pi_1^\perp)$ with $|\alpha| \neq 1$. Then, for $\Phi_\lambda \in \mathcal{M}^n(G)$, there exist $\widetilde{\pi}_i : \Omega \to \mathcal{P}(2N)$, $i = 1, 2$, such that*

$$f^\#\Phi_\lambda = f(\lambda)\Phi_\lambda(\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp) \tag{2.7}$$

with $\widetilde{\pi}_1\widetilde{\pi}_2^\perp = (\widetilde{\pi}_1^\perp\widetilde{\pi}_2)_*$. Moreover, we have $\widetilde{\pi}_i : \Omega \times \mathbb{C}^{2N} \to \eta_i$, $i = 1, 2$, where

$$\eta_1 = \Phi_\alpha^{-1}V_1, \quad \eta_2 = \widetilde{\Phi}_\alpha^* V_2, \quad \widetilde{\Phi}_\alpha = \Phi_\alpha\widetilde{\pi}_1 + \pi_1^\perp\Phi_\alpha + \widetilde{\beta}\pi_1^\perp\dot{\Phi}_\alpha\widetilde{\pi}_1 \tag{2.8}$$

with $\widetilde{\beta} = (1 - |\alpha|^2)(1 - \alpha)/(1 - \overline{\alpha})$ and $\dot{\Phi}_\alpha = (d\Phi_\lambda/d\lambda)\,|_{\lambda=\alpha}$.

Proof. Set

$$\begin{aligned}
\widetilde{\Phi}_\lambda &= (\pi_1 + \xi_\alpha(\lambda)^{-1}\pi_1^\perp)\Phi_\lambda(\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)\\
&= (\pi_1 + \xi_{\sigma(\alpha)}(\lambda)\pi_1^\perp)\Phi_\lambda(\widetilde{\pi}_1 + \xi_{\sigma(\alpha)}(\lambda)^{-1}\widetilde{\pi}_1^\perp).
\end{aligned} \tag{2.9}$$

By Theorem 6.1 and Corollary 6.2 in [1], one can see that $\widetilde{\Phi}_\lambda$ is an extended $2n$-uniton and $\eta_1 = \Phi_{\sigma(\alpha)}^* V_1 = \Phi_\alpha^{-1}V_1$. Then, by (2.9), $f^\#\Phi_\lambda = (\pi_2 + \xi_\alpha(\lambda)\pi_2^\perp)\widetilde{\Phi}_\lambda(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp)$. By the same reason, $f^\#\Phi_\lambda$ is an extended $2n$-uniton and $\eta_2 = \widetilde{\Phi}_\alpha^* V_2$. Noting that $\pi_1^\perp\Phi_\alpha\widetilde{\pi}_1 = 0$ (see [1, (20)]), from (2.4) and (2.9) we have

$$\begin{aligned}
\widetilde{\Phi}_\alpha &= \lim_{\lambda\to\alpha}\widetilde{\Phi}_\lambda = \pi_1\Phi_\alpha\widetilde{\pi}_1 + \pi_1^\perp\Phi_\alpha\widetilde{\pi}_1^\perp + \lim_{\lambda\to\alpha}\xi_{\sigma(\alpha)}(\lambda)^{-1}\pi_1^\perp\Phi_\lambda\widetilde{\pi}_1\\
&= \Phi_\alpha\widetilde{\pi}_1 + \pi_1^\perp\Phi_\alpha + \widetilde{\beta}\pi_1^\perp\Big(\lim_{\lambda\to\alpha}\frac{\Phi_\lambda}{\lambda - \alpha}\Big)\widetilde{\pi}_1 = \Phi_\alpha\widetilde{\pi}_1 + \pi_1^\perp\Phi_\alpha + \widetilde{\beta}\pi_1^\perp\dot{\Phi}_\alpha\widetilde{\pi}_1.
\end{aligned}$$

Since $\pi_2\pi_1^\perp = (\pi_2^\perp\pi_1)_*$, we have, by Lemma 2.1 and Theorem 2.1, $f^\#\Phi_\lambda \in \mathcal{M}^n(G)$ and $R_\lambda = (\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp) : \Omega \to \mathcal{B}(S^2, G)$, so that $\widetilde{\pi}_1\widetilde{\pi}_2^\perp = (\widetilde{\pi}_1^\perp\widetilde{\pi}_2)_*$.

Corollary 2.1. *Let $\Phi_\lambda \in \mathcal{M}^n(G)$ satisfy the initial condition that $\Phi_\lambda(p) = I$ for a fixed point $p \in \Omega$. Then $\widetilde{\pi}_i, \pi_i$ $(i = 1, 2)$ described as in Theorem 2.2 satisfy $\widetilde{\pi}_i(p) = \pi_i$, $i = 1, 2$. Thus, if $R_\lambda = (\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp)$ is of symplectic simplest type at a point $p \in \Omega$, then so is it everywhere on Ω.*

Now consider actions of S^1 on $\mathcal{M}^n(G)$.

Proposition 2.2. *If $\gamma \in S^1$ and $\Phi_\lambda \in \mathcal{M}^n(G)$, then $\gamma^{\#}\Phi_\lambda = \Phi_{\lambda\gamma}\Phi_\gamma^{-1} \in \mathcal{M}^n(G)$, i.e., $\gamma^{\#} : \mathcal{M}^n(G) \to \mathcal{M}^n(G)$.*

Proof. By Theorem 7.1 of [1], $\gamma^{\#}\Phi_\lambda$ is an extended $2n$-uniton. It is easy to see that $(\gamma^{\#}\Phi_{\sigma(\lambda)})_* = (\Phi_{\sigma(\lambda)\gamma})_*(\Phi_\gamma^{-1})_* = \Phi_{\lambda\gamma}\Phi_\gamma^{-1} = \gamma^{\#}\Phi_\lambda$. Hence, $\gamma^{\#}\Phi_\lambda \in \mathcal{M}^n(G)$.

Theorem 2.2 and Proposition 2.2 give two ways to construct new symplectic extended unitons from known ones.

§3. Symplectic Bäcklund Transformations

Proposition 3.1. *Let $\varphi : \Omega \to Sp(N)$ be a harmonic map, and $A = \frac{1}{2}\varphi^{-1}d\varphi$ as usual. Let $\pi_i \in \mathcal{P}(2N)$, $i = 1, 2$, satisfying $\pi_1\pi_2^\perp = (\pi_1^\perp\pi_2)_*$. Then a family of new harmonic maps into $Sp(N)$ parametrized by $\alpha \in \mathbb{C}^*$ can be found by solving the following system of ordinary differential equations for $\widetilde{\pi}_i : \Omega \to \mathcal{P}(2N)$ with $\widetilde{\pi}_i(p) = \pi_i$ for a point $p \in \Omega$:*

$$
\begin{aligned}
&\overline{\partial}\widetilde{\pi}_1 = (1-\sigma(\alpha))\widetilde{\pi}_1 A_{\overline{z}}\widetilde{\pi}_1^\perp - (1-\alpha)\widetilde{\pi}_1^\perp A_{\overline{z}}\widetilde{\pi}_1,\\
&\overline{\partial}\widetilde{\pi}_2 = (1-\alpha)\widetilde{\pi}_2(A_{\overline{z}} + \beta\overline{\partial}\widetilde{\pi}_1)\widetilde{\pi}_2^\perp - (1-\sigma(\alpha))\widetilde{\pi}_2^\perp(A_{\overline{z}} + \beta\overline{\partial}\widetilde{\pi}_1)\widetilde{\pi}_2,
\end{aligned}
\tag{3.1}
$$

where $\beta = \frac{1-|\alpha|^2}{(1-\alpha)(1-\overline{\alpha})}$. The new harmonic maps can be written as

$$
\widetilde{\varphi} = \widetilde{Q}\varphi(\widetilde{\pi}_1 - \gamma\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 - \overline{\gamma}\widetilde{\pi}_2^\perp),
\tag{3.2}
$$

where $\gamma = \frac{(1-\overline{\alpha})(1+\alpha)}{(1+\overline{\alpha})(1-\alpha)} \in S^1$ and $\widetilde{Q} \in Sp(N)$.

Proof. Let Φ_λ be the symplectic extended solution of φ, so that $\Phi_{-1} = Q\varphi$ for some $Q \in Sp(N)$. By using Theorem 6.3 of [1] and noting that $\xi_\alpha = \xi_{\sigma(\alpha)}^{-1}$, we know that

$$
\Psi_\lambda = (\pi_1 + \xi_\alpha(\lambda)^{-1}\pi_1^\perp)\Phi_\lambda(\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)
$$

is a new extended solution if and only if $\widetilde{\pi}_1$ satisfies the first equation in (3.1), i.e., $B_{\overline{z}} := (1-\lambda)^{-1}\Psi_\lambda^{-1}\overline{\partial}\Psi_\lambda$ is independent of λ. A straightforward computation gives $B_{\overline{z}} = A_{\overline{z}} + \beta\overline{\partial}\widetilde{\pi}_1$ with $\beta = \frac{1-|\alpha|^2}{(1-\alpha)(1-\overline{\alpha})}$. By the same reason, we see that

$$
\begin{aligned}
\widetilde{\Phi}_\lambda &= (\pi_2 + \xi_\alpha(\lambda)\pi_2^\perp)\Psi_\lambda(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp)\\
&= (\pi_2 + \xi_\alpha(\lambda)\pi_2^\perp)(\pi_1 + \xi_\alpha(\lambda)^{-1}\pi_1^\perp)\Phi_\lambda(\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp)
\end{aligned}
\tag{3.3}
$$

is also a new extended solution if and only if $\widetilde{\pi}_2$ satisfies the second equation in (3.1).

On the other hand, by Corollary 2.2, we have $\widetilde{\pi}_1\widetilde{\pi}_2^\perp = (\widetilde{\pi}_1^\perp\widetilde{\pi}_2)_*$ on Ω everywhere. So, $\widetilde{\Phi}_\lambda \in \mathcal{M}(G)$. Now, (3.2) follows directly from taking $\lambda = -1$ in $\widetilde{\Phi}_\lambda$.

Proposition 3.1 gives a description of a Bäcklund transformation for harmonic maps into $Sp(N)$. The factor $(\widetilde{\pi}_1 + \xi_\alpha(\lambda)\widetilde{\pi}_1^\perp)(\widetilde{\pi}_2 + \xi_\alpha(\lambda)^{-1}\widetilde{\pi}_2^\perp)$ of symplectic simplest type where $\widetilde{\pi}_i$ satisfy (3.1) will be called the symplectic Bäcklund factor. Clearly, the factor of symplectic simplest type constructed by $\widetilde{\pi}_i$ $(i = 1, 2)$ described as in Theorem 2.2 is just a symplectic Bäcklund factor.

It is known in Lemma 4.1 of [7] that every extended n-uniton can be written as a product of n factors of the form $(\pi + \lambda\pi^\perp)$. Thus, every symplectic extended n-uniton may be expressed as a product of n factors of the form $(\pi_1 + \lambda\pi_1^\perp)(\pi_2 + \lambda^{-1}\pi_2^\perp)$. We now consider the symplectic one-uniton.

Proposition 3.2. *$\varphi :\to Sp(N)$ is a symplectic one-uniton if and only if $\varphi = Q(\pi_1 - \pi_1^\perp)(\pi_2 - \pi_2^\perp)$ for $Q \in Sp(N)$, where $\pi_i : \Omega \to \mathcal{P}(2N)$, $i = 1, 2$, satisfy the following*: (1) $\pi_1^\perp\pi_2 = (\pi_1\pi_2^\perp)_*$; (2) $\pi_2(\overline{\partial}\pi_2 - (\overline{\partial}\pi_1)\pi_2^\perp) = 0$; (3) $\pi_2\pi_1^\perp(\overline{\partial}\pi_1) = 0$; (4) $\pi_1^\perp(\overline{\partial}\pi_1)\pi_2^\perp = 0$; (5) $\pi_2^\perp\pi_1(\overline{\partial}\pi_1)\pi_2 = 0$.

Proof. By Theorem 2.3 of [1], $\Phi_\lambda = (\pi_1 + \lambda\pi_1^\perp)(\pi_2 + \lambda^{-1}\pi_2^\perp)$ is an extended solution of Φ_{-1} if and only if $(1-\lambda)^{-1}\Phi_\lambda^{-1}\overline{\partial}\Phi_\lambda$ is independent of λ, i.e.,

$$(\pi_2 + \lambda\pi_2^\perp)\Big((\pi_1 + \lambda^{-1}\pi_1^\perp)\overline{\partial}\pi_1(\pi_2 + \lambda^{-1}\pi_2^\perp) + \lambda^{-1}\overline{\partial}\pi_2^\perp\Big) \tag{3.4}$$

is independent of λ. Hence, by (3.4), it is necessary and sufficient that

$$\begin{aligned} &\pi_2^\perp\pi_1(\overline{\partial}\pi_1)\pi_2 = 0, \qquad \pi_2\pi_1^\perp(\overline{\partial}\pi_1)\pi_2^\perp = 0, \\ &\pi_2\overline{\partial}\pi_2^\perp + \pi_2\pi_1(\overline{\partial}\pi_1)\pi_2^\perp + \pi_2\pi_1^\perp(\overline{\partial}\pi_1)\pi_2 + \pi_2^\perp\pi_1^\perp(\overline{\partial}\pi_1)\pi_2^\perp = 0. \end{aligned} \tag{3.5}$$

By multiplying $(3.5)_2$ on the left by $\pi_2^\perp$ and on the right by π_2, we can get respectively

$$\pi_2^\perp\pi_1^\perp(\overline{\partial}\pi_1)\pi_2^\perp = 0, \quad \pi_2\pi_1^\perp(\overline{\partial}\pi_1)\pi_2 = 0, \quad \pi_2\overline{\partial}\pi_2^\perp + \pi_2\pi_1(\overline{\partial}\pi_1)\pi_2^\perp = 0.$$

It is easy to verify that these conditions is equivalent to (2)–(5) in the proposition. By the proof of Lemma 2.1, we can see that the extended solution Φ_λ is symplectic if and only if the condition (1) in the proposition holds.

Corollary 3.1. *If $\pi:\Omega\to\mathcal{P}(2N)$ satisfies that $\pi^{\perp}\pi_*^\perp = 0$, then $\varphi = Q(\pi-\pi^\perp)(\pi_*-\pi_*^\perp)$ for some $Q\in Sp(N)$ is a symplectic one-uniton if and only if $\pi^\perp\overline{\partial}\pi = 0$ and $\pi_*^\perp\overline{\partial}\pi = 0$.*

Example 3.1. Let $\pi' : \Omega\times\mathbb{C}^N \to \eta'$ where η' is a holomorphic subbundle of $\Omega\times\mathbb{C}^N$, namely, $\pi'^\perp\overline{\partial}\pi' = 0$. Consider $\pi:\Omega\to\mathcal{P}(2N)$ defined by $\pi = \begin{pmatrix}\pi' & 0\\ 0 & I_N\end{pmatrix}$ and $\pi_*^\perp = \begin{pmatrix}0 & 0\\ 0 & \pi'^\perp\end{pmatrix}$. Clearly, such a π satisfies the conditions in Corollary 3.1. Hence, for $Q\in Sp(N)$,

$$\varphi = Q(\pi-\pi^\perp)(\pi_*-\pi_*^\perp) = Q\begin{pmatrix}\pi'-\pi'^\perp & 0\\ 0 & \overline{\pi}'-\overline{\pi}'^\perp\end{pmatrix} \tag{3.6}$$

is a symplectic one-uniton with the symplectic extended solution

$$\Phi_\lambda = (\pi+\lambda\pi^\perp)(\pi_* + \lambda^{-1}\pi_*^\perp).$$

More general, by Lemma 1.2, any extended n-uniton $\Phi'_\lambda = (\pi'_1 + \lambda\pi_1'^\perp)\cdots(\pi'_n + \lambda\pi_n'^\perp)$ can be used to construct a symplectic extended n-uniton

$$\Phi_\lambda = \prod_{i=1}^n(\pi_i+\lambda\pi_i^\perp)(\pi_{i*}+\lambda^{-1}\pi_{i*}^\perp) = \prod_{i=1}^n(\pi_i+\lambda\pi_i^\perp)\prod_{i=1}^n(\pi_{i*}+\lambda^{-1}\pi_{i*}^\perp),$$

where $\pi_i = \begin{pmatrix}\pi'_i & 0\\ 0 & I_N\end{pmatrix}$ for $i = 1,\cdots,n$.

It is known that the symplectic Bäcklund transformations described as in Theorem 3.1 is degenerate as $\alpha\to 0$. By Theorem 12.1 of [1], we can get the following singular Bäcklund transformations.

Proposition 3.3. *Let $\Phi_\lambda\in\mathcal{M}(G)$ and $A = \frac{1}{2}\Phi_{-1}^{-1}d\Phi_{-1}$. For $\pi_i:\Omega\to\mathcal{P}(2N)$, $i = 1.2$, satisfying $\pi_1^\perp\pi_2 = (\pi_1\pi_2^\perp)_*$, if*

$$\begin{aligned} &\pi_1 A_{\overline{z}}\pi_1^\perp = 0, \quad \pi_1^\perp(\overline{\partial}\pi_1 + A_{\overline{z}}\pi_1) = 0; \\ &\pi_2^\perp(A_{\overline{z}} + \overline{\partial}\pi_1)\pi_2 = 0, \quad \pi_2(\overline{\partial}\pi_2 - A_{\overline{z}}\pi_2^\perp - (\overline{\partial}\pi_1)\pi_2^\perp) = 0, \end{aligned} \tag{3.7}$$

then

$$\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi_1+\lambda\pi_1^\perp)(\pi_2+\lambda^{-1}\pi_2^\perp)\in\mathcal{M}(G). \tag{3.8}$$

The proof is similar to that of Proposition 3.1. So, we omit it here.

Clearly, (3.7) is the limit of (3.1) as $\alpha\to 0$. The factor $(\pi_1+\lambda\pi_1^\perp)(\pi_2+\lambda^{-1}\pi_2^\perp)$ satisfying (3.7) will be called the symplectic flag factor. For example, let Φ_λ be the symplectic extended solution given by (1.10), and $(\pi' + \lambda\pi'^{\perp})$ be the flag factor of Φ'_λ in (1.10). Then $(\pi + \lambda\pi^{\perp})(\pi_* + \lambda^{-1}\pi_*^{\perp})$ is a symplectic flag factor of Φ_λ, where $\pi = \begin{pmatrix} \pi' & 0 \\ 0 & I_N \end{pmatrix}$.

§4. Darboux Transformations

In [6,7] the Darboux transformation and the renormalization procedure are used to construct new unitons from a known one via a purely algebraic algorithm. We now will apply this method to the construction of symplectic Bäcklund factors and symplectic flag factors via double Darboux transformations.

Let $\varphi : \Omega \to Sp(N)$ be a symplectic uniton which has the extended solution of the form $\Phi_\lambda = \sum\limits_{\alpha=-n}^{n} T_\alpha \lambda^\alpha$ with $(T_\alpha)_* = T_{-\alpha}$ (see (1.11)). Let $\varepsilon \in \mathbb{C}^*$ satisfy $|\varepsilon| \neq 1$, and let L_1, L_2 be $k \times 2N$ and $(2N-k) \times 2N$ constant matrices respectively such that

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \in GL(2N, \mathbb{C}), \quad L_1 L_2^* = 0, \quad L_1 J L_1^t = 0. \tag{4.1}$$

We set $H_\varepsilon = \begin{pmatrix} L_1 \Phi_\varepsilon \\ L_2 \Phi_{\sigma(\varepsilon)} \end{pmatrix}$, $\Lambda_\varepsilon = \begin{pmatrix} \omega_1 I_k & 0 \\ 0 & \omega_2 I_{2N-k} \end{pmatrix}$ for $\omega_1 = 1 - \varepsilon$, $\omega_2 = 1 - \sigma(\varepsilon)$, $S_\varepsilon = H_\varepsilon^{-1} \Lambda_\varepsilon^{-1} H_\varepsilon$. It has been proved in [6,7] that

$$\Phi_\lambda^{(1)}(\varepsilon) = \Phi_\lambda (I - (1-\lambda) S_\varepsilon) \tag{4.2}$$

is a new extended solution and

$$S_\varepsilon = \frac{1}{\omega_1} \pi_\varepsilon^{\perp} + \frac{1}{\omega_2} \pi_\varepsilon, \tag{4.3}$$

where

$$\pi_\varepsilon = H_\varepsilon^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_{2N-k} \end{pmatrix} H_\varepsilon, \quad \pi_\varepsilon^{\perp} = I - \pi_\varepsilon \tag{4.4}$$

are Hermitian projections satisfying the first equation of (3.1) with $\alpha = \sigma(\varepsilon)$. Substituting (4.3) into (4.2) yields

$$\Phi_\lambda^{(1)}(\varepsilon) = \Phi_\lambda (\pi_\varepsilon + \xi_\varepsilon(\lambda) \pi_\varepsilon^{\perp}) \zeta_\varepsilon(\lambda), \tag{4.5}$$

where

$$\xi_\varepsilon(\lambda) = \frac{(\lambda - \varepsilon)(\bar{\varepsilon} - 1)}{(\bar{\varepsilon}\lambda - 1)(1 - \varepsilon)}, \quad \zeta_\varepsilon(\lambda) = \frac{\lambda - \sigma(\varepsilon)}{1 - \sigma(\varepsilon)}. \tag{4.6}$$

Following [7], π_ε and $\pi_\varepsilon^{\perp}$ can be expressed explicitly as

$$\begin{aligned} \pi_\varepsilon &= \Phi_\varepsilon^{-1} L_2^* (L_2 \Phi_{\sigma(\varepsilon)} \Phi_\varepsilon^{-1} L_2^*)^{-1} L_2 \Phi_{\sigma(\varepsilon)}, \\ \pi_\varepsilon^{\perp} &= \Phi_\varepsilon^* L_1^* (L_1 \Phi_\varepsilon \Phi_\varepsilon^* L_1^*)^{-1} L_1 \Phi_\varepsilon. \end{aligned} \tag{4.7}$$

It follows from (4.5) and (4.6) that $\Phi_\lambda^{(1)}(\varepsilon)$ is degenerate as $\lambda = \varepsilon$ or $\lambda = \sigma(\varepsilon)$. So, we put

$$\widetilde{\Phi}_\lambda(\varepsilon) = (\zeta_\varepsilon(\lambda)^{-1} \rho + \zeta_{\sigma(\varepsilon)}(\lambda)^{-1} \rho^{\perp}) \Phi_\lambda^{(1)}(\varepsilon), \tag{4.8}$$

where

$$\rho = L_2^* (L_2 L_2^*)^{-1} L_2, \quad \rho^{\perp} = L_1^* (L_1 L_1^*)^{-1} L_1. \tag{4.9}$$

Clearly, $\widetilde{\Phi}_\lambda(\varepsilon)$ is still an extended solution. From $(4.1)_2$, (4.7) and (4.9) we know that

$$\rho^{\perp} \Phi_\varepsilon \pi_\varepsilon = 0, \quad \rho \Phi_{\sigma(\varepsilon)} \pi_\varepsilon^{\perp} = 0. \tag{4.10}$$

Thus, we have

$$\widetilde{\Phi}_\varepsilon(\varepsilon) = \lim_{\lambda\to\varepsilon} \widetilde{\Phi}_\lambda(\varepsilon) = \Phi_\varepsilon\pi_\varepsilon + \rho^\perp\Phi_\varepsilon + \frac{(1-\varepsilon)(\varepsilon-\sigma(\varepsilon))}{(\overline{\varepsilon}-1)\sigma(\varepsilon)}\rho^\perp\dot{\Phi}_\varepsilon\pi_\varepsilon,$$

$$\widetilde{\Phi}_{\sigma(\varepsilon)}(\varepsilon) = \rho\Phi_{\sigma(\varepsilon)} + \Phi_{\sigma(\varepsilon)}\pi_\varepsilon^\perp + \frac{(1-\overline{\varepsilon})(\varepsilon-\sigma(\varepsilon))}{\overline{\varepsilon}(1-\varepsilon)}\rho\dot{\Phi}_{\sigma(\varepsilon)}\pi_\varepsilon^\perp, \tag{4.11}$$

where $\dot{\Phi}_{\sigma(\varepsilon)} = (d\Phi_\lambda/d\lambda)\mid_{\lambda=\sigma(\varepsilon)}$. One then can verify directly that $\widetilde{\Phi}_\varepsilon\widetilde{\Phi}^*_{\sigma(\varepsilon)} = I$, which implies that $\widetilde{\Phi}_\lambda(\varepsilon)$ defined by (4.8) is nondegenerate for $\lambda \in \mathbb{C}^*$. Here and from now on, we denote simply $\widetilde{\Phi}_\varepsilon(\varepsilon)$ and $\widetilde{\Phi}_{\sigma(\varepsilon)}(\varepsilon)$ by $\widetilde{\Phi}_\varepsilon$ and $\widetilde{\Phi}_{\sigma(\varepsilon)}$.

Set $\widetilde{H}_\varepsilon = \begin{pmatrix} \overline{L}_1 J\widetilde{\Phi}_{\sigma(\varepsilon)} \\ \overline{L}_2 J\widetilde{\Phi}_\varepsilon \end{pmatrix}$, $\widetilde{\Lambda}_\varepsilon = \begin{pmatrix} \omega_2 I_k & 0 \\ 0 & \omega_1 I_{2N-k} \end{pmatrix}$, $\widetilde{S}_\varepsilon = \widetilde{H}_\varepsilon^{-1}\widetilde{\Lambda}_\varepsilon^{-1}\widetilde{H}_\varepsilon$. Then, by the same reason as above,

$$\Phi_\lambda^{(2)}(\varepsilon) = \widetilde{\Phi}_\lambda(\varepsilon)(I-(1-\lambda)\widetilde{S}_\varepsilon) \tag{4.12}$$

is a new extended solution and

$$\widetilde{S}_\varepsilon = \frac{1}{\omega_2}\widetilde{\pi}_\varepsilon^\perp + \frac{1}{\omega_1}\widetilde{\pi}_\varepsilon, \tag{4.13}$$

where

$$\widetilde{\pi}_\varepsilon = \widetilde{H}_\varepsilon^{-1}\begin{pmatrix} 0 & 0 \\ 0 & I_{2N-k}\end{pmatrix}\widetilde{H}_\varepsilon, \quad \widetilde{\pi}_\varepsilon^\perp = I - \widetilde{\pi}_\varepsilon \tag{4.14}$$

are Hermitian projections satisfying the second equation of (3.1) with $\alpha = \varepsilon$ related to Φ_λ. Moreover, they can be expressed explicitly as

$$\widetilde{\pi}_\varepsilon = \widetilde{\Phi}_\varepsilon^* J L_2^t(\overline{L}_2 J\widetilde{\Phi}_\varepsilon\widetilde{\Phi}_\varepsilon^* J L_2^t)^{-1}\overline{L}_2 J\widetilde{\Phi}_\varepsilon,$$

$$\widetilde{\pi}_\varepsilon^\perp = \widetilde{\Phi}_\varepsilon^{-1} J L_1^t(\overline{L}_1 J\widetilde{\Phi}_{\sigma(\varepsilon)}\widetilde{\Phi}_\varepsilon^{-1} J L_1^t)^{-1}\overline{L}_1 J\widetilde{\Phi}_{\sigma(\varepsilon)}. \tag{4.15}$$

By inserting (4.5), (4.8) and (4.13) into (4.12), we obtain finally

$$\Phi_\lambda^{(2)}(\varepsilon) = (\zeta_\varepsilon(\lambda)\rho^\perp + \zeta_{\sigma(\varepsilon)}(\lambda)\rho)\Phi_\lambda R_\varepsilon(\lambda), \tag{4.16}$$

where

$$R_\varepsilon(\lambda) = (\pi_\varepsilon + \xi_\varepsilon(\lambda)\pi_\varepsilon^\perp)(\widetilde{\pi}_\varepsilon + \xi_\varepsilon(\lambda)^{-1}\widetilde{\pi}_\varepsilon^\perp) \tag{4.17}$$

is just the symplectic Bäcklund factor described as in Theorem 2.2. In fact, it follows from (4.7) and (4.15) that

$$\operatorname{range}\pi_\varepsilon = \operatorname{span}\{\Phi_\varepsilon^{-1}L_2^*\}, \quad \operatorname{range}\widetilde{\pi}_\varepsilon = \operatorname{span}\{\widetilde{\Phi}_\varepsilon^* J L_2^t\},$$

$$\operatorname{range}\rho = \operatorname{span}\{L_2^*\}, \quad \operatorname{range}\rho_* = \operatorname{span}\{JL_2^t\},$$

which imply that π_ε and $\widetilde{\pi}_\varepsilon$ are respectively $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ described as in Theorem 2.2, where $\pi_1 = \rho$ and $\pi_2 = \rho_*$. On the other hand, $(4.1)_3$ yields that $\rho^\perp\rho_*^\perp = 0$. So, by Theorem 2.2, we have $\pi_\varepsilon^\perp\widetilde{\pi}_\varepsilon = (\pi_\varepsilon\widetilde{\pi}_\varepsilon^\perp)_*$, which implies that $R_\varepsilon(\lambda)$ defined by (4.17) is symplectic. Hence, we have proved the following

Theorem 4.1. *Let $\Phi_\lambda \in \mathcal{M}^n(G)$, and π_ε and $\widetilde{\pi}_\varepsilon$ be defined by (4.7) and (4.15) respectively. Then, $R_\varepsilon(\lambda)$ defined by (4.17) is a symplectic Bäcklund factor, so that $(\zeta_\varepsilon(\lambda)^{-1}\rho_*^\perp + \zeta_{\sigma(\varepsilon)}(\lambda)^{-1}\rho_*)\Phi_\lambda^{(2)}(\varepsilon) \in \mathcal{M}^n(G)$, where $\Phi_\lambda^{(2)}(\varepsilon)$ is defined by (4.16).*

We now consider singular Darboux transformations. Since $\Phi_\lambda = \sum_{\alpha=-n}^{n} T_\alpha\lambda^\alpha$, by a renormalization procedure for $\varepsilon^n\Phi^*_{\sigma(\varepsilon)}L_2^*$ as in [7], we can show $\pi_\varepsilon \to \pi$ as $\varepsilon \to 0$. Moreover,

$\operatorname{rank}\pi = \operatorname{rank}(\lim_{\varepsilon\to 0}\pi_\varepsilon) = 2N-k$ as in (4.4). Similarly, $\pi_\varepsilon^\perp \to \pi^\perp$ as $\varepsilon \to 0$. It follows from (4.10) that

$$\rho^\perp T_{-n}\pi = 0, \quad \rho T_n \pi^\perp = 0. \tag{4.18}$$

Thus, (4.11) together with (4.18) yields

$$\lim_{\varepsilon\to 0}\varepsilon^{n+1}\widetilde{\Phi}_\varepsilon = 0, \quad \lim_{\varepsilon\to 0}\overline{\varepsilon}^{n+1}\widetilde{\Phi}_{\sigma(\varepsilon)} = 0. \tag{4.19}$$

(4.19) implies that we have the Laurent expression $\widetilde{\Phi}_\varepsilon = \sum_{\alpha=-n}^{n}\widetilde{T}_\alpha\varepsilon^\alpha$ for $\varepsilon(\neq 0)$ small enough.

Similarly, by a renormalization procedure for $\overline{\varepsilon}^n\widetilde{\Phi}_\varepsilon^* J L_2^t$, we can show $\widetilde{\pi}_\varepsilon \to \widetilde{\pi}$ as $\varepsilon \to 0$ and $\operatorname{rank}\widetilde{\pi} = 2N-k$. Hence, from (4.16) and (4.17) we have

$$\lim_{\varepsilon\to 0}\Phi_\lambda^{(2)}(\varepsilon) = (\rho^\perp + \lambda\rho)\Phi_\lambda(\pi + \lambda\pi^\perp)(\widetilde{\pi} + \lambda^{-1}\widetilde{\pi}^\perp) \tag{4.20}$$

with $\pi^\perp\widetilde{\pi} = (\pi\widetilde{\pi}^\perp)_*$. Thus, we have proved the following

Theorem 4.2. *Let $\Phi_\lambda \in \mathcal{M}^n(G)$, and π and $\widetilde{\pi}$ be Darboux limits of π_ε and $\widetilde{\pi}_\varepsilon$, repectively, as $\varepsilon \to 0$. Then $R(\lambda) = (\pi + \lambda\pi^\perp)(\widetilde{\pi} + \lambda^{-1}\widetilde{\pi}^\perp)$ is a symplectic flag factor of Φ_λ, so that $\Phi_\lambda R(\lambda) \in \mathcal{M}(G)$.*

The constant matrices L_1, L_2 as above always exist. In fact, the condition $(4.1)_3$ implies that $\rho^\perp\rho_*^\perp = 0$ so that $k \le N$. On putting $L_1 = (C \quad D)$ where C and D are $k\times N$ matrices, we see that $L_1 J L_1^t = 0$ only if CD^t is a symmetric $k \times k$ matrix. For example, we may take $C = D$. On the other hand, the condition $(4.1)_3$ can be replaced by $L_2 J L_2^t = 0$. In such a case, we have $\rho\rho_* = 0$ so that $k \ge N$. Therefore, by choosing L_1 and L_2 suitably, we can construct symplectic flag factors of arbitrary rank via a purely algebraic algorithm. We think that such symplectic flag factors may be used to factorize the symplectic extended unitons. This problem will be studied in a forthcoming paper.

References

[1] Uhlenbeck, K., Harmonic maps into Lie groups (classical solutions of the chiral model) [J], *J. Diff. Geom.*, **30**(1989), 1–50.

[2] Valli, G., On the energy spectrum of harmonic 2-spheres in unitary groups [J], *Topology*, **27**(1988), 129–136.

[3] Wood, J. C., Explicit construction and parametrization of harmonic two-spheres in the unitary group [J], *Proc. London Math. Soc.*, **59**(1989), 608–624.

[4] Dong, Y. X. & Shen, Y. B., Factorization and uniton numbers for harmonic maps into the unitary group $U(N)$ [J], *Science in China* (*Ser. A*), **39**(1996), 589–597.

[5] Burstall, F. E. & Guest, M. A., Harmonic two-spheres in compact symmetric spaces, revised [J], *Math. Ann.*, **309**(1997), 541–572.

[6] Gu, C. H. & Hu, H. S., Explicit construction of harmonic maps from R^2 to $U(N)$ [J], *Chin. Ann. of Math.*, **16B**(1995), 139–152.

[7] Gu, C. H. & Hu, H. S., Construction of unitons via purely algebraic algorithm [J], *Chin. Ann. of Math.*, **20B**(1999), 1–6.

[8] Gu, C. H., Dong, Y. X. & Shen, Y. B., Constructive factorizations of unitons via singular Darboux transformations [J], *Science in China* (*Ser. A*), **29**(1999), 961–968.